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Fixed point results for multivalued operators with respect to a c - distance

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ABSTRACT. In this paper we present a fixed point theorem for contractive type multivalued operators in cone metric spaces by using the concept of c-distance.

1. INTRODUCTION

Banach contraction principle, which appeared around 1922, is a very useful tool in the theory of metric spaces, with several applications to differential and integral equations, fractal theory and several other topics. In 1969, Nadler [23] extended the Banach contraction principle from singlevalued to multivalued mappings.

The existence of fixed points for various classes of multivalued contractive mappings has been studied also by many authors under different conditions. See, for instance, [9] and [10]. Nadler Theorem has been modified and generalized by many authors in metric fixed point theory. These generalizations often consist in weaker forms of the assumption of contractivity of the involved mappings, although quite often with some additional requirements, as for instance to take compact values. See for example the fixed point results for multivalued mappings of generalized contractive type of Reich (1972) [25], L. Ćirić (1972) [9], V. M Sehgal and R. E. Smithson (1980) [28].

In 1976 J. Caristi proved the following famous generalization of the Banach contraction theorem: Let (M, d) be a complete metric space, $f : M \to M$, and $\phi : M \to [0, \infty)$ lower semi-continuous. If $d(x, f(x)) \leq \phi(x) - \phi(f(x))$ for all $x \in M$, then f has a fixed point. The relevance of this theorem is their close relationship with several important results in optimization theory as, for instance, the famous Ekeland variational principle. Mizoguchi and Takahashi (1989) [22] gave a set valued version of Caristi theorem.

Y. Feng and S. Liu defined in [12] a kind of contractivity for multivalued mappings, which again focuses the requirements on some orbits of the mapping under consideration. The main fixed point theorem is also a proper generalization of Nadler's Theorem.

In 2007 D. Klim and D. Wardowski [18] inspired by Mizoguchi-Takahashi and Feng-Liu work, obtained a further generalization of the previous fixed point results given in [12], [22], [25].

O. Kada, T. Suzuki and W. Takahashi [16] introduced in 1996 the concept of ω-distance on a metric space and by using this notion they got an improvement of the Takahashi nonconvex optimization Theorem, as well as generalizations of Caristi fixed point theorem and Ekeland variational principle. They also gave fixed point theorems for singlevalued mappings of ω-contractive type.

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In 2009 J. García-Falset, L. Guran and E. Llorens-Fuster [11] obtained a generalization of the fixed point results presented by D. Klim and D. Wardowski (Theorem 2.1 of [18]), for multivalued mappings of contractive type in complete metric spaces, but using the concept of ω-distance. Similar results can be found in [20] (2001).

The fixed point theory for cone metric spaces was introduced by Huang and Zhang [13] in 2007 and become a subject of interest for many authors. Cone metric spaces are generalizations of metric spaces where the metric is replaced by a mapping taking values in an ordered Banach space. For some more results regarding fixed point theory and applications in cone metric spaces see for example [1]-[8], [14], [15], [17], [24], [29]-[31]. Huang and Zhang [13] introduced the basic definitions, (although this structure already existed under the name of K-metric or K-normed spaces, see [15]), and proved several properties of sequences in cone metric spaces.

Y. J. Cho, R. Saadati and S. Wang [7] introduced in 2011 a new concept of c-distance in cone metric spaces, which is a cone version of ω-distance of O. Kada, T. Suzuki and W. Takahashi [16]. They proved in [7] some fixed point theorems for contractive type mappings in partially ordered cone metric spaces using c-distance.

In these notes we will use c-distances in order to obtain a fixed point theorem for multivalued mappings which allows us to give a generalization of Theorem 3.3 presented in [11]. An example is presented which exhibits that a mapping which is not contractive with respect to the ordinary distances, becomes of contractive type in terms of c-distances.

2. Preliminaries

Here we introduce some notions which will be used in the next section.

Let $(E, \|\cdot\|)$ be a real Banach space and θ denote the zero vector in E. The set $P \subset E$ is called a cone if the following conditions are satisfied:

(i) P is closed and $P \neq \{\theta\}$,

(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$,

(iii) $x \in P \cap (-P)$ implies $x = \theta$.

We will always assume that if $P \neq \emptyset$ is such a cone, then int $P \neq \emptyset$. In other words, we only deal with the so called *solid* cones.

