CARPATHIAN J. MATH. 31 (2015), No. 2, 233 - 240
Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Ulam-Hyers-Rassias stability of pseudoparabolic partial differential equations

NICOLAIE LUNGU and SORINA ANAMARIA CIPLEA

ABSTRACT. The aim of this paper is to give some types of Ulam stability for a pseudoparabolic partial differential equation. In this case we consider Ulam-Hyers stability and generalized Ulam-Hyers-Rassias stability. We investigate some new applications of the Gronwall lemmas to the Ulam stability of a nonlinear pseudoparabolic partial differential equations.

1. INTRODUCTION

In 1940, on a talk given at Wisconsin University, S. M. Ulam has formulated the following problem: Under what conditions does there exist near every approximately homomorphism of a given metric group an homomorphism of the group? If the answer to the previous question is affirmative, we say that the equation of the homomorphism is stable. Generally, we say that a differential equation is stable (Ulam) if for every approximate solution of the differential equation, there exists an exact solution near it ([1], [5], [6], [10], [15], [16]).

The Ulam stability is an important concept in the theory of nonlinear partial differential equations (see [13], [14], [15], [16]). Following I. A. Rus - N. Lungu ([13]), I. A. Rus ([15], [16]), in this paper we shall present two types of Ulam stability for a pseudoparabolic partial differential equation: Ulam-Hyers stability and generalized Ulam-Hyers-Rassias stability.

More precisely we shall consider the following equation ([2], [3], [4], [17], [18], [19]):

$$\frac{\partial^3 u(x,y)}{\partial x^2 \partial y} = f\left(x, y, u(x,y), \frac{\partial u(x,y)}{\partial y}, \frac{\partial^2 u(x,y)}{\partial x^2}\right), \tag{1.1}$$

$0 \le x < a,\ 0 \le y < b$, where $f \in C([0,a) \times [0,b) \times \mathbb{B}^3, \mathbb{B})$ and $(\mathbb{B}, |\cdot|)$ is a (real or complex) Banach space.

A function $u : [0,a) \times [0,b) \to \mathbb{B}$ is a solution of (1.1) if $u \in C([0,a) \times [0,b)) \cap C^1([0,a) \times [0,b))$, $\dfrac{\partial^2 u}{\partial x^2} \in C([0,a) \times (0,b))$, $\dfrac{\partial^3 u}{\partial x^2 \partial y} \in C([0,a) \times [0,b))$ and u satisfies the equation (1.1).

The equations of this type arise in the theory of the consolidation of clay and in the theory of seepage of homogeneous fluid through fissured rocks.

2. DEFINITIONS AND NOTIONS

Let $a, b \in (0, \infty]$, $\varepsilon > 0$ and $\varphi \in C([0,a) \times [0,b), \mathbb{R}_+)$.

Received: 20.09.2013; In revised form: 19.11.2014; Accepted: 01.12.2014
2010 *Mathematics Subject Classification.* 45G10, 45M10, 45N05, 47N10.
Key words and phrases. *Pseudoparabolic partial differential equations, Ulam-Hyers stability, Gronwall lemma, inequalities.*
Corresponding author: Nicolaie Lungu; nlungu@math.utcluj.ro

We consider the following inequalities:

$$\left| \frac{\partial^3 v(x,y)}{\partial x^2 \partial y} - f\left(x, y, v(x,y), \frac{\partial v(x,y)}{\partial y}, \frac{\partial^2 v(x,y)}{\partial x^2}\right) \right| \leq \varepsilon \tag{2.2}$$

$$\left| \frac{\partial^3 v(x,y)}{\partial x^2 \partial y} - f\left(x, y, v(x,y), \frac{\partial v(x,y)}{\partial y}, \frac{\partial^2 v(x,y)}{\partial x^2}\right) \right| \leq \varphi(x,y) \tag{2.3}$$

for $x \in [0,a)$, $y \in [0,b)$.

