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A new approach to fixed point theorems for multivalued contractive maps

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ABSTRACT. In the present paper, considering the Wardowski's technique we give many fixed point results for multivalued maps on complete metric space without using the Hausdorff metric. Our results are real generalization of some related fixed point theorems including the famous Feng and Liu's result in the literature. We also give some examples to both illustrate and show that our results are proper generalizations of the mentioned theorems.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $C(X)$ denotes the family of all nonempty, closed subsets of X, $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that, $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. For $A, B \in C(X)$, let

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is called generalized Pompeiu-Hausdorff distance on $C(X)$. It is well known that, H is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [4, 9]. Let $T : X \to CB(X)$ be a map, then T is called multivalued contraction (see [13]) if for all $x, y \in X$ there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y).$$

In 1969, Nadler [13] proved that every multivalued contraction on a complete metric space has a fixed point.

Then various fixed point theorems concerning with for multivalued contractions appeared in the last decades; see, for instance, [3, 5, 6, 7, 10, 11, 12, 15, 16]. Concerning these, the following theorem was given by Feng and Liu [8].

Theorem 1.1 ([8]). *Let (X, d) be a complete metric space, $T : X \to C(X)$ be a multi-valued mapping. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^x$, where*

$$I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\} \text{ and } b \in (0, 1),$$

satisfying

$$d(y, Ty) \leq cd(x, y)$$

then T has a fixed point in X provided $c < b$ and the function $x \to d(x, Tx)$ lower semi-continuous.

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In the present paper, we give some fixed point results, which extend and generalize many fixed point theorems including Theorem 1.1 in the literature, for multivalued mappings without using the Pompeiu-Hausdorff metric. Our results are based on the new approach to contraction mapping, which is called F-contraction. The concept of F-contraction for single valued maps on complete metric space was introduced by Wardowski [17]. First, we recall this new concept and some related results.

Let $F : (0, \infty) \to \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$,

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by $\mathcal{F}$ and $\mathcal{F}_*$ be the set of all functions F satisfying (F1)-(F3) and (F1)-(F4), respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$ and some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$. If we define $F_5(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F}\backslash\mathcal{F}_*$.

Remark 1.1. If F satisfies (F1), then it satisfies (F4) if and only if it is right continuous.

Definition 1.1 ([17]). Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then, we say that T is an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X\ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{1.1}$$

If we take $F(\alpha) = \ln \alpha$ in Definition 1.1, the inequality (1.1) turns to

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.2}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Thus T is an ordinary contraction with contractive constant $L = e^{-\tau}$. Therefore every ordinary contraction is also F-contraction, but the converse may not be true as shown in the Example 2.5 of [17]. If we choose $F(\alpha) = \alpha + \ln \alpha$, the inequality (1.1) turns to

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.3}$$

In addition, Wardowski showed that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous map. Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction. He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and the mapping $F_2 - F_1$ is strictly increasing. Hence, every Banach contraction satisfies the contractive condition (1.3). On the other side, Example 2.5 in [17] shows that the mapping T is not F_1-contraction (Banach Contraction), but still is an F_2-contraction. Thus, the following theorem is a proper generalization of Banach Contraction Principle.

Theorem 1.2 ([17]). *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

By combining the ideas of Wardowski and Nadler, Altun et al [2] introduced the concept of multivalued F-contractions and obtained some fixed point results for these type mappings on complete metric space.

Definition 1.2 ([2])**.** Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then we say that T is a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X \ [H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(d(x, y))].$$

By the considering $F(\alpha) = \ln \alpha$, then every multivalued contraction in the sense of Nadler is also multivalued F-contraction.

Theorem 1.3 ([2])**.** *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

Here, the following question may come to mind: Can we take $CB(X)$ instead of $K(X)$ in Theorem 1.3? By adding the condition (F4) on F, this question has been solved as follows:

Theorem 1.4 ([2])**.** *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. Suppose $F \in \mathcal{F}_*$, then T has a fixed point in X.*

2. Main results

Let $T : X \to P(X)$ be a multivalued map, $F \in \mathcal{F}$ and $\sigma > 0$. For $x \in X$ with $d(x, Tx) > 0$, define the set $F_\sigma^x \subseteq X$ as

$$F_\sigma^x = \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\}.$$

We need to consider the following cases:

If $T : X \to K(X)$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, since Tx is compact, for every $x \in X$ we have $y \in Tx$ such that $d(x, y) = d(x, Tx)$. Therefore, for every $x \in X$ with $d(x, Tx) > 0$, we have $F(d(x, y)) = F(d(x, Tx))$. Thus $y \in F_\sigma^x$ for all $\sigma > 0$.