Example 2.1. Let ℓ_2 be the the classical real sequence space

$$\ell_2 := \{(x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

endowed with the (Euclidean) norm

$$\|(x_n)\|_2 := \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$$

It can be easily checked that the set

$$P := \{(x_n) \in \ell_2 : x_n \geq 0 \ \ (n = 1, 2, \dots)\}$$

is a cone in the Hilbert space ℓ_2, but $\mathrm{int}P = \emptyset$.

Notice that, if P is a cone, from (ii) it follows that $\theta \in P$ and that $P + P \subset P$. For a cone $P \subset E$, a *partial ordering* $\preceq$ with respect to P is defined on E by putting $v \preceq w$ if and only if $w - v \in P$. The notation $v \prec w$ stands for $v \preceq w$, but $v \neq w$. Also, we use $v \ll w$ to indicate that $w - v \in \mathrm{int}P$. It is straightforward to check that, if $v_1, v_2, w_1, w_2 \in E$, $v_1 \ll v_2$ and $w_1 \ll w_2$, then $v_1 + w_1 \ll v_2 + w_2$.

A cone P is called *normal* if there exists a number $K > 0$ such that for all $v, w \in E$,

$$\theta \preceq v \preceq w \Longrightarrow \|v\| \leq K \|w\| .$$

The smallest positive number K satisfying the above condition is called the normal constant of P.

The cone P is called regular if every increasing sequence in P which is bounded from above is convergent. That is, if (d_n) is a sequence in P such that

$$d_1 \preceq d_2 \preceq \cdots \preceq d_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\|d_n - x\| \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that every regular cone is a normal cone. (See, for instance, [26, Lemma 1.1.]).

Definition 2.1. (See [13]).

Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Notice that, in particular, from (1) it follows that for every $x, y \in X$ $d(x, y) - \theta = d(x, y) \in P$.

Now we give an example of cone metric space.

Example 2.2. Let $X = \mathbb{R}^2$, $E = (\mathbb{R}^3, \| \cdot \|_2)$, where $\| \cdot \|_2$ stands for the Euclidean ordinary norm in $\mathbb{R}^3$. Let $P = \{(z_1, z_2, z_3) \in E : z_i \geq 0, i = 1, 2, 3\}$, and $d : X \times X \to E$ defined as

$$d(x, y) = (d_\infty(x, y), d_2(x, y), d_1(x, y)),$$

where d_∞ is the Chebyshev metric, d_2 is the Euclidean metric and d_1 is the Minkowski metric in $\mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a cone metric space. Notice that $d((1, 2)(-1, 0)) = (2, 2\sqrt{2}, 4)$, while $d((1, 2)(1, -1)) = (3, 3, 3)$. Hence, $d((1, 2)(-1, 0)) \not\preceq d((1, 2)(1, -1))$ and $d((1, 2)(1, -1)) \not\preceq d((1, 2)(-1, 0))$.

On the other hand, if $z = (z_1, z_2, z_3)$ and $z' = (z_1', z_2', z_3')$ are vectors in $\mathbb{R}^3$ such that

$$(0, 0, 0) \preceq (z_1, z_2, z_3) \preceq (z_1', z_2', z_3')$$

then

$$\begin{array}{c} 0 \leq z_1 \leq z_1' \\ 0 \leq z_2 \leq z_2' \\ 0 \leq z_3 \leq z_3' \end{array}$$

and hence

$$\|z\|_2 \leq \|z'\|_2.$$

Thus, the cone P is normal, with normal constant $M = 1$. Notice that, according with [26, Lemma 2.1.], there is no normal cone with normal constant $M < 1$. It is also straightforward to check that the cone P is regular.

From now, we will assume that a real Banach space E and a solid cone P are given, as well as that a cone metric space (X, d) is also given with respect to E and P. For the sake of brevity we will omit hereafter to mention E and P if no confusion arises.

Definition 2.2. (See [13]).

Let (X, d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$.

(1) If, for any $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then the sequence (x_n) is said to be **convergent** to the point $x \in X$ and x is the (d-)limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) If, for any $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then (x_n) is called a **Cauchy sequence** in X.

(3) The space (X, d) is called a complete cone metric space if every Cauchy sequence is convergent to a point of X.

Lemma 2.1. [13, Lemmata 1, 4 and 5]

Let (X, d) be a cone metric space and P be a normal cone with normal constant K.

1. *Let (x_n) be a sequence in X. Then (x_n) converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ in E.*
2. *Let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ in E $(m, n \to \infty)$.*
3. *Let (x_n) and (y_n) be two sequences in X with $x_n \to x$ and $y_n \to y$. Then $d(x_n, y_n) \to d(x, y)$.*

Indeed, in this setting some more can be said.