Definition 2.1. The equation (1.1) is Ulam-Hyers stable if there exists the real numbers $c_f^1, c_f^2, c_f^3 > 0$ such that for any $\varepsilon > 0$ and for each solution v of (2.2), there exists a solution u of (1.1) with

$$\begin{cases} |v(x,y) - u(x,y)| \leq c_f^1 \cdot \varepsilon \\ \left| \dfrac{\partial v(x,y)}{\partial y} - \dfrac{\partial u(x,y)}{\partial y} \right| \leq c_f^2 \cdot \varepsilon \\ \left| \dfrac{\partial^2 v(x,y)}{\partial x^2} - \dfrac{\partial^2 u(x,y)}{\partial x^2} \right| \leq c_f^3 \cdot \varepsilon, \ \forall\, x \in [0,a),\ y \in [0,b). \end{cases} \tag{2.4}$$

Definition 2.2. The equation (1.1) is generalized Ulam-Hyers-Rassias stable if there exists the real numbers $c_{f,\varphi}^1, c_{f,\varphi}^2, c_{f,\varphi}^3 > 0$ such that for any $\varepsilon > 0$ and for each solution v of (2.3), there exists a solution of (1.1) with

$$\begin{cases} |v(x,y) - u(x,y)| \leq c_{f,\varphi}^1 \cdot \varphi(x,y) \\ \left| \dfrac{\partial v(x,y)}{\partial y} - \dfrac{\partial u(x,y)}{\partial y} \right| \leq c_{f,\varphi}^2 \cdot \varphi(x,y) \\ \left| \dfrac{\partial^2 v(x,y)}{\partial x^2} - \dfrac{\partial^2 u(x,y)}{\partial x^2} \right| \leq c_{f,\varphi}^3 \cdot \varphi(x,y), \ \forall\, x \in [0,a),\ y \in [0,b). \end{cases} \tag{2.5}$$

Remark 2.1. A function v is a solution of (2.2) if and only if there exists a function $g \in C([0,a) \times [0,b), \mathbb{B})$ such that

$$(i) \quad |g(x,y)| \leq \varepsilon,\ \forall\, x \in [0,a),\ \forall\, y \in [0,b);$$

$$(ii) \quad \frac{\partial^3 v(x,y)}{\partial x^2 \partial y} = f\left(x, y, v(x,y), \frac{\partial v(x,y)}{\partial y}, \frac{\partial^2 v(x,y)}{\partial x^2}\right) + g(x,y), \tag{2.6}$$

$$\forall\, x \in [0,a),\ y \in [0,b).$$

Remark 2.2. A function v is a solution of (2.3) if and only if there exists a function $g \in C([0,a) \times [0,b), \mathbb{B})$ such that

$$(i) \quad |g(x,y)| \leq \varphi(x,y),\ \forall\, x \in [0,a),\ \forall\, y \in [0,b);$$

$$(ii) \quad \frac{\partial^3 v(x,y)}{\partial x^2 \partial y} = f\left(x, y, v(x,y), \frac{\partial v(x,y)}{\partial y}, \frac{\partial^2 v(x,y)}{\partial x^2}\right) + g(x,y), \tag{2.7}$$

$$\forall\, x \in [0,a),\ y \in [0,b).$$

Remark 2.3. If v is a solution of the inequality (2.2), then (v, v_1, v_2) is a solution of the following system of integral inequalities:

$$\begin{cases} \Big| v(x,y) - v(x,0) - v(0,y) - xv_x(0,y) \\ - \int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t))dtdzds \Big| \le \frac{1}{2}\varepsilon x^2 y, \\ \\ \Big| v_1(x,y) - v_1(0,y) - xv_{1x}(0,y) \\ - \int_0^x \int_0^s f(z,y,v(z,y),v_1(z,y),v_2(z,y))dzds \Big| \le \frac{1}{2}\varepsilon x^2, \\ \\ \Big| v_2(x,y) - v_2(x,0) - \int_0^y f(x,t,v(x,t),v_1(x,t),v_2(x,t))dt \Big| \le \varepsilon y, \end{cases} \tag{2.8}$$

$\forall\, x \in [0,a),\ y \in [0,b)$ and $v_1 = \frac{\partial v}{\partial y}$, $v_2 = \frac{\partial^2 v}{\partial x^2}$.

