CARPATHIAN J. MATH. **31** (2015), No. 2, 255 - 260
Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Towards a new bound for a matrix norm

P. G. POPESCU, E. I. SLUŞANSCHI and V. PREDA

ABSTRACT. In this paper are given refinements of several classical inequalities like Jensen, Young and Heinz which are then applied to obtain new improvements of recent results. We also give a new bound for a matrix norm expression, related to a matrix inequality of Bhatia and Davis.

1. INTRODUCTION

It is well known [4, 7] that a continous function, f, convex in a real interval $I \subseteq \mathbb{R}$ has the property $f\left(\frac{1}{P_n}\sum_{k=1}^n p_k x_k\right) \leq \frac{1}{P_n}\sum_{k=1}^n p_k f(x_k)$, the Jensen inequality, where $x_k \in I, 1 \leq k \leq n$ are given data points and $p_1, p_2, \cdots, p_n$ is a set of nonnegative real numbers constrained by $\sum_{k=1}^j p_k = P_j$. If f is concave the inequality is reversed.
The classical Young inequality for scalars, which states that for any nonnegative real numbers $a, b \geq 0$ and $0 \leq v \leq 1$ we have $a^v b^{1-v} \leq va + (1-v)b$, can also be deduced as a direct consequence of Jensen inequality, for $f = \ln x$.
Furthermore from the Young inequality one can deduce the Heinz inequality, which states that for any nonnegative real numbers $a, b \geq 0$ and $0 \leq v \leq 1$ we have the following $H_v(a,b) \leq \frac{a+b}{2}$, where $2H_v(a,b) = a^v b^{1-v} + a^{1-v} b^v$.
Let $\mathbb{M}_n$ be the space of $n \times n$ matrices and $||\cdot||$ any unitary invariant norm on $\mathbb{M}_n$. Also for all $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$, we have that $||UAV|| = ||A||$. Now if $A = [a_{ij}] \in \mathbb{M}_n$, then the following norm is called the Hilbert-Schmidt norm, $||A||_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$. (see [3])

2. REFINEMENT OF JENSEN INEQUALITY AND APPLICATIONS

Many refinements of Jensen inequality have been presented in the recent literature, see [2, 8, 9, 10]. We present here another one, which will then be applied to refine the Young and Heinz inequalities.

Theorem 2.1. *Let f be a convex function on interval $[0, \infty)$, $a, b \geq 0$ and $1 \geq v, u \geq 0$, then if*

$$i.\ 0 \leq v \leq u: \quad f(va + (1-v)b) \leq \frac{v}{u} f(ua + (1-u)b) + \left(1 - \frac{v}{u}\right) f(b) \leq v f(a) + (1-v) f(b),$$

$$ii.\ u \leq v \leq 1: \quad f(va + (1-v)b) \leq \frac{v-u}{1-u} f(a) + \left(1 - \frac{v-u}{1-u}\right) f(ua + (1-u)b) \leq v f(a) + (1-v) f(b),$$

Received: 04.12.2013; In revised form: 13.06.2014; Accepted: 15.06.2014
2010 *Mathematics Subject Classification.* 26B25, 26D15, 15A60.
Key words and phrases. *Jensen inequality, Young inequality, Heinz inequality, refinement, Hilbert-Schmidt norm .*
Corresponding author: P. G. Popescu; pgpopescu@yahoo.com

Proof. $i.\ 0 \le v \le u$. We express $va + (1-v)b$ like $\frac{v}{u}(ua + (1-u)b) + \left(1 - \frac{v}{u}\right) b$, where $\frac{v}{u} \le 1$. Then by applying the Jensen inequality for the nonnegative numbers $ua + (1-u)b$ and b, we obtain

$$f(va+(1-v)b) = f\left(\frac{v}{u}(ua+(1-u)b) + \left(1-\frac{v}{u}\right)b\right) \le \frac{v}{u}f(ua+(1-u)b) + \left(1-\frac{v}{u}\right)f(b)$$

and appling again Jensen inequality for a and b, we get

$$\frac{v}{u}f(ua+(1-u)b) + \left(1-\frac{v}{u}\right)f(b) \le \frac{v}{u}(uf(a) + (1-u)f(b)) + \left(1-\frac{v}{u}\right)f(b),$$

which is equal to $vf(a) + (1-v)f(b)$ and we are done.