Lemma 2.2. See [15, Sect. 4].

Let (X, d) be a cone metric space and P be a normal cone with normal constant $K \geq 1$. Then the mapping $\delta : X \times X \to [0, \infty)$ defined as $\delta(x, y) = \|d(x, y)\|$ has the following properties:

1. *For all $x, y \in X$, $\delta(x, y) = 0$ if and only if $x = y$;*
2. *For all $x, y \in X$, $\delta(x, y) = \delta(y, x)$;*
3. *For all $x, y, z \in X$, $\delta(x, y) \leq K(\delta(x, z) + \delta(z, y))$.*

If (X, d) is a cone metric space, a set $A \subset X$ is called *closed* if for any sequence $\{x_n\} \subset A$ convergent to x we have $x \in A$.

A set $A \subset X$ is said to be sequentially compact if for any sequence $\{x_n\} \subset A$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is convergent to an element of A.

We denote $N(X)$ the collection of all nonempty subsets of X and $C(X)$ the collection of all nonempty closed subsets of X and $K(X)$ the collection of all nonempty sequentially compact subsets of X.

For further definitions and properties of cone metric spaces see [13, 26] and [24].

The concept of c-distance on a cone metric space (X, d), which was introduced by Y. J. Cho, R. Saadati and S. Wang in [7], is a generalization of the concept of ω-distance given by O. Kada, T. Suzuki and W. Takahashi in [16].

Definition 2.3. (See [7]). Let (X, d) be a cone metric space. Then a function $q : X \times X \to E$ is called a c-distance on X if the following conditions are satisfied:

(q1) For all $x, y \in X$, $\theta \preceq q(x, y)$.

(q2) For all $x, y, z \in X$, $q(x, z) \preceq q(x, y) + q(y, z)$.

(q3) For each $x \in X$ and $n \geq 1$, if there exists $u = u_x \in P$ such that $q(x, y_n) \preceq u$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to $y \in X$.

(q4) For any $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Now we give some examples of c-distances on the cone metric space $(\mathbb{R}^2, d)$ which we considered in Example 2.2.

Example 2.3. Let $q : X \times X \to \mathbb{R}^3$, defined by $q(x,y) = d(x,y)$, for all $x, y \in X$. Then q is a c-distance. Indeed, q obviously satisfies conditions (q1) and (q2). From Lemma 2.1 it follows that q satisfies (q3). Indeed, if (y_n) is a sequence in X with $y_n \to y \in X$ and for $x, u \in X$ one has that

$$q(x, y_n) \preceq u\ ,$$

that is, $u - d(x, y_n) \in P$ for $n \geq 1$. Taking $x_n = x$ for $n = 1, 2, \ldots$, according Lemma 2.1 one has that $d(x_n, y_n) \to d(x, y)$, and hence

$$u - d(x_n, y_n) \to u - d(x, y)$$

in $(\mathbb{R}^3, \|\cdot\|)$. Since $q(x, y_n) \preceq u$, then for each positive integer n

$$u - q(x, y_n) = u - d(x_n, y_n) \in P.$$

Therefore, tacking into account that P is a closed set, one follows that $u - d(x,y) \in P$, that is, $q(x,y) \preceq u$.

Finally, let $c := (c_1, c_2, c_3) \in \mathbb{R}^3$ with $(0,0,0) \ll (c_1, c_2, c_3)$. This means that $(c_1, c_2, c_3) \in \text{int}P$, and hence $e := (e_1, e_2, e_3) := (\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_3}{2}) \in \text{int}P$. Let us suppose that, for $x, y, z \in X$, $q(z, x) \ll e$ and $q(z, y) \ll e$.

Then $q(x,y) \preceq q(x,z) + q(z,y) \ll (e_1, e_2, e_3) + (e_1, e_2, e_3) = c$. So (q4) is satisfied. Hence q is a c-distance.

Example 2.4. Let $q : X \times X \to \mathbb{R}^3$, defined by

$$q(x,y) = d(\theta, x) + d(\theta, y)$$

for $x, y \in X$, and $\theta = (0,0) \in \mathbb{R}^2$. Hence

$$\begin{aligned} q(x,y) &= (d_\infty(\theta,x), d_2(\theta,x), d_1(\theta,x)) + (d_\infty(\theta,y), d_2(\theta,y), d_1(\theta,y)) \\ &= (\|x\|_\infty + \|y\|_\infty, \|x\|_2 + \|y\|_2, \|x\|_1 + \|y\|_1). \end{aligned}$$

Then q is a c-distance. Indeed, it immediately satisfies condition (q1). Since

$$q(x,z) := d(\theta,x) + d(\theta,z) \leq d(\theta,x) + d(\theta,y) + d(\theta,y) + d(\theta,z) = q(x,y) + q(y,z),$$

also (q2) holds for q.

