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The metric dimension of strong product graphs

JUAN A. RODRÍGUEZ-VELÁZQUEZ[1], DOROTA KUZIAK[1], ISMAEL G. YERO[2] and JOSÉ M. SIGARRETA[3]

ABSTRACT. For an ordered subset $S = \{s_1, s_2, \dots s_k\}$ of vertices in a connected graph G, the metric representation of a vertex u with respect to the set S is the k-vector $r(u|S) = (d_G(v, s_1), d_G(v, s_2), \dots, d_G(v, s_k))$, where $d_G(x, y)$ represents the distance between the vertices x and y. The set S is a metric generator for G if every two different vertices of G have distinct metric representations with respect to S. A minimum metric generator is called a metric basis for G and its cardinality, $dim(G)$, the metric dimension of G. It is well known that the problem of finding the metric dimension of a graph is NP-Hard. In this paper we obtain closed formulae and tight bounds for the metric dimension of strong product graphs.

1. INTRODUCTION

A *generator* of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the metric $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ is the length of a shortest path between x and y. (V, d_G) is clearly a metric space. A vertex $v \in V$ is said to distinguish two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A metric generator of minimum cardinality is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $dim(G)$.

If $S = \{s_1, s_2, \dots s_k\}$ and u is a vertex of G, then the *metric representation of u with respect to S* is the k-vector $r(u|S) = (d_G(v, s_1), d_G(v, s_2), \dots, d_G(v, s_k))$. Hence, the set S is a metric generator for G if every two different vertices of G have distinct metric representations with respect to S.

The concept of metric dimension was introduced by Slater in [16], where the metric generators were called *locating sets*, and studied independently by Harary and Melter [5], where the metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [9], and applications to chemistry in [7, 8]. This invariant was studied further in a number of other papers, including for example [2, 3, 4, 14, 17, 18]. Several variations of metric generators have been appearing in the literature, like those about resolving dominating sets [1], local metric sets [14], resolving partitions [4, 17], and strong metric generators [11, 15].

It was shown in [9] that the problem of computing $dim(G)$ is NP-complete. This suggests finding the metric dimension for special classes of graphs, or obtaining good bounds on this invariant. Metric basis have been studied, for instance, for digraphs [13], Cartesian product graphs [2, 17], corona product graphs [11, 18], distance-hereditary graphs

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Corresponding author: Dorota Kuziak; dorota.kuziak@urv.cat

[12], and Hamming graphs [10]. In this paper we study the problem of finding exact values or sharp bounds for the metric dimension of strong product graphs, and express these in terms of invariants of the factor graphs.

The strong product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \boxtimes H = (V, E)$, such that $V = V_1 \times V_2$ and two vertices $(a, b), (c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if ($a = c$ and $bd \in E_2$) or ($b = d$ and $ac \in E_1$) or ($ac \in E_1$ and $bd \in E_2$).

One of our tools will be a well-known result, which states the relationship between the vertex distances in $G \boxtimes H$ and the vertex distances in the factor graphs.

Remark 1.1. [6] Let G and H be two connected graphs. Then
$d_{G\boxtimes H}((a, b), (c, d)) = \max\{d_G(a, c), d_H(b, d)\}$.

2. Results

We begin with a general upper bound for the metric dimension of strong product graphs.

Theorem 2.1. *Let G and H be two connected graphs of order $n_1 \geq 2$ and n_2, respectively. Then $dim(G \boxtimes H) \leq n_1 \cdot dim(H) + n_2 \cdot dim(G) - dim(G) \cdot dim(H)$.*

Proof. Let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$ be the set of vertices of G and H, respectively. Let $S = (V_1 \times S_2) \cup (S_1 \times V_2)$, where S_1 and S_2 are metric basis for G and H, respectively. Let (u_i, v_j) and (u_k, v_l) be two different vertices of $G \boxtimes H$. Let $u_\alpha \in S_1$ such that u_i, u_k are distinguished by u_α and let $v_\beta \in S_2$ such that v_j, v_l are distinguished by v_β. If $i = k$, then (u_i, v_j) and (u_k, v_l) are distinguished by $(u_i, v_\beta) \in (V_1 \times S_2) \subset S$. Analogously, if $j = l$, then (u_i, v_j) and (u_k, v_l) are distinguished by $(u_\alpha, v_j) \in (S_1 \times V_2) \subset S$. If $i \neq k$ and $j \neq l$, then we suppose that neither (u_i, v_β) nor (u_k, v_β) distinguishes the pair $(u_i, v_j), (u_k, v_l)$, *i.e.*,