$ii.\ u \le v \le 1$. We consider $va + (1-v)b = \frac{v-u}{1-u}a + \left(1 - \frac{v-u}{1-u}\right)(ua + (1-u)b)$, so applying Jensen inequality for nonnegative numbers a and $ua + (1-u)b$, yields

$$\begin{aligned} f(va+(1-v)b) &= f\left(\frac{v-u}{1-u}a + \left(1-\frac{v-u}{1-u}\right)(ua+(1-u)b)\right) \\ &\le \frac{v-u}{1-u}f(a) + \left(1-\frac{v-u}{1-u}\right)f(ua+(1-u)b) \end{aligned}$$

and applying again Jensen inequality for a and b, we conclude

$$\frac{v-u}{1-u}f(a) + \frac{1-v}{1-u}f(ua+(1-u)b) \le \frac{v-u}{1-u}f(a) + \frac{1-v}{1-u}(uf(a) + (1-u)f(b)),$$

which is equal to $vf(a) + (1-v)f(b)$ and we are done. □

As can be noticed, the results form the previous Theorem can be considered as two different refinements of the Jensen inequality. However, they only differ by the choosing of u. Also both inequalities reverse when f is concave.

Using the previous result we refine the Young and Heinz inequalities, as follows

Theorem 2.2. (Refinements of Young Inequality) *Let* $a, b \ge 0$ *and* $1 \ge v, u \ge 0$, *then if*

$$\begin{aligned} i.\ 0 \le v \le u:&\quad va + (1-v)b \ge (ua + (1-u)b)^{\frac{v}{u}}\, b^{1-\frac{v}{u}} \ge a^v b^{1-v}, \\ ii.\ u \le v \le 1:&\quad va + (1-v)b \ge a^{\frac{v-u}{1-u}}\, (ua + (1-u)b)^{1-\frac{v-u}{1-u}} \ge a^v b^{1-v}. \end{aligned}$$

Proof. Considering the previous Theorem applied for function $f = ln(x)$, which is concave, so both inequalities reverse and thus follows the conclusion. □

Theorem 2.3. (Refinements of Heinz Inequality) *Let* $a, b \ge 0$ *and* $1 \ge v, u \ge 0$, *then if*

$$i.\ 0 \le v \le u: \quad \frac{a+b}{2} \ge \frac{(ua+(1-u)b)^{\frac{v}{u}}\, b^{1-\frac{v}{u}} + (ub+(1-u)a)^{\frac{v}{u}}\, a^{1-\frac{v}{u}}}{2} \ge H_v(a,b),$$

$$ii.\ u \le v \le 1: \quad \frac{a+b}{2} \ge \frac{a^{\frac{v-u}{1-u}}\, (ua+(1-u)b)^{1-\frac{v-u}{1-u}} + b^{\frac{v-u}{1-u}}\, (ub+(1-u)a)^{1-\frac{v-u}{1-u}}}{2} \ge H_v(a,b).$$

Proof. It follows directly from the previous Theorem by summating the inequalities obtained for a, b with the ones obtained for b, a and divided by two. □

In order to delineate the impact of the results we choose a numerical example for the concave function $ln(x)$ where $a, b \ge 0$ and $1 \ge v \ge 0$. We have the folowing results

(1) $Jensen\ inequality: \quad va + (1-v)b \ge a^v b^{1-v}$.

(2) $Inequality\ from$ [6] $: \quad va + (1-v)b \ge a^v b^{1-v} + \min\{v, 1-v\}(\sqrt{a} - \sqrt{b})^2$.