If (y_n) is a sequence in X with $y_n \to y \in X$ and for $x, u \in X$ one has that

$$q(x, y_n) \preceq u\ ,$$

that is,

$$d(\theta, x) + d(\theta, y_n) \preceq u\ ,$$

Then $u - (d(\theta,x) + d(\theta,y_n)) \in P$ for $n \geq 1$. Taking $x_n = \theta$ for $n = 1, 2, \ldots$, according Lemma 2.1 one has that $d(\theta, y_n) \to d(\theta, y)$, and hence

$$u - (d(\theta,x) + d(\theta,y_n)) \to u - (d(\theta,x) + d(\theta,y)$$

in $(\mathbb{R}^3, \|\cdot\|)$. Tacking into account that P is a closed set, one follows that $u - (d(\theta,x) + d(\theta,y) \in P$, that is, $q(x,y) \preceq u$.

Let $c := (c_1, c_2, c_3) \in \mathbb{R}^3$ with $(0,0,0) \ll (c_1, c_2, c_3)$. As in the above examples, this means that $(c_1, c_2, c_3) \in \text{int}P$, and hence $e := (e_1, e_2, e_3) := (\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_3}{2}) \in \text{int}P$. Let us suppose that, for $x, y, z \in X$, $q(z,x) \ll e$ and $q(z,y) \ll e$. Then

$$\begin{aligned} d(x,y) &\preceq d(x,\theta) + d(\theta,y) \\ &\preceq d(\theta,x) + d(\theta,z) + d(\theta,y) + d(\theta,z) \\ &= q(x,z) + q(y,z) \\ &\ll e + e = c. \end{aligned}$$

This shows that the mapping q satisfies (q4). Thus, we have obtained that q is a c-distance.

Example 2.5. Let $q : X \times X \to \mathbb{R}^3$, defined by

$$q(x,y) = d(\theta, y) = (\|y\|_\infty, \|y\|_2, \|y\|_1).$$

for all $x, y \in X$. Indeed, q immediately satisfies condition (q1). Since for all $x, y, z \in X$

$$q(x,z) = q(y,z) \leq q(x,y) + q(y,z),$$

one has that q satisfies condition (q2). Let $c := (c_1, c_2, c_3) \in \mathbb{R}^3$ with $(0,0,0) \ll (c_1, c_2, c_3)$. As in the above examples, this means that $(c_1, c_2, c_3) \in \mathrm{int}P$, and hence $e := (e_1, e_2, e_3) := (\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_3}{2}) \in \mathrm{int}P$. Let us suppose that, for $x, y, z \in X$, $q(z,x) \ll e$ and $q(z,y) \ll e$. Then

$$\begin{aligned} d(x,y) &\preceq d(x,\theta) + d(\theta,y) = d(\theta,x) + d(\theta,y) = q(z,x) + q(z,y) \\ &\ll e + e = c. \end{aligned}$$

We get that q satisfies (q4), and hence it is a c-distance.

In [7] some further nontrivial examples of c-distances are given. The above example can be used to check the following warnings, regarding the concept of c-distance, as Y. J. Cho, R. Saadati and S. Wang [7] remarked.

Remark 2.1. Let q be a c-distance on a cone metric space (X,d). Then
(1) $q(x,y) = q(y,x)$ does not necessarily hold for all $x, y \in X$,
(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

An important result in order to obtain fixed point theorems by using c-distances is the following.

Lemma 2.3. (See [7, Lemma 2.12]).
Let (X,d) be a cone metric space, and let q be a c-distance on X. Let (x_n) and (y_n) be sequences in X and $x, y, z \in X$. Suppose that (u_n) is a sequence in P converging to θ. Then the following hold:
(1) if $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$;
(2) if $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then (y_n) converges to a point $z \in X$;
(3) if $q(x_n, x_m) \preceq u_n$ for each $m > n$, then (x_n) is a Cauchy sequence in X;
(4) if $q(y, x_n) \preceq u_n$, then (x_n) is a Cauchy sequence in X.