Proof. From (2.3) we have that

$$v(x,y) = v(x,0) + v(0,y) + xv_x(0,y) \tag{2.9}$$

$$- \int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t))dtdzds + \int_0^x \int_0^s \int_0^y g(z,t)dtdzds.$$

Then we have

$$\Big| v(x,y) - v(x,0) - v(0,y) - xv_x(0,y) \tag{2.10}$$

$$+ \int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t))dtdzds \Big|$$

$$\le \int_0^x \int_0^s \int_0^y |g(z,t)|dtdzds \le \frac{1}{2}\varepsilon x^2 y.$$

By analogous method, we have the inequalities

$$\Big| v_1(x,y) - v_1(0,y) - xv_{1x}(0,y) - \int_0^x \int_0^s f(z,y,v(z,y),v_1(z,y),v_2(z,y))dzds \Big|$$

$$\le \frac{1}{2}\varepsilon x^2,$$

and

$$\Big| v_2(x,y) - v_2(x,0) - \int_0^y f(x,t,v(x,t),v_1(x,t),v_2(x,t))dt \Big| \le \varepsilon y.$$

□

In a similar way we have

Remark 2.4. If v is a solution of (2.3), then (v, v_1, v_2) is a solution of the following system of integral inequalities:

$$
\left\{
\begin{array}{l}
\Big| v(x,y) - v(x,0) - v(0,y) - x v_x(0,y) \\
- \displaystyle\int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t)) dt dz ds \Big| \\
\leq \displaystyle\int_0^x \int_0^s \int_0^y \varphi(z,t) dt dz ds, \\
\\
\Big| v_1(x,y) - v_1(0,y) - x v_{1x}(0,y) \\
- \displaystyle\int_0^x \int_0^s f(z,y,v(z,y),v_1(z,y),v_2(z,y)) dz ds \Big| \\
\leq \displaystyle\int_0^x \int_0^s \varphi(z,y) dz ds, \\
\\
\Big| v_2(x,y) - v_2(x,0) - \displaystyle\int_0^y f(x,y,v(x,t),v_1(x,t),v_2(x,t)) dt \Big| \\
\leq \displaystyle\int_0^y \varphi(x,t) dt.
\end{array}
\right. \tag{2.11}
$$

3. Ulam-Hyers stability

In this paragraph, we will consider Ulam-Hyers stability of the equation (1.1).

For this problem we have the following result:

Lemma 3.1. *(Gronwall Lemma)* ([3], [8], [14], [15]) *let $\alpha, \beta \in \mathbb{R}_+$ and $\psi \in C([a,b], \mathbb{R}_+)$ be given. Then if $x \in C[a,b]$ is a solution of the inequality*

$$x(t) \leq \alpha + \beta \int_a^t \psi(s) x(s) ds, \ \forall\, t \in [a,b]$$

then

$$x(t) \leq \alpha \exp\left(\beta \int_a^t \psi(s) ds\right), \ \forall\, t \in [a,b].$$

In what follows, we consider:

Theorem 3.1. *We suppose that*

(i) $a < \infty, \ b < \infty$;

(ii) $f \in C(D \times \mathbb{B}^3, \mathbb{B})$;

(iii) $\exists\, L_f > 0$ such that

$$|f(x,y,u_1,u_2,u_3) - f(x,y,v_1,v_2,v_3)| \leq L_f \max\{|u_i - v_i|, \ i = 1,2,3\};$$

for all $x \in [0,a], \ y \in [0,b]$ and $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{B}$.