(3) $Theorem\ 2.1 (0 \le v \le u \le 1): \quad va + (1-v)b \ge (ua + (1-u)b)^{\frac{v}{u}}\, b^{1-\frac{v}{u}}$.

$$(4)\, Theorem\, 2.1 (0 \le u \le v \le 1): \quad va + (1-v)b \ge a^{\frac{v-u}{1-u}} \left(ua + (1-u)b\right)^{1-\frac{v-u}{1-u}}.$$

It is simple to observe that the result from [6], inequality (2) is a refinement of Jensen inequality (1). In the next section we prove that inequalities (3) and (4) are refinements of inequality (2).

Numerically, the inequalities (1),(2) and (3), for $a = 1.1, b = 2.3$ and $v = 0.35, u = 0.4$ are

$$(1)\, Jensen\, inequality: \quad 0.35*1.1+(1-0.35)*2.3 > 1.1^{0.35}*2.3^{1-0.35} \Leftrightarrow 1.88 > 1.77669...$$

$$(2)\, Inequality\, from\, [6]: \quad 0.35*1.1+(1-0.35)*2.3 > 1.1^{0.35}*2.3^{1-0.35}+0.35*(\sqrt{1.1}-\sqrt{2.3})^2$$
$$\Leftrightarrow 1.88 > 1.85327...$$

$$(3)\, Theorem\, 2.1: \quad 0.35*1.1+(1-0.35)*2.3 > (0.4*1.1+(1-0.4)*2.3)^{\frac{0.35}{0.4}}*2.3^{1-\frac{0.35}{0.4}}$$
$$\Leftrightarrow 1.88 > 1.87404...$$

Now if we consider $a = 1.1, b = 2.3$ and $v = 0.35, u = 0.32$, we have

$$(1)\, Jensen\, inequality: \quad 0.35*1.1+(1-0.35)*2.3 > 1.1^{0.35}*2.3^{1-0.35} \Leftrightarrow 1.88 > 1.77669...$$

$$(2)\, Inequality\, from\, [6]: \quad 0.35*1.1+(1-0.35)*2.3 > 1.1^{0.35}*2.3^{1-0.35}+0.35*(\sqrt{1.1}-\sqrt{2.3})^2$$
$$\Leftrightarrow 1.88 > 1.85327...$$

$$(4)\, Theorem\, 2.1: \quad 0.35*1.1+(1-0.35)*2.3 > 1.1^{\frac{0.35-0.32}{1-0.32}}(0.32*1.1+(1-0.32)*2.3)^{1-\frac{0.35-0.32}{1-0.32}}$$
$$\Leftrightarrow 1.88 > 1.86966...$$

3. Refinements of some recent results

We continue with a refinement of a result presented in [6] as Theorem 2.1, which states

Theorem 3.4. (F. Kittaneh and Y. Manasrah) *If $a, b \ge 0$ and $0 \le v \le 1$, then*

$$a^v b^{1-v} + r_0(\sqrt{a} - \sqrt{b})^2 \le va + (1-v)b,$$

where $r_0 = \min\{v, 1-v\}$.

Considering the refinements of the Young inequality presented in Theorem 2.2, one obtains

Theorem 3.5. *If $a, b \ge 0$ and $0 \le v, u \le 1$, then*

$$va + (1-v)b \ge r_0(\sqrt{a} - \sqrt{b})^2 + R_0 \ge r_0(\sqrt{a} - \sqrt{b})^2 + a^v b^{1-v},$$