Y. Feng and S. Liu [12] obtained an extension of Nadler's fixed point theorem in complete metric spaces in the following way:
Let (X,d) be a metric space. Let $T : X \to N(X)$ be a multivalued mapping. Define the function $f : X \to \mathbb{R}$ as $f(x) = d(x, T(x))$.
For a constant $b \in (0,1)$ and $x \in X$, define the set $I_b^x \subset X$ as

$$I_b^x = \{y \in T(x) : bd(x,y) \leq d(x, T(x))\}.$$

Theorem 2.1. (See [12]).
Let (X,d) be a complete metric space, $T : X \to C(X)$ be a multivalued mapping. If there exists a constant $c \in (0,1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying

$$d(y, T(y)) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and f is lower semi-continuous.

D. Wardowski, in [33], inspired by the work of Y. Feng and S. Liu [12], introduced the concept of set-valued contractions in cone metric spaces and obtained a fixed point theorem, considering the distance between a point and a set in the following way:

Let (X, d) be a cone metric space. Let $T : X \to C(X)$. For $x \in X$, we denote

$$\begin{aligned} D(x, Tx) &= \{d(x, z) \in E : z \in Tx\}, \\ S(x, Tx) &= \{u \in D(x, Tx) : \|u\| = \inf\{\|v\| : v \in D(x, Tx)\}\}. \end{aligned}$$

Recall that a mapping $f : X \to \mathbb{R}$ is said to be lower semi-continuous at $x \in X$ (lsc for short), with respect to d, if for any sequence (x_n) in X and $x \in X$ with $x_n \to x$, the inequality $f(x) \leq \lim_{n\to\infty} \inf f(x_n)$ holds. We say that the mapping $f : X \to \mathbb{R}$ is lsc on X if it is lcs at each point $x \in X$.

Theorem 2.2. (See [32]).

Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K, and let $T : X \to K(X)$. Assume that the function $I : X \to \mathbb{R}$ defined by $I(x) = \inf_{y \in Tx} \|d(x, y)\|$, $x \in X$, is lower semi-continuous. Then, the following statements hold:

(i) If there exists $\lambda \in [0, 1), b \in (\lambda, 1]$ such that

$$\forall_{x\in X} \exists_{y\in Tx} \exists_{v\in S(y,Ty)} \forall_{u\in S(x,Tx)} \ \{[bd(x, y) \preceq u] \wedge [v \preceq \lambda d(x, y)]\},$$

then $Fix(T) \neq \emptyset$.

(ii) If there exist $\lambda \in [0, 1), b \in (\lambda, 1]$ such that

$$\forall_{x\in X} \exists_{y\in Tx} \exists_{v\in S(y,Ty)} \forall_{u\in S(x,Tx)} \ \{[bd(x, y) \preceq u] \wedge [v \preceq \lambda d(x, y)]\},$$

then $Fix(T) = End(T) = \emptyset$.

Let $T : X \to K(X)$, $b \in (0, 1]$ and $x \in X$. We will consider the following set:

$$I_b^x := \{y \in T(x) : bd(x, y) \leq S(x, T(x))\},$$

3. Fixed point results

We present now a fixed point theorem for multivalued operators on cone metric spaces endowed with a c-distance.

We need the following notation:

Let $T : X \to K(X)$ be a multivalued mapping, and let q be a c-distance on X. Define the function $f : X \to \mathbb{R}$ by

$$f(x) := D_q(x, T(x)),$$

where

$$D_q(x, T(x)) = \inf_{y\in T(x)} \|q(x, y)\|.$$

Given $x \in X$, for each $b \in [0, 1]$

$$I_{b,q}^x := \{y \in T(x) : b \|q(x, y)\| \leq D_q(x, T(x))\}.$$

Remark 3.2. If $T : X \to K(X)$ is a multivalued mapping and $0 < b < 1$, it is clear that, for every $x \in X$, the set $I_{b,q}^x$ is nonempty.

Theorem 3.3. *Let (X, d) be a complete cone metric space, P be a regular cone and let q be a c-distance on X and let $T : X \to K(X)$ be a multivalued mapping. Assume that the mapping $g : X \to \mathbb{R}$ defined by $g(x) = \|q(x, y)\|$, $x \in X$, is lower semicontinuous. If the following conditions hold:*

1. There exist $b \in (0, 1)$ and $\varphi : [0, \infty[\to [0, b[$ such that

(1i) for each $t \in [0, \infty[$,
$$\lim_{r \to t^+} \sup \varphi(r) < b;$$
(1ii) for every $x \in X$, *there exists* $y \in I^x_{b,q}$ *such that*
$$D_q(y, T(y)) \leq \varphi(\|q(x,y)\|)\, \|q(x,y)\|\,;$$
2. for every $y \in X$ *with* $y \notin T(y)$
$$\inf\{\|q(x,y)\| + D_q(x, T(x)) : x \in X\} > 0$$
3. If (ε_n) *is a sequence of non negative real numbers with* $\varepsilon_n \to 0$, *and* (x_n) *is a sequence in* X *with* $\|q(x_n, x_m)\| \leq \varepsilon_n$ *for* $m > n$ *and* n *large enough, then* (x_n) *is a Cauchy sequence in* (X, d).

