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On some convergences to the constant e and improvements of Carleman's inequality

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ABSTRACT. We present sharp inequalities related to the sequence $(1 + 1/n)^n$ and some applications to Kellers' limit and Carleman's inequality.

1. INTRODUCTION AND MOTIVATION

The starting point of this paper is the following well-known double inequality

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1}, \quad n \geq 1. \tag{1.1}$$

This inequality was highly discussed and extended in the recent past, since it was used to improve inequalities of Hardy-Carleman type. See for example [2], [5], [7], [8], [9].

As (1.1) is equivalent to

$$\frac{2n}{2n+1} < \frac{1}{e}\left(1+\frac{1}{n}\right)^n < \frac{2n+1}{2n+2},$$

we prove that the best approximation of the form

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx \frac{n+a}{n+b}, \quad \text{as } n \to \infty \tag{1.2}$$

is obtained for $a = 5/12$ and $b = 11/12$. Then we prove the following

Theorem 1.1. *For every real number $x \in [1, \infty)$, the following inequalities hold:*

$$\begin{aligned} & \frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5} \\ < \quad & \frac{1}{e}\left(1+\frac{1}{x}\right)^x \\ < \quad & \frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5} - \frac{2621}{41\,472x^5} + \frac{300\,901}{3483\,648x^6}. \end{aligned}$$

As application, we give a new proof of the limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right) = e. \tag{1.3}$$

This limit is also known as Keller's limit. See e.g. [6], *where a different proof of (1.3) is presented.*

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Moreover, the estimates from Theorem 1.1 are strong enough to prove

$$\lim_{n\to\infty} n^2 \left(\left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \right) - e \right) = \frac{e}{24}, \tag{1.4}$$

which is a new result, according to the best of our knowledge.

Finally, improvements of Carleman's inequality are given.

2. THE PROOFS

In order to find the best approximation (1.2), we associate the relative error sequence w_n by the relations

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n = \frac{n+a}{n+b} \cdot \exp w_n, \quad n \geq 1$$

and we consider an approximation (1.2) to be better when w_n converges faster to zero. We have

$$w_n = n \ln\left(1+\frac{1}{n}\right) - 1 - \ln \frac{n+a}{n+b},$$

but using a mathematical software such as Maple, we get

$$w_n = \left(-a+b-\frac{1}{2}\right)\frac{1}{n} + \left(\frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{3}\right)\frac{1}{n^2} + \left(\frac{1}{3}b^3 - \frac{1}{3}a^3 - \frac{1}{4}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

This form can be also obtained by direct computation.

Evidently, the fastest sequence w_n is obtained when the first two coefficients in this expansion vanish, that is $a = 5/12$ and $b = 11/12$. Our first aim is now attained.

Proof of Theorem 1.1. The requested inequalities can be written as $f > 0$ and $g < 0$, where

$$f(x) = x \ln\left(1+\frac{1}{x}\right) - 1 - \ln\left(\frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5}\right)$$

$$g(x) = x \ln\left(1+\frac{1}{x}\right) - 1 - \ln\left(\frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5} + \frac{300\,901}{3483\,648x^6}\right).$$

We have

$$f''(x) = \frac{A(x-1)}{x^2(x+1)^2(12x+11)^2 P^2(x)} > 0$$

and

$$g''(x) = -\frac{B(x-1)}{x^2(x+1)^2(12x+11)^2 Q^2(x)} < 0,$$

where

$$P(x) = 59\,184x^2 - 66\,708x - 43\,200x^3 + 1036\,800x^5 + 2488\,320x^6 - 144\,155$$

$$\begin{aligned} Q(x) \;&=\; 5945\,040x - 5603\,472x^2 + 4971\,456x^3 - 3628\,800x^4 \\ &\quad +87\,091\,200x^6 + 209\,018\,880x^7 + 16\,549\,555 \end{aligned}$$