where $r_0 = \min\{v, 1-v\}$ and

$$R_0 = \begin{cases} (u\sqrt{ab} + (1-u)b)^{\frac{2v}{u}} b^{1-\frac{2v}{u}} & \text{if } 0 \le 2v \le u \text{ and } r_0 = v, \\ (ua + (1-u)\sqrt{ab})^{\frac{2v-1}{u}} \sqrt{ab}^{1-\frac{2v-1}{u}} & \text{if } 0 \le 2v-1 \le u \text{ and } r_0 = 1-v, \\ \sqrt{ab}^{\frac{2v-u}{1-u}} (u\sqrt{ab} + (1-u)b)^{1-\frac{2v-u}{1-u}} & \text{if } u \le 2v \le 1 \text{ and } r_0 = v, \\ a^{\frac{2v-1-u}{1-u}} (ua + (1-u)\sqrt{ab})^{1-\frac{2v-1-u}{1-u}} & \text{if } u \le 2v-1 \le 1 \text{ and } r_0 = 1-v. \end{cases}$$

Proof. The first two inequalities (concerning the first two expressions of R_0) correspond to the first refinement of the Young inequality presented in Theorem 2.2, as follows, for $0 \le v \le u$,

$$va + (1-v)b \ge (ua + (1-u)b)^{\frac{v}{u}} b^{1-\frac{v}{u}} \ge a^v b^{1-v}.$$

So if $r_0 = v$ $(v < 1/2)$ then

$$va + (1-v)b - v(\sqrt{a} - \sqrt{b})^2 = 2v\sqrt{ab} + (1-2v)b$$

and applying the previous inequality, where $a = \sqrt{ab}, b = b, v = 2v$, yields

$$va + (1-v)b - v(\sqrt{a} - \sqrt{b})^2 \geq (u\sqrt{ab} + (1-u)b)^{\frac{2v}{u}} b^{1-\frac{2v}{u}} \geq \sqrt{ab}^{2v} b^{1-2v} = a^v b^{1-v},$$

where $0 \leq 2v \leq u$.
For the other case, when $r_0 = 1 - v$ $(v > 1/2)$, we consider

$$va + (1-v)b - (1-v)(\sqrt{a} - \sqrt{b})^2 = (2v-1)a + 2(1-v)\sqrt{ab}$$

and applying again the previous inequality (first refinement of the Young inequality from Theorem 2.2), this time for $a = a, b = \sqrt{ab}, v = 2v - 1$, we obtain

$$va+(1-v)b-(1-v)(\sqrt{a}-\sqrt{b})^2 \geq (ua+(1-u)\sqrt{ab})^{\frac{2v-1}{u}} \sqrt{ab}^{1-\frac{2v-1}{u}} \geq a^{2v-1}\sqrt{ab}^{2(1-v)} = a^v b^{1-v},$$

where $0 \leq 2v - 1 \leq u$.
The other two inequalities can be deduced following the above steps, this time using the second refinement of the Young inequality, as shown in Theorem 2.2, for $u \leq v \leq 1$,

$$va + (1-v)b \geq a^{\frac{v-u}{1-u}} (ua + (1-u)b)^{1-\frac{v-u}{1-u}} \geq a^v b^{1-v}.$$

□

In [11], Theorem 2.1 presents a refinement of the Heinz inequality, as follows

Theorem 3.6. (L. Zou and Y. Jiang) *Let* $a, b \geq 0$ *and* $0 \leq v \leq 1$. *If* $r_0 = \min\{v, 1-v\}$, *then*

$$2H_v(a,b) \leq \begin{cases} (1-4r_0)(a+b) + 4r_0(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [0,1/4] \cup [3/4,1], \\ 2(4r_0-1)\sqrt{ab} + 2(1-2r_0)(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [1/4,3/4]. \end{cases}$$

This result is in fact a refinement of an inequality presented in [6], based on Theorem 3.4, which states that for $a, b \geq 0$, $0 \leq v \leq 1$ and $r_0 = \min\{v, 1-v\}$,

$$H_v(a,b) + r_0(\sqrt{a} - \sqrt{b})^2 \leq \frac{a+b}{2}.$$

The demonstration of the previous Theorem is based on the following

Lemma 3.1. *Let f be a real valued convex function on an interval $[a,b]$. For any $x_1 \leq x_2 \in [a,b]$, we have*

$$f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}, \quad x \in (x_1, x_2).$$