$$d_{G\boxtimes H}((u_i, v_j), (u_i, v_\beta)) = d_{G\boxtimes H}((u_k, v_l), (u_i, v_\beta)) \tag{2.1}$$

and

$$d_{G\boxtimes H}((u_i, v_j), (u_k, v_\beta)) = d_{G\boxtimes H}((u_k, v_l), (u_k, v_\beta)). \tag{2.2}$$

By (2.1) we have $d_H(v_j, v_\beta) = \max\{d_G(u_k, u_i), d_H(v_l, v_\beta)\}$ and since $d_H(v_j, v_\beta) \neq d_H(v_l, v_\beta)$, we obtain that

$$d_H(v_j, v_\beta) = d_G(u_k, u_i). \tag{2.3}$$

Also, by (2.2) we have $d_H(v_l, v_\beta) = \max\{d_G(u_i, u_k), d_H(v_j, v_\beta)\}$ and since $d_H(v_j, v_\beta) \neq d_H(v_l, v_\beta)$, we obtain that

$$d_H(v_l, v_\beta) = d_G(u_i, u_k). \tag{2.4}$$

From (2.3) and (2.4) we have that $d_H(v_j, v_\beta) = d_H(v_l, v_\beta)$, which is a contradiction with the statement that v_j, v_l are distinguished by v_β in H. □

Since $K_{n_1} \boxtimes K_{n_2} \cong K_{n_1 \cdot n_2}$ and for any complete graph K_n, $dim(K_n) = n-1$, we deduce

$$\begin{aligned} dim(K_{n_1} \boxtimes K_{n_2}) &= n_1 \cdot n_2 - 1 \\ &= n_1(n_2 - 1) + n_2(n_1 - 1) - (n_1 - 1)(n_2 - 1) \\ &= n_1 \cdot dim(K_{n_2}) + n_2 \cdot dim(K_{n_1}) - dim(K_{n_1}) \cdot dim(K_{n_2}). \end{aligned}$$

Therefore, the above bound is tight. Examples of non-complete graphs, where the above bound is attained, can be derived from Theorem 2.3.

Given two vertices x and y in a connected graph $G = (V, E)$, the interval $I[x, y]$ between x and y is defined as the collection of all vertices which lie on some shortest $x - y$ path. Given a nonnegative integer k, we say that G *is self k-resolved* if for every two different vertices $x, y \in V$, there exists $w \in V$ such that ($d_G(y, w) \geq k$ and $x \in I[y, w]$) or ($d_G(x, w) \geq k$ and $y \in I[x, w]$). For instance, the path and the cycle graphs of order n ($n \geq 2$) are self $\lceil \frac{n}{2} \rceil$-resolved, the two-dimensional grid graphs $P_n \Box P_m$ are self $\left(\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil\right)$-resolved, and the hypercube graphs Q_k are self k-resolved.

Theorem 2.2. *Let H be a self k-resolved graph of order n_2 and let G be a graph of diameter $D(G) < k$. Then $dim(G \boxtimes H) \leq n_2 \cdot dim(G)$.*

Proof. Let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$ be the set of vertices of G and H, respectively. Let S_1 be a metric generator for G. We will show that $S = S_1 \times V_2$ is a metric generator for $G \boxtimes H$. Let $(u_i, v_j), (u_r, v_l)$ be two different vertices of $G \boxtimes H$. We differentiate the following two cases.

Case 1. $j = l$. Since $i \neq r$ and S_1 is a metric generator for G, there exists $u \in S_1$ such that $d_G(u_i, u) \neq d_G(u_r, u)$. Hence,

$$d_{G \boxtimes H}((u_i, v_j), (u, v_j)) = d_G(u_i, u) \neq d_G(u_r, u) = d_{G \boxtimes H}((u_r, v_j), (u, v_j)).$$

Case 2. $j \neq l$. Since H is self k-resolved, there exists $v \in V_2$ such that ($d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l]$) or ($d_H(v, v_j) \geq k$ and $v_l \in I[v, v_j]$). Say $d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l]$. In such a case, for every $u \in S$ we have

$$\begin{aligned} d_{G \boxtimes H}((u_i, v_j), (u, v)) &= \max\{d_G(u_i, u), d_H(v_j, v)\} \\ &< d_H(v, v_l) \\ &= \max\{d_G(u, u_r), d_H(v, v_l)\} \\ &= d_{G \boxtimes H}((u_r, v_l), (u, v)). \end{aligned}$$

Therefore, S is a metric generator for $G \boxtimes H$. □

Now we derive some consequences of the above result.