If $T : X \to C(X)$, then F_σ^x may be empty for some $x \in X$ and $\sigma > 0$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$ and let $X = \{0\} \cup (1, 2)$ with the usual metric. Define $T : X \to C(X)$ by $T0 = (1, 2)$ and $Tx = \{0\}$ for $x \in (1, 2)$. Then, for $x = 0$ we have (note that $d(0, T0) = 1 > 0$)

$$\begin{aligned} F_1^0 &= \{y \in T0 : F(d(0, y)) \leq F(d(0, T0)) + 1\} \\ &= \{y \in (1, 2) : F(y) \leq F(1) + 1\} \\ &= \{y \in (1, 2) : 2y \leq 1\} \\ &= \emptyset. \end{aligned}$$

If $T : X \to C(X)$ (even if $T : X \to P(X)$) and $F \in \mathcal{F}_*$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, by (F4), we have

$$\begin{aligned} F_\sigma^x &= \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\} \\ &= \{y \in Tx : F(d(x, y)) \leq F(\inf\{d(x, y) : y \in Tx\}) + \sigma\} \\ &= \{y \in Tx : F(d(x, y)) \leq \inf\{F(d(x, y)) : y \in Tx\} + \sigma\} \\ &\neq \emptyset. \end{aligned}$$

By considering the above facts we give the following theorems:

Theorem 2.5. *Let (X,d) be a complete metric space, $T : X \to K(X)$ be a multivalued map and $F \in \mathcal{F}$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y,Ty)) \leq F(d(x,y)),$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x,Tx)$ is lower semi-continuous.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $d(x,Tx) > 0$. Since $Tx \in K(X)$ for every $x \in X$, the set F_σ^x is nonempty for any $\sigma > 0$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in F_\sigma^{x_0}$ such that

$$\tau + F(d(x_1,Tx_1)) \leq F(d(x_0,x_1))$$

and for $x_1 \in X$, there exists $x_2 \in F_\sigma^{x_1}$ satisfying

$$\tau + F(d(x_2,Tx_2)) \leq F(d(x_1,x_2)).$$

Continuing this process we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in F_\sigma^{x_n}$ and

$$\tau + F(d(x_{n+1},Tx_{n+1})) \leq F(d(x_n,x_{n+1})). \tag{2.4}$$

We will verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in F_\sigma^{x_n}$ we have

$$F(d(x_n,x_{n+1})) \leq F(d(x_n,Tx_n)) + \sigma. \tag{2.5}$$

From (2.4) and (2.5) we have

$$F(d(x_{n+1},Tx_{n+1})) \leq F(d(x_n,Tx_n)) + \sigma - \tau \tag{2.6}$$

and

$$F(d(x_{n+1},x_{n+2})) \leq F(d(x_n,x_{n+1})) + \sigma - \tau. \tag{2.7}$$

By the way we can obtain

$$F(d(x_n,x_{n+1})) \leq F(d(x_0,x_1)) + n(\sigma - \tau) \tag{2.8}$$

and

$$F(d(x_n,Tx_n)) \leq F(d(x_0,Tx_0)) + n(\sigma - \tau). \tag{2.9}$$

From (2.8), we get $\lim_{n\to\infty} F(d(x_n,x_{n+1})) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Therefore, from (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} [d(x_n,x_{n+1})]^k F(d(x_n,x_{n+1})) = 0.$$

Again, by (2.8), the following holds for all $n \in \mathbb{N}$

$$\begin{aligned} & [d(x_n,x_{n+1})]^k F(d(x_n,x_{n+1})) - [d(x_n,x_{n+1})]^k F(d(x_0,x_1)) \\ \leq \; & [d(x_n,x_{n+1})]^k n(\sigma - \tau) \leq 0. \end{aligned} \tag{2.10}$$