Then T *has a fixed point.*

Proof. For $x_0 \in X$, by (1ii) there exists $x_1 \in I^{x_0}_{b,q} \subset T(x_0)$ which satisfies the following two conditions:
$$b\,\|q(x_0, x_1)\| \leq D_q(x_0, T(x_0)) \tag{3.1}$$
and
$$D_q(x_1, T(x_1)) \leq \varphi(\|q(x_0, x_1)\|)\, \|q(x_0, x_1)\| \tag{3.2}$$
Inequalities (3.1) and (3.2) yield
$$\begin{aligned} D_q(x_0, T(x_0)) - D_q(x_1, T(x_1)) &\geq b\,\|q(x_0, x_1)\| - \varphi(\|q(x_0, x_1)\|)\, \|q(x_0, x_1)\| \\ &= (b - \varphi(\|q(x_0, x_1)\|))\, \|q(x_0, x_1)\| \geq 0 \end{aligned}$$
Given x_1, there exists $x_2 \in I^{x_1}_{b,q} \subset T(x_1)$ such that
$$b\,\|q(x_1, x_2)\| \leq D_q(x_1, T(x_1)) \tag{3.3}$$
and
$$D_q(x_2, T(x_2)) \leq \varphi(\|q(x_1, x_2)\|)\, \|q(x_1, x_2)\| \tag{3.4}$$
From (3.3) and (3.4) we have
$$\begin{aligned} D_q(x_1, T(x_1)) - D_q(x_2, T(x_2)) &\geq b\,\|q(x_1, x_2)\| - \varphi(\|q(x_1, x_2)\|)\, \|q(x_1, x_2)\| \\ &= (b - \varphi(\|q(x_1, x_2)\|))\, \|q(x_1, x_2)\| \geq 0 \end{aligned}$$
From (3.3) and (3.2) we get the following inequality:
$$\|q(x_1, x_2)\| \leq \frac{1}{b} D_q(x_1, T(x_1)) \leq \frac{1}{b} \varphi(\|q(x_0, x_1)\|)\, \|q(x_0, x_1)\|$$
After an inductive process we get a sequence (x_n) of elements of X satisfying the following conditions:

(i) for every $n \in \mathbb{N}$, $x_{n+1} \in T(x_n)$;
(ii) $b\,\|q(x_n, x_{n+1})\| \leq D_q(x_n, T(x_n))$;
(iii) $D_q(x_{n+1}, T(x_{n+1})) \leq \varphi(\|q(x_n, x_{n+1})\|)\, \|q(x_n, x_{n+1})\|$.

Applying these conditions (i), (ii), (iii) we have that for each $n \in \mathbb{N}$, the following inequalities:
$$\begin{cases} D_q(x_n, T(x_n)) \geq D_q(x_{n+1}, T(x_{n+1})), \\ \qquad \|q(x_n, x_{n+1})\| \leq \|q(x_{n-1}, x_n)\| \end{cases} \tag{3.5}$$
$$D_q(x_n, T(x_n)) - D_q(x_{n+1}, T(x_{n+1})) \geq (b - \varphi(\|q(x_n, x_{n+1})\|))\, \|q(x_n, x_{n+1})\| \geq 0$$
hold.

Inequalities (3.5) in turn implies that $(\|(q(x_n, x_{n+1}))\|)$ is convergent to $t \in [0, \infty[$.

By assumption (1i) we may find $p \in [0, b)$ such that

$$\lim_{n\to\infty} \sup \varphi(\|q(x_n, x_{n+1})\|) \leq \lim_{r\to t^+} \sup \varphi(r) = p.$$

Thus, given $b_0 \in (p, b)$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$\varphi(\|q(x_n, x_{n+1})\|) < b_0. \tag{3.6}$$

Consequently, for any $n \geq n_0$ we have:

$$\begin{aligned} & D_q(x_n, T(x_n)) - D_q(x_{n+1}, T(x_{n+1})) \\ \geq\ & [b - \varphi(\|q(x_n, x_{n+1})\|)] \,\|q(x_n, x_{n+1})\| \geq \alpha \,\|q(x_n, x_{n+1})\| \end{aligned} \tag{3.7}$$

where $\alpha = b - b_0$.