Then:

(a) for $h \in C^2([0,a], \mathbb{B})$, $g_1, g_2 \in C^1([0,b], \mathbb{B})$ the equation (1.1) has a unique solution with

$$
\left\{
\begin{array}{ll}
u(x,0) = h(x), & x \in [0,a] \\
\\
u(0,y) = g_1(y), & y \in [0,b] \\
\\
u_x(0,y) = g_2(y), & y \in [0,b].
\end{array}
\right. \tag{3.12}
$$

(b) the equation (1.1) is Ulam-Hyers stable.

Proof. (a) This is a known result, consequence of the Banach fixed point theorem (see [4], [7]).

(b) Let v be a solution of the inequality (2.2). Let u be the unique solution of the equation (1.1) which satisfies the conditions:

$$\begin{cases} u(x,0) = v(x,0), & x \in [0,a] \\ u(0,y) = v(0,y), & y \in [0,b] \\ u_x(0,y) = v_x(0,y), & y \in [0,b]. \end{cases} \tag{3.13}$$

From Remark 2.3, the condition (iii) and Lemma 3.1 (Gronwall lemma), we have that

$$|v(x,y) - u(x,y)| \leq \frac{1}{2}\varepsilon a^2 b \exp\left(\frac{1}{2}L_f a^2 b\right) = c_f^1 \cdot \varepsilon$$

where

$$c_f^1 = \frac{1}{2}a^2 b \exp\left(\frac{1}{2}L_f a^2 b\right),$$

analogous we have

$$|v_1(x,y) - u_1(x,y)| \leq \frac{1}{2}\varepsilon a^2 \exp\left(\frac{1}{2}L_f a^2\right) = c_f^2 \cdot \varepsilon, \quad c_f^2 = \frac{1}{2}a^2 \exp\left(\frac{1}{2}L_f a^2\right),$$

and

$$|v_2(x,y) - u_2(x,y)| \leq \varepsilon b \exp(L_f b) = c_f^3 \cdot \varepsilon, \quad c_f^3 = b \exp(L_f b).$$

So, the equation (1.1) is Ulam-Hyers stable. □

4. Generalized Ulam-Hyers-Rassias stability

In what follows we consider the equation (1.1) and the inequality (2.3) in the case $a = \infty$, $b = \infty$.

Theorem 4.2. *We suppose that:*

(i) $f \in C([0,\infty) \times [0,\infty) \times \mathbb{B}^3, \mathbb{B})$;

(ii) there exists $l_f \in C^1([0,\infty) \times [0,\infty), \mathbb{R}_+)$ such that

$$|f(x,y,u_1,u_2,u_3) - f(x,y,v_1,v_2,v_3)| \leq l_f(x,y) \max\{|u_i - v_i|,\ i = 1,2,3\},$$

for all $x, y \in [0,\infty)$, $u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{B}$;

(iii) there exists $\lambda_\varphi^1, \lambda_\varphi^2, \lambda_\varphi^3 > 0$ such that

$$\int_0^x \int_0^s \int_0^y \varphi(z,t)dtdzds \leq \lambda_\varphi^1 \varphi(x,y), \quad \forall\, x,y \in [0,\infty),$$

$$\int_0^x \int_0^s \varphi(z,y)dzds \leq \lambda_\varphi^2 \varphi(x,y), \quad \forall\, x,y \in [0,\infty),$$

$$\int_0^y \varphi(x,t)dt \leq \lambda_\varphi^3 \varphi(x,y), \quad \forall\, x,y \in [0,\infty),$$

(iv) $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing.

Then the equation (1.1) ($a = \infty,\ b = \infty$) is generalized Ulam-Hyers-Rassias stable.