$$\begin{aligned} A(x) &= 387\,888\,768\,643\,091\,163x + 1374\,068\,561\,183\,884\,363x^2 \\ &+2856\,411\,438\,418\,498\,368x^3 + 3861\,333\,058\,156\,847\,712x^4 \\ &+3547\,125\,026\,642\,062\,080x^5 + 2242\,448\,726\,942\,859\,264x^6 \\ &+963\,345\,615\,805\,707\,264x^7 + 269\,162\,452\,894\,408\,704x^8 \\ &+44\,174\,729\,709\,158\,400x^9 + 3234\,548\,057\,702\,400x^{10} \\ &+48\,685\,659\,681\,079\,707 \end{aligned}$$

$$\begin{aligned} B(x) &= 5495\,336\,279\,092\,271\,136\,793x + 22\,015\,820\,845\,590\,210\,733\,374x^2 \\ &+52\,587\,526\,363\,654\,958\,754\,048x^3 + 83\,107\,983\,906\,845\,638\,539\,984x^4 \\ &+91\,197\,790\,053\,279\,643\,410\,048x^5 + 70\,886\,916\,929\,730\,329\,339\,904x^6 \\ &+39\,022\,307\,420\,738\,572\,320\,768x^7 + 14\,907\,444\,982\,230\,536\,515\,584x^8 \\ &+3763\,807\,019\,677\,591\,584\,768x^9 + 565\,244\,311\,814\,774\,194\,176x^{10} \\ &+38\,255\,330\,631\,116\,390\,400x^{11} + 621\,810\,333\,384\,191\,039\,953. \end{aligned}$$

Evidently, g is strictly concave, f is strictly convex, with $f(\infty) = g(\infty) = 0$, so $g < 0$ and $f > 0$ on $[1, \infty)$. The proof is completed. □

The complete expansion of $(1 + 1/x)^x$ was recently discovered by Yue and Mortici in [10].

3. KELLERS' LIMIT

Let us rewrite Theorem 1.1 in the form

$$u(n) < \frac{1}{e}\left(1 + \frac{1}{n}\right)^n < v(n),$$

where

$$u(x) = \frac{x + \frac{5}{12}}{x + \frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5}$$

and

$$v(x) = \frac{x + \frac{5}{12}}{x + \frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8640x^4} - \frac{2621}{41\,472x^5} - \frac{2621}{41\,472x^5} + \frac{300\,901}{3483\,648x^6}.$$

We prove (1.3) using policemen lemma. As the sequence

$$x_n = \frac{1}{e}\left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}}\right)$$

can be written as

$$x_n = (n+1)\frac{1}{e}\left(1 + \frac{1}{n}\right)^n - n\frac{1}{e}\left(1 + \frac{1}{n-1}\right)^{n-1},$$

we use Theorem 1.1 to obtain

(3.5) $$(n+1)\,u(n) - nv(n-1) < x_n < (n+1)\,v(n) - nu(n-1).$$

The extreme-side sequences are rational functions of n and they tends together to 1, as n approaches infinity. Indeed,

$$(n+1)\,u(n) - nv(n-1) = \frac{2508\,226\,560n^{13} - 12\,959\,170\,560n^{12} + \cdots}{17\,418\,240n^5\,(n-1)^6\,(12n-1)\,(12n+11)}$$

and

$$(n+1)\,v(n) - nu(n-1) = \frac{2508\,226\,560n^{13} - 10\,450\,944\,000n^{12} + \cdots}{17\,418\,240n^6\,(n-1)^5\,(12n-1)\,(12n+11)}.$$

It results that x_n tends to 1, as n approaches infinity, so (1.3) is proved.

Further, by (3.5), we get

$$\begin{aligned} & n^2\left(\left((n+1)\,u(n) - nv(n-1)\right) - 1\right) \\ < \; & n^2\,(x_n - 1) \\ < \; & n^2\left(\left((n+1)\,v(n) - nu(n-1)\right) - 1\right) \end{aligned}$$

and again the extreme-side sequences are rational functions of n and they tends together to $1/24$, as n approaches infinity. Indeed,

$$n^2\left(\left((n+1)\,u(n) - nv(n-1)\right) - 1\right) = \frac{104\,509\,440n^{11} - 539\,965\,440n^{10} + \cdots}{17\,418\,240n^3\,(n-1)^6\,(12n-1)\,(12n+11)}$$

$$n^2\left(\left((n+1)\,v(n) - nu(n-1)\right) - 1\right) = \frac{104\,509\,440n^{11} + -435\,456\,000n^{10} \cdots}{17\,418\,240n^4\,(n-1)^5\,(12n-1)\,(12n+11)}.$$

In consequence, $n^2\,(x_n - 1)$ tends to $1/24$, which is (1.4).