For each $n > n_0$, inequalities (ii), (iii) and (3.6) yield.

$$\begin{aligned} D_q(x_{n+1}, T(x_{n+1})) &\leq \varphi(\|q(x_n, x_{n+1})\|) \,\|q(x_n, x_{n+1})\| \\ &\leq b_0 \frac{1}{b} D_q(x_n, T(x_n)) \leq \ldots \leq \left(\frac{b_0}{b}\right)^{n-n_0} D_q(x_0, T(x_0)). \end{aligned} \tag{3.8}$$

Since $b_0 < b$, $\lim_{n\to\infty} \left(\frac{b_0}{b}\right)^{n-n_0} = 0$. This means that $\lim_{n\to\infty} D_q(x_n, T(x_n)) = 0$.

If $m > n > n_0$, by (3.7) we get that

$$\begin{aligned} \|q(x_n, x_m)\| &= \sum_{s=n}^{m-1} \|q(x_s, x_{s+1})\| \leq \frac{1}{\alpha} \sum_{s=n}^{m-1} (D_q(x_s, T(x_s)) - D_q(x_{s+1}, T(x_{s+1}))) \\ &= \frac{1}{\alpha}(D_q(x_n, T(x_n)) - D_q(x_m, T(x_m))) \\ &\leq \frac{1}{\alpha} D_q(x_n, T(x_n)). \end{aligned}$$

We obtain that $\|q(x_n, x_m)\| \to 0$. From assumption 3 we get that (x_n) is a Cauchy sequence in (X, d).

Since (X, d) is complete, then (x_n) is a convergent sequence.

Let $z \in X$ be the limit of the sequence (x_n). Assume that $z \notin T(z)$. Since for each $x \in X$, $g(x) = \|q(x, y)\|$ is lower semicontinuous, for every $n > n_0$ we have that

$$\|q(x_n, z)\| \leq \lim_{m\to\infty} \inf \|q(x_n, x_m)\| \leq \frac{1}{\alpha} D_q(x_n, T(x_n)) \tag{3.9}$$

By asumption (2) and using (3.9) we get that

$$\begin{aligned} 0 &< \inf \{\|q(x, z)\| + D_q(x, T(x)) : x \in X\} \\ &\leq \inf \{\|q(x_n, z)\| + D_q(x_n, T(x_n)) : n > n_0\} \\ &\leq \inf \left\{(1 + \frac{2}{\alpha}) D_q(x_n, T(x_n)) : n > n_0\right\} = \lim_{n\to\infty} (1 + \frac{2}{\alpha}) D_q(x_n, T(x_n)) = 0 \end{aligned}$$

This is a contradiction, so we conclude that $z \in T(z)$. □

Example 3.6. Consider the set $X := [0, 1] \times [0, 1]$. On this set we assume defined the cone metric d introduced in Example 2.2, i.e. $d : X \times X \to \mathbb{R}^3$ defined for $x, y \in X$ as

$$d(x, y) = (d_\infty(x, y), d_2(x, y), d_1(x, y)) = (\|x - y\|_\infty, \|x - y\|_2, \|x - y\|_1).$$

On $\mathbb{R}^3$ we suppose defined the Euclidean norm $\|\cdot\|$, and the cone P under consideration is just the positive octant $P = \{(u_1, u_2, u_3) : u_1 \geq 0, u_2 \geq 0, u_3 \geq 0\}$. For a sequence (x_n) in X, and $x \in$ it is easy to check that $x_n \to x$ in the cone metric space (X, d) if and only if

$\|x_n - x\|_2 \to 0$. Hence a subset A of X is closed (sequentially compact) if and only if A is closed (compact) in the usual topology of X.

With respect to the cone metric space (X, d) we define the c-distance considered in Example 2.4, that is $q : X \to \mathbb{R}^3$

$$q(x, y) = (\|x\|_\infty + \|y\|_\infty, \|x\|_2 + \|y\|_2, \|x\|_1 + \|y\|_1).$$

From the continuity of the norms in $\mathbb{R}^2$ and $\mathbb{R}^3$ it follows that (for each $y \in X$) the functions $x \mapsto g(x) := \|q(x, y)\|$ are continuous and hence lsc.