Proof. The inequality is in fact the Jensen inequality considering that for any $x \in (x_1, x_2)$, there exists $\alpha \in (0,1)$ such that $x = \alpha x_1 + (1-\alpha)x_2$ and then the inequality transforms into

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}(\alpha x_1 + (1-\alpha)x_2) - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1},$$

where the right hand side equals $\alpha f(x_1) + (1-\alpha)f(x_2)$, after calculations. □

So applying the refinements for the Jensen inequality presented in Theorem 2.1, to the previous Lemma, yields

Lemma 3.2. *Let f be a real valued convex function on an interval $[a,b]$ and $x_1 \leq x_2 \in [a,b]$, then for any $x \in (x_1, x_2)$, i.e. $\exists \alpha \in (0,1)$, such that $x = \alpha x_1 + (1-\alpha)x_2$,*

$$f(x) \leq R \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1},$$

where

$$R = \begin{cases} \frac{\alpha}{\beta} f(\beta x_1 + (1-\beta)x_2) + \left(1 - \frac{\alpha}{\beta}\right) f(x_2), & \forall \beta \in (0,1), with\ \alpha \leq \beta, \\ \frac{\alpha-\beta}{1-\beta} f(x_1) + \left(1 - \frac{\alpha-\beta}{1-\beta}\right) f(\beta x_1 + (1-\beta)x_2), & \forall \beta \in (0,1), with\ \beta \leq \alpha. \end{cases}$$

Proof. It follows from Theorem 2.1 and Lemma 3.1. □

We continue presenting a refinement of Theorem 3.6, as follows

Theorem 3.7. *Let* $a, b \geq 0$ *and* $0 \leq v \leq 1$. *If* $r_0 = \min\{v, 1-v\}$, *then*

$$2H_v(a,b) \leq \begin{cases} R_1 \leq (1-4r_0)(a+b) + 4r_0(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [0, 1/4], \\ R_2 \leq 2(4r_0 - 1)\sqrt{ab} + 2(1-2r_0)(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [1/4, 1/2], \\ R_3 \leq 2(4r_0 - 1)\sqrt{ab} + 2(1-2r_0)(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [1/2, 3/4], \\ R_4 \leq (1-4r_0)(a+b) + 4r_0(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [3/4, 1], \end{cases}$$

with

$$\frac{R_1}{2} = \begin{cases} \frac{1-4v}{u} H_{(1-u)/4}(a,b) + \left(1 - \frac{1-4v}{u}\right) H_{1/4}(a,b), & \forall u \in (0,1), with\ 1 \leq 4v + u, \\ \frac{1-4v-u}{1-u} H_0(a,b) + \left(1 - \frac{1-4v-u}{1-u}\right) H_{(1-u)/4}(a,b), & \forall u \in (0,1), with\ 4v + u \leq 1, \end{cases}$$

$$\frac{R_2}{2} = \begin{cases} \frac{2-4v}{u} H_{(2-u)/4}(a,b) + \left(1 - \frac{2-4v}{u}\right) H_{1/2}(a,b), & \forall u \in (0,1), with\ 2 \leq 4v + u, \\ \frac{2-4v-u}{1-u} H_{1/4}(a,b) + \left(1 - \frac{2-4v-u}{1-u}\right) H_{(2-u)/4}(a,b), & \forall u \in (0,1), with\ 4v + u \leq 2, \end{cases}$$

$$\frac{R_3}{2} = \begin{cases} \frac{3-4v}{u} H_{(3-u)/4}(a,b) + \left(1 - \frac{3-4v}{u}\right) H_{3/4}(a,b), & \forall u \in (0,1), with\ 3 \leq 4v + u, \\ \frac{3-4v-u}{1-u} H_{1/2}(a,b) + \left(1 - \frac{3-4v-u}{1-u}\right) H_{(3-u)/4}(a,b), & \forall u \in (0,1), with\ 4v + u \leq 3, \end{cases}$$