Proof. Let v be a solution of the inequality (2.3) and $u(x,y)$ the unique solution of the problem:

$$(4.14)\qquad \begin{cases} \dfrac{\partial^3 u(x,y)}{\partial x^2 \partial y} = f\left(x, y, u(x,y), \dfrac{\partial u(x,y)}{\partial y}, \dfrac{\partial^2 u(x,y)}{\partial x^2}\right),\ x, y \in [0,\infty) \\ u(x,0) = v(x,0),\ \forall\, x \in [0,\infty), \\ u(0,y) = v(0,y),\ \forall\, y \in [0,\infty), \\ u_x(0,y) = v_x(0,y),\ \forall\, y \in [0,\infty), \end{cases}$$

then (u, u_1, u_2) is a solution of the system:

$$(4.15)\qquad \begin{cases} u(x,y) &= v(x,0) + v(0,y) + x v_x(0,y) \\ &\quad + \displaystyle\int_0^x \int_0^s \int_0^y f(z,t,u(z,t),u_1(z,t),u_2(z,t))dtdzds \\ u_1(x,y) &= v_1(0,y) + x v_{1x}(0,y) \\ &\quad + \displaystyle\int_0^x \int_0^s f(z,y,u(z,y),u_1(z,y),u_2(z,y))dzds \\ u_2(x,y) &= v_2(x,0) + \displaystyle\int_0^y f(x,t,u(x,t),u_1(x,t),u_2(x,t))dt. \end{cases}$$

From Remark 2.4 and the condition (iii), we have that

$$(4.16)\qquad \begin{cases} \Big| v(x,y) - v(x,0) - v(0,y) - x v_x(0,y) \\ - \displaystyle\int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t))dtdzds \Big| \\ \le \displaystyle\int_0^x \int_0^s \int_0^y \varphi(z,t)dtdzds \le \lambda_\varphi^1 \varphi(x,y),\ \forall\, x,y \in [0,\infty), \\ \Big| v_1(x,y) - v_1(0,y) - x v_{1x}(0,y) \\ - \displaystyle\int_0^x \int_0^s f(z,y,v(z,y),v_1(z,y),v_2(z,y))dzds \Big| \\ \le \displaystyle\int_0^x \int_0^s \varphi(z,y)dzds \le \lambda_\varphi^2 \cdot \varphi(x,y),\ x,y \in [0,\infty), \\ \Big| v_2(x,y) - v_2(x,0) - \displaystyle\int_0^y f(x,t,v(x,t),v_1(x,t),v_2(x,t))dt \Big| \\ \le \lambda_\varphi^3 \cdot \varphi(x,y),\ x \in [0,\infty). \end{cases}$$

From (4.16), we have

$$|v(x,y) - u(x,y)| \le \Big| v(x,y) - v(x,0) - v(0,y) - x v_x(0,y)$$

$$- \int_0^x \int_0^s \int_0^y f(z,t,v(z,t),v_1(z,t),v_2(z,t))dtdzds \Big|$$

$$+\int_0^x\int_0^s\int_0^y |f(z,t,v(z,t),v_1(z,t),v_2(z,t))-f(z,t,u(z,t),u_1(z,t),u_2(z,t))|dtdzds$$
$$\leq \lambda_\varphi^1\cdot\varphi(x,y)+\int_0^x\int_0^s\int_0^y l_f(z,t)\max_{i\in\{1,2,3\}}(|v_i(z,t)-u_i(z,t)|)dtdzds.$$

From Lemma 3.1 (Gronwall lemma) it follows:

$$|v(x,y)-u(x,y)|\leq \lambda_\varphi^1\cdot\varphi(x,y)\exp\left(\int_0^\infty\int_0^s\int_0^\infty l_f(z,t)dtdzds\right),$$

and $|v(x,y)-u(x,y)|\leq c_{f,\varphi}^1\cdot\varphi(x,y)$ where

$$c_{f,\varphi}^1=\lambda_\varphi^1\exp\left(\int_0^\infty\int_0^s\int_0^\infty l_f(z,t)dtdzds\right),\ \forall\, x,y\in[0,\infty).$$