Letting $n \to \infty$ in (2.10), we obtain that

$$\lim_{n\to\infty} n\,[d(x_n,x_{n+1})]^k = 0. \tag{2.11}$$

From (2.11), there exits $n_1 \in \mathbb{N}$ such that $n\,[d(x_n,x_{n+1})]^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{1/k}}. \tag{2.12}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.12), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. On the other hand, from (2.9) and (F2) we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

This is a contradiction. Hence T has a fixed point. □

In the following theorem we replace $C(X)$ by $K(X)$, but we need to take $F \in \mathcal{F}_*$.

Theorem 2.6. *Let (X, d) be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F(d(y, Ty)) \leq F(d(x, y))$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. Since $F \in \mathcal{F}_*$, for any $x \in X$ the set F_σ^x is nonempty for any $\sigma > 0$. The rest of the proof can be completed as in the proof of Theorem 2.5 by considering the closedness of Tz. □

Corollary 2.1 (Theorem 1.1)**.** *Let (X, d) be a complete metric space, $T : X \to C(X)$. If there exists $c \in (0, 1)$ such that for any $x \in X$, there exists $y \in I_b^x$ ($b \in (0, 1)$) satisfying*

$$d(y, Ty) \leq c d(x, y),$$

then T has a fixed point in X provided $c < b$ and $x \to d(x, Tx)$ is lower semi-continuous.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $d(x, Tx) > 0$. If we define $F(\alpha) = \ln \alpha$, $\tau = -\ln c$ and $\sigma = -\ln b$ in Theorem 2.6, then T has a fixed point, which is a contradiction. □

Remark 2.2. Theorem 2.5 is a generalization of Theorem 1.3. In fact, let T satisfies the conditions of Theorem 1.3. Since every multivalued F-contractions are multivalued nonexpansive and every multivalued nonexpansive maps are upper semi-continuous, then T is upper semi-continuous. Therefore, the function $x \to d(x, Tx)$ is lower semi-continuous (see the Proposition 4.2.6 of [1]). On the other hand, for any $x \in X$ with $d(x, Tx) > 0$ and $y \in F_\sigma^x$, we have

$$\tau + F(d(y, Ty)) \leq \tau + F(H(Tx, Ty)) \leq F(d(x, y)).$$

Hence T satisfies conditions of Theorem 2.5, the existence of a fixed point has been proved. There is the similar relation between Theorem 1.4 and Theorem 2.6.

The following example shows that Theorem 2.5 (resp. Theorem 2.6) is a proper generalization of Theorem 1.3 (resp. Theorem 1.4).

Example 2.1. Let $X = \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric d, then (X, d) is complete metric space. Define a mapping $T : X \to C(X)$ as

$$Tx = \begin{cases} \{\frac{1}{2^n}, 1\} & , \quad x = \frac{1}{2^{n-1}} \\ \{0, \frac{1}{2}\} & , \quad x = 0 \end{cases}.$$

Since $H(T\frac{1}{2}, T0) = \frac{1}{2} = d(\frac{1}{2}, 0)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T\frac{1}{2}, T0)) > F(d(\frac{1}{2}, 0)).$$

Thus T is not multivalued F-contraction. Therefore Theorem 1.3 and Theorem 1.4 can not be applied to this example.

On the other hand, it is easy to compute that

$$d(x, Tx) = \begin{cases} \frac{1}{2^n} & , \quad x = \frac{1}{2^{n-1}}, n > 1 \\ 0 & , \quad x = 0, 1 \end{cases},$$

hence $x \to d(x, Tx)$ is lower semi-continuous. Now, let $F(\alpha) = \ln \alpha$. If $d(x, Tx) > 0$ then $x = \frac{1}{2^{n-1}}, n > 1$. Thus for $y = \frac{1}{2^n} \in Tx$ we have

$$F(d(x, y)) - F(d(x, Tx)) = 0$$

and

$$\begin{aligned} F(d(y, Ty)) - F(d(x, y)) &= \ln(\frac{1}{2^{n+1}}) - \ln(\frac{1}{2^n}) \\ &= \ln(\frac{2^n}{2^{n+1}}) \\ &= \ln \frac{1}{2} \\ &= -\ln 2. \end{aligned}$$

Therefore, $y \in F_\sigma^x$ and

$$\tau + F(d(y, Ty)) \le F(d(x, y))$$

is satisfied for $0 < \sigma < \tau \le \ln 2$. Hence all conditions of Theorem 2.5 and Theorem 2.6 are satisfied and so T has a fixed point.