$$\frac{R_4}{2} = \begin{cases} \frac{4(1-v)}{u} H_{(4-u)/4}(a,b) + \left(1 - \frac{4(1-v)}{u}\right) H_1(a,b), & \forall u \in (0,1), with\ 4 \leq 4v + u, \\ \frac{4(1-v)-u}{1-u} H_{3/4}(a,b) + \left(1 - \frac{4(1-v)-u}{1-u}\right) H_{(4-u)/4}(a,b), & \forall u \in (0,1), with\ 4v + u \leq 4. \end{cases}$$

Proof. For each of the three double inequalities we will apply Lemma 3.2, for $f(v) = 2H_v(a,b),\ 0 \leq v \leq 1$, which is obviously convex.
For $v \in [0, 1/4]$, there exists $\alpha \in [0,1]$ such that $v = \alpha 0 + (1-\alpha)1/4$, whence $\alpha = 1 - 4v$. So applying Lemma 3.2 for $f \equiv 2H_v(a,b), \alpha = 1 - 4v, \beta = u$, yields the wanted result.
For $v \in [1/4, 1/2]$, there exists $\alpha \in [0,1]$ such that $v = \alpha 1/4 + (1-\alpha)1/2$, whence $\alpha = 2 - 4v$. So applying Lemma 3.2 for $f \equiv 2H_v(a,b), \alpha = 2 - 4v, \beta = u$, we get the wanted result.
For $v \in [1/2, 3/4]$, there exists $\alpha \in [0,1]$ such that $v = \alpha 1/2 + (1-\alpha)3/4$, whence $\alpha = 3 - 4v$. So applying Lemma 3.2 for $f \equiv 2H_v(a,b), \alpha = 3 - 4v, \beta = u$, we obtain the result.
And finally for $v \in [3/4, 1]$, there exists $\alpha \in [0,1]$ such that $v = \alpha 3/4 + (1-\alpha)1$, whence $\alpha = 4(1-v)$. Using Lemma 3.2 for $f \equiv 2H_v(a,b), \alpha = 4(1-v), \beta = u$ we are done. □

4. A NEW MATRIX NORM INEQUALITY

Some new refinements of a matrix inequality of R. Bhatia and C. Davis from [1], have been presented in [6, 5]. These state that for $A, B, X \in \mathbb{M}_n$, with A, B positive semidefinite and $0 \leq v \leq 1$, then

$$2||A^{1/2}XB^{1/2}|| \leq ||A^v X B^{1-v} + A^{1-v} X B^v|| \leq ||AX + XB||.$$

We are interested to obtain a new bound for the expression $||A^v X B^{1-v} + A^{1-v} X B^v||_2$, for which we will make use of the first refinement of the Heinz inequality presented in Theorem 2.3 as follows for suitable positive semidefinite matrices C and D

Theorem 4.8. *Let $A, X, B \in \mathbb{M}_n$ such that A and B are positive semidefinite and $0 \leq v \leq 1$, then*

$$||A^v X B^{1-v} + A^{1-v} X B^v||_2^2 \leq ||A^{1-v/u} X C^{v/u} + D^{v/u} X B^{1-v/u}||_2^2,$$

for all $0 < u \leq v \leq 1$, where $C, D \in \mathbb{M}_n$ are suitable positive semidefinite matrices.