Analogous we have:

$$|v_1(x,y)-u_1(x,y)|\leq c_{f,\varphi}^2\cdot\varphi(x,y)$$

where

$$c_{f,\varphi}^2=\lambda_\varphi^2\exp\left(\int_0^\infty\int_0^s l_f(z,y)dzds\right),\ \forall\, x,y\in[0,\infty),$$

and $|v_2(x,y)-u_2(x,y)|\leq c_{f,\varphi}^3\cdot\varphi(x,y)$ where

$$c_{f,\varphi}^3=\lambda_\varphi^3\exp\left(\int_0^\infty l_f(z,t)dt\right),\ \forall\, x,y\in[0,\infty).$$

Then the equation (1.1) is generalized Ulam-Hyers-Rassias stable. □

For other applications see N. Lungu, D. Popa [9], D. Popa [10], [11], [12].

References

[1] Brillouët-Belluot, N., Brzdęk, J. and Cieplinski, K., *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal., 2012 (2012), Article ID 716936, 41 pages

[2] Colton, D. L., *Pseudoparabolic equations in one space variable*, J. Diff. Equations, **12** (1972), 559–565

[3] Crăciun, C. and Lungu, N., *Abstract and concrete Gronwall lemmas*, Fixed Point Theory, **10** (2009), No. 2, 221–228

[4] Crăciun, C. and Lungu, N., *Pseudo-Parabolic Inequalities*, Carpathian J. Math., **27** (2011), No. 2, 201–207

[5] Lungu, N. and Popa, D., *Hyers-Ulam stability of a first order partial differential equation*, J. Math. Anal. Appl., **385** (2012), 86–91

[6] Lungu, N. and Popa, D., *On the Ulam-Hyers stability of first order partial differential equations*, Carpathian J. of Math., **28** (2012), No. 1, 77–82

[7] Lungu, N. and Rus, I. A., *Hyperbolic differential inequalities*, Libertas Math., (2001), 35–40

[8] Lungu, N. and Rus, I. A., *Gronwall inequality via Picard operators*, Scientific Annals of Al. I. Cuza Iaşi, Tomul LVIII, 2012, f. 2, 269–278

[9] Lungu, N. and Popa, D., *Hyers-Ulam stability of some partial differential equations*, Carpathian J. Math., **30** (2014), No. 3, 327–334

[10] Popa, D., *A generalization of Jensen equation for set-valued maps*, Seminar on Fixed Point Theory, **3** (2002), 317–322

[11] Popa, D., *Hyers-Ulam-Rassias stability of a linear recurrence*, J. Math. Anal. Appl., **309** (2005), 591–597

[12] Popa, D. and Raşa, I., *On Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl., **381** (2011), 530–537

[13] Rus, I. A. and Lungu, N., *Ulam stability of a nonlinear hyperbolic partial differential equation*, Carpathian J. Math., **24** (2008), No. 3, 403–408

[14] Rus, I. A., *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation*, in: Nonlinear Analysis and Variational Problems, Springer, 2009, 147–152

[15] Rus, I. A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10** (2009), No. 2, 305–320

[16] Rus, I. A., *Ulam stability of ordinary differential equations*, Studia Univ. Babeş-Bolyai, Mathematica, **54** (2009), No. 4, 125–133

[17] Rundell, W. and Stecher, M., *Remarks concerning the supports of solutions of pseudoparabolic equations*, Proc. Amer. Math. Soc., **63** (1977), No. 1, 77–81

[18] Showalter, R. E. and Ting, T. W., *Pseudoparabolic partial differential equations*, Siam J. Math. Anal., **1** (1970), 1–26

[19] Vodahova, V. A., *Limit problems for a pseudoparabolic differential equations*, Dif. Urav., **18** (1982), No. 2, 280–285

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
MEMORANDUMULUI 28, 400114 CLUJ-NAPOCA, ROMANIA
AND
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA, DEPARTMENT OF MANAGEMENT
MEMORANDUMULUI 28 , 400114 CLUJ-NAPOCA, ROMANIA
E-mail address: nlungu@math.utcluj.ro
E-mail address: sorina.ciplea@ccm.utcluj.ro