Notice also that if $x, y \in X$

$$\begin{aligned}\|d(x, y)\| &= \|(\|x - y\|_\infty, \|x - y\|_2, \|x - y\|_1)\| \\ &\leq \|(\|x\|_\infty + \|y\|_\infty, \|x\|_2 + \|y\|_2, \|x\|_1 + \|y\|_1)\| \\ &= \|q(x, y)\|.\end{aligned}$$

Thus, q satisfies Assumption 3 of the above theorem.

We define the multivalued operator $T : X \to K(X)$ given by

$$T(x) = T((x_1, x_2) = \text{conv}\{(0, 0), (x_1, 0), (0, x_2)\}.$$

Notice that the set $T(x)$ is the solid triangle of vertices $(0, 0)$, $(x_1, 0)$ and $(0, x_2)$. It is obvious to check that the set of fixed points of T is just $Fix(T) = \{(x_1, x_2) \in X : x_1x_2 = 0\}$, which is not convex in $\mathbb{R}^2$.

If $x \in X$, bearing in mind that $y = (0, 0) \in T(x)$ and that the Euclidean norm is monotone,

$$\begin{aligned}D_q(x, T(x)) &:= \inf\{\|q(x, y)\| : y \in T(x)\} \\ &= \inf\{\|(\|x\|_\infty + \|y\|_\infty,\ \|x\|_2 + \|y\|_2,\ \|x\|_1 + \|y\|_1)\| : y \in T(x)\} \\ &= \|(\|x\|_\infty,\ \|x\|_2,\ \|x\|_1)\|\,.\end{aligned}$$

Given $x \in X$, for each $b \in [0, 1]$ we have that:

$$\begin{aligned}I^x_{b,q} &:= \{y \in T(x) : b\,\|q(x, y)\| \leq D_q(x, T(x))\} \\ &= \{y \in T(x) : b\,\|(\|x\|_\infty + \|y\|_\infty, \|x\|_2 + \|y\|_2, \|x\|_1 + \|y\|_1)\| \leq \|(\|x\|_\infty,\ \|x\|_2,\ \|x\|_1)\|\}\end{aligned}$$

Therefore, $y = (0, 0) \in I^x_{b,q}$. In summary, taking, for instance $\varphi(t) = \frac{1}{2}t$ and $b = \frac{1}{2}$, given $x \in X$, there exists $y = (0, 0) \in I^x_{b,q}$ such that

$$D_q(y, T(y)) \leq \varphi(\|q(x, y)\|)\,\|q(x, y)\|$$

because $T(0, 0) = \{(0, 0)\}$ and we know that, $D_q((0, 0), T((0, 0))) = \|q((0, 0), (0, 0))\| = \|(0, 0, 0)\| = 0$.

Finally we will check that the mapping T satisfies assumption 2 of the above theorem, that is, for every $y \in X$ with $y \notin T(y)$

$$\inf\{\|q(x, y)\| + D_q(x, T(x)) : x \in X\} > 0$$

Indeed it holds because, if $y \in X$ such that $y \notin T(y)$ then we know that $y \neq (0, 0)$ and hence

$$\begin{aligned}\|q(x, y)\| + D_q(x, T(x)) &= \|(\|x\|_\infty + \|y\|_\infty, \|x\|_2 + \|y\|_2, \|x\|_1 + \|y\|_1)\| \\ &\qquad\qquad\qquad\qquad + \|(\|x\|_\infty,\ \|x\|_2,\ \|x\|_1)\| \\ &\geq \|(\|y\|_\infty, \|y\|_2, \|y\|_1)\|\,.\end{aligned}$$

Thus,

$$\inf\{\|q(x, y)\| + D_q(x, T(x)) : x \in X\} \geq \|(\|y\|_\infty, \|y\|_2, \|y\|_1)\| > 0.$$

Remark 3.3. The set-valued mapping T considered in the above example was first defined by K. M. Ko in [19]. Let us point out that this mapping is nonexpansive on X, that is, it satisfies that for every $x, y \in X$

$$H_2(T(x), T(y)) \leq d_2(x, y)$$

where H_2 stands for the well known Hausdorff-Pompeiu metric on $C(X)$ associated to the Euclidean metric d_2. Therefore, the operator T falls into the scope of the corresponding fixed point theorems for nonexpansive set-valued operators, for instance those which are due to Markin (1965) [21].

However, is should be noted that our main theorem lies upon the notion of contractivity, which is deeply different than the notion of nonexpansivity. Thus, in this example it is exhibited that an operator which is not contractive with respect to the ordinary notions of distance, can become of contractive type under c-distances.

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