4. Improvements of Carleman's inequality

While Swedish mathematician Torsten Carleman was studying quasi-analytical functions, he discovered an important inequality, now known as Carleman's inequality. If $\sum a_n$ is a convergent series of nonneagtive reals, then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{n=1}^{\infty} a_n. \tag{4.6}$$

This inequality was proven to be of great independent interest, since many authors improved it in the recent past.

The main tool for studying and improving (4.6) was the proof of Pólya (see [3]-[4]), who started from AM-GM inequality in the form

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{c_1 a_1 + c_2 a_2 + \cdots + c_n a_n}{n\,(c_1 c_2 \cdots c_n)^{1/n}}, \tag{4.7}$$

where $c_1, c_2, ..., c_n > 0$. The proof of the following result is based on Pólya's idea.

Theorem 4.2. Let$a_n > 0$ such that $\sum a_n$ is convergent and $c_n > 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k\,(c_1 c_2 \cdots c_k)^{1/k}} = l \in \mathbb{R}.$$

Denote

$$x_n = \sum_{k=n}^{\infty} \frac{1}{k\,(c_1 c_2 \cdots c_k)^{1/k}}.$$

Then

$$\sum_{n=1}^{\infty}(a_1...a_n)^{1/n}\le\sum_{n=1}^{\infty}c_nx_na_n. \tag{4.8}$$

Proof. Using (4.7), we have

$$\begin{aligned}\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} &\le \sum_{n=1}^{\infty}\left(\frac{1}{n\,(c_1c_2\cdots c_n)^{1/n}}\sum_{m=1}^{n}c_ma_m\right)\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{c_ma_m}{n\,(c_1c_2\cdots c_n)^{1/n}}\\ &= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{c_ma_m}{n\,(c_1c_2\cdots c_n)^{1/n}}\\ &= \sum_{m=1}^{\infty}c_ma_m\sum_{n=m}^{\infty}\frac{1}{n\,(c_1c_2\cdots c_n)^{1/n}}\\ &= \sum_{m=1}^{\infty}c_ma_m\sum_{n=m}^{\infty}(x_n-x_{n+1})\\ &= \sum_{m=1}^{\infty}c_mx_ma_m.\square\end{aligned}$$

Pólya took $c_n=(n+1)^n/n^{n-1}$ and (4.8) becomes

$$\sum_{n=1}^{\infty}(a_1...a_n)^{1/n}\le\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n.$$

Now (4.7) follows from $(1+1/n)^n<e$.

Almost all improvements stated in the recent past use upper bounds for $(1+1/n)^n$, stronger than $(1+1/n)^n<e$. See [2], [6]-[9].

We use Theorem 1.1 to establish the following improvement of Carleman's inequality. □

Theorem 4.3. Let $a_n>0$ such that $\sum a_n<\infty$. Then

$$\sum_{n=1}^{\infty}(a_1...a_n)^{1/n}\le e\sum_{n=1}^{\infty}\frac{12n+5}{12n+11}a_n.$$

It also holds good

$$\sum_{n=1}^{\infty}(a_1...a_n)^{1/n}\le e\sum_{n=1}^{\infty}\left(\frac{12n+5}{12n+11}-\varepsilon_n\right)a_n,$$

where

$$\varepsilon_n=\frac{5}{288n^3}-\frac{343}{8640n^4}+\frac{2621}{41\,472n^5}+\frac{2621}{41\,472n^5}-\frac{300\,901}{3483\,648n^6}.$$

Being very accurate, we are convinced that the inequalities presented in Theorem 1.1 can be succesfully used to obtain other new results.

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