Proof. Because A and B are positive semidefinite, exists unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_2 V^*$, with $\Lambda_1 = diag(\lambda_1, \lambda_2, ..., \lambda_n), \Lambda_2 = diag(\mu_1, \mu_2, ..., \mu_n)$ and $\lambda_i, \mu_i \geq 0,\ i = 1, ..., n$. Let $C = V\Lambda_3 V^*,\ D = U\Lambda_4 U^*$, with $\Lambda_3 = u\Lambda_2 + (1-u)\Lambda_1$, $\Lambda_4 = u\Lambda_1 + (1-u)\Lambda_2$. Considering $Y = U^* X V = [y_{ij}]$ and applying the first refinement of Heinz inequality from Theorem 2.3, yields

$$||A^v X B^{1-v} + A^{1-v} X B^v||_2^2 = ||U\left(\Lambda_1^v Y \Lambda_2^{1-v} + \Lambda_1^{1-v} Y \Lambda_2^v\right) V^*||_2^2 = \sum_{i,j=1}^n \left[\lambda_i^v \mu_j^{1-v} + \lambda_i^{1-v} \mu_j^v\right]^2 |y_{ij}|^2$$
$$\leq \sum_{i,j=1}^n \left[(u\lambda_i + (1-u)\mu_j)^{v/u} \mu_j^{1-v/u} + (u\mu_j + (1-u)\lambda_i)^{v/u} \lambda_i^{1-v/u}\right]^2 |y_{ij}|^2$$
$$= \sum_{i,j=1}^n \left[\lambda_i^{1-v/u} (u\mu_j + (1-u)\lambda_i)^{v/u} + (u\lambda_i + (1-u)\mu_j)^{v/u} \mu_j^{1-v/u}\right]^2 |y_{ij}|^2$$
$$= ||U\left(\Lambda_1^{1-v/u} Y \Lambda_3^{v/u} + \Lambda_4^{v/u} Y \Lambda_2^{1-v/u}\right) V^*||_2^2 = ||A^{1-v/u} X C^{v/u} + D^{v/u} X B^{1-v/u}||_2^2,$$

because $\Lambda_3 = u\Lambda_2 + (1-u)\Lambda_1$, $\Lambda_4 = u\Lambda_1 + (1-u)\Lambda_2$ and we are done. □

Acknowledgements. The work has been funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Ministry of European Funds through the Financial Agreement POSDRU/159/1.5/S/134398.

References

[1] Bhatia, R. and Davis, C., *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. A., **14** (1993), No. 1, 132–136
[2] Dragomir, S. S., *A new refinement of Jensens inequality in linear spaces with applications*, Math. Comput. Model., **52** (2010), No. 9-10, 1497–1505
[3] Horn, R. A. and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, 2th edition, 2012
[4] Jensen, J. L. W. V., *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Math., **30** (1906), 175–193
[5] Kittaneh, F., *On the convexity of the Heinz means*, Integr. Equ. Oper. Theory, **68(2010)**, 519–527
[6] Kittaneh, F. and Manasrah, Y., *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361** (2010), 262–269
[7] Rassias, T. M., *Survey on Classical Inequalities*, Kluwer Academic Publishers, Dordrecht, 2000
[8] Simic, S., *Best possible global bounds for Jensens inequality*, Appl. Math. Comput., **215** (2009), No. 6, 2224–2228
[9] Tapus, N. and Popescu, P. G., *A new entropy upper bound*, Appl. Math. Lett., **25** (2012), No. 11, 1887–1890
[10] Xiao, Z.-G., Srivastava, H. M. and Zhang, Z.-H., *Further refinements of the Jensen inequalities based upon samples with repetitions*, Math. Comput. Model., **51** (2010), No. 5-6, 592–600
[11] Zou, L. and Jiang, Y., *Improved Heinz inequality and its application*, J. Inequal. Appl., **2012:113**

University Politehnica of Bucharest
Computer Science and Engineering Departament
Splaiul Independenţei 313, 060042 Bucharest, Romania
E-mail address: pgpopescu@yahoo.com

University Politehnica of Bucharest
Computer Science and Engineering Departament
Splaiul Independenţei 313, 060042 Bucharest, Romania
E-mail address: emil.slusanschi@cs.pub.ro

Mathematics Department
University of Bucharest
Academiei 14, 010014 Bucharest, Romania
E-mail address: preda@fmi.unibuc.ro