In the following theorem we replace $P(X)$ by $C(X)$, but we need to add an extra condition.

Theorem 2.7. *Let (X, d) be a complete metric space, $T : X \to P(X)$ and $F \in \mathcal{F}_*$. Suppose there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $d(y, Ty) > 0$ and*

$$\tau + F(d(y, Ty)) \le F(d(x, y)).$$

If there exists $x_0 \in X$ with $d(x_0, Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

Proof. Since $d(x_0, Tx_0) > 0$, then there exists $x_1 \in F_\sigma^{x_0}$ such that $d(x_1, Tx_1) > 0$ and

$$\tau + F(d(x_1, Tx_1)) \leq F(d(x_0, x_1)).$$

Also, since $d(x_1, Tx_1) > 0$, there exists $x_2 \in F_\sigma^{x_1}$ satisfying $d(x_2, Tx_2) > 0$ and

$$\tau + F(d(x_2, Tx_2)) \leq F(d(x_1, x_2)).$$

Continuing this process we get an iterative sequence $\{x_n\}$ as in the proof of Theorem 2.5 such that $x_{n+1} \in Tx_n$ and $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges to a point of X, say z. By the hypotheses, we have Tz is closed. On the other hand from (2.9) and (F2) we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \leq d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0$$

and so $z \in Tz$. Hence T has a fixed point. □

Corollary 2.2. *Let (X, d) be a complete metric space, $T : X \to P(X)$. Suppose there exists $c \in (0, 1)$ such that for any $x \in X$ with $d(x, Tx) > 0$ there exists $y \in I_b^x$ ($b \in (0, 1)$) satisfying*

$$0 < d(y, Ty) \leq cd(x, y). \tag{2.13}$$

If there exists $x_0 \in X$ with $d(x_0, Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $c < b$ and $x \to d(x, Tx)$ is lower semi-continuous.

Proof. If we take $F(\alpha) = \ln\alpha$, $\tau = -\ln c$ and $\sigma = -\ln b$ in Theorem 2.7, then the proof is complete. □

Example 2.2. Let $X = [0, 2]$ with the usual metric. Define $T : X \to P(X)$ as

$$Tx = \begin{cases} (\frac{x}{4}, \frac{x}{2}] & , \quad x \in (0, 1] \\ \{\frac{x}{2}\} & , \quad x \in \{0\} \cup (1, 2] \end{cases}.$$

Since Tx is not closed for some $x \in X$, both Nadler's result and Theorem 1.1 can not be applied to this example. On the other hand if we take $\frac{1}{2} \leq c < b$ and $x_0 \in (0, 2]$, then all conditions of Corollary 2.2 are satisfied. Indeed, if $d(x, Tx) > 0$, then $x \in (0, 2]$ and so, for $y = \frac{x}{2} \in Tx$, we have

$$bd(x, y) = bd(x, \frac{x}{2}) = b\frac{x}{2} \leq \frac{x}{2} = d(x, Tx)$$

and

$$d(y, Ty) = d(\frac{x}{2}, T\frac{x}{2}) = \frac{x}{4} \leq c\frac{x}{2} = cd(x, y).$$

That is, $y \in I_b^x$ for any $x \in X$ with $d(x, Tx) > 0$ and (2.13) is satisfied. Now, let $x_0 \in (0, 2]$, then we have, for all $n \in \mathbb{N}$, $0 < x_n \leq \frac{x_0}{2^n}$ for the sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$. Therefore $\{x_n\}$ converges to 0 and $T0$ is closed. Finally, the function $f(x) = d(x, Tx) = \frac{x}{2}$ is lower semi-continuous. Therefore all conditions of Corollary 2.2 are satisfied and so T has a fixed point.

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