Corollary 2.1. *Let $n_1 \geq 2$ be an integer.*

- *For any integer $n_2 \geq 4$ such that $n_1 - 1 < \lfloor \frac{n_2}{2} \rfloor$, $dim(P_{n_1} \boxtimes C_{n_2}) \leq n_2$.*
- *Let $k \geq 2$ be an integer. For any self k-resolved graph H of order n_2, $dim(K_{n_1} \boxtimes H) \leq (n_1 - 1)n_2$.*

Given a vertex v of a graph $G = (V, E)$, we denote by $N_G(v)$ the open neighborhood of v, *i.e.*, the set of neighbors of v, and by $N_G[v]$ the closed neighborhood of v, *i.e.*, $N_G[v] = N_G(v) \cup \{v\}$. Two vertices u and v are false twins if $N_G(u) = N_G(v)$, while they are true twins if $N_G[u] = N_G[v]$. Note that if two vertices u and v of a graph $G = (V, E)$ are (true or false) twins, then $d_G(x, u) = d_G(x, v)$, for every $x \in V - \{u, v\}$. We define the true twin equivalence relation $\mathcal{R}$ on $V(G)$ as follows: $x\mathcal{R}y \leftrightarrow N_G[x] = N_G[y]$. If the true twin equivalence classes are $U_1, U_2, ..., U_t$, then every metric generator of G must contain at least $|U_i| - 1$ vertices from U_i, for each $i \in \{1, ..., t\}$. Therefore, since $U_1, U_2, ..., U_t$ form a partition of $V(G)$, it follows $dim(G) \geq \sum_{i=1}^{t}(|U_i| - 1) = n - t$, where n is the order of G.

Theorem 2.3. *Let G and H be two nontrivial connected graphs of order n_1 and n_2, having t_1 and t_2 true twin equivalent classes, respectively. Then $dim(G \boxtimes H) \geq n_1 n_2 - t_1 t_2$. Moreover, if $dim(G) = n_1 - t_1$ and $dim(H) = n_2 - t_2$, then $dim(G \boxtimes H) = n_1 n_2 - t_1 t_2$.*

Proof. Let $U_1, U_2, ..., U_{t_1}$ and $U'_1, U'_2, ..., U'_{t_2}$ be the true twin equivalence classes of G and H, respectively. Since each U_i (and U'_j) induces a clique and its vertices have identical closed neighborhoods, for every $a, c \in U_i$ and $b, d \in U'_j$,

$$\begin{aligned} N_{G\boxtimes H}[(a,b)] &= \{(x,y): \ x \in N_G[a], y \in N_H[b]\} \\ &= \{(x,y): \ x \in N_G[c], y \in N_H[d]\} \\ &= N_{G\boxtimes H}[(c,d)]. \end{aligned}$$

Hence, $V(G)\times V(H)$ is partitioned as $V(G)\times V(H) = \bigcup_{j=1}^{t_2}\left(\bigcup_{i=1}^{t_1} U_i \times U'_j\right)$, where $U_i \times U'_j$ induces a clique in $G \boxtimes H$ and its vertices have identical closed neighborhoods. Therefore, the metric dimension of $G \boxtimes H$ is at least $\sum_{j=1}^{t_2}\left(\sum_{i=1}^{t_1}(|U_i||U_j|-1)\right) = n_1n_2 - t_1t_2$.

Finally, if $dim(G) = n_1 - t_1$ and $dim(H) = n_2 - t_2$, then the above bound and Theorem 2.1 lead to $dim(G \boxtimes H) = n_1n_2 - t_1t_2$. □

As an example of non-complete graph G of order n having t true twin equivalence classes, where $dim(G) = n - t$, we take $G = K_1 + (\bigcup_{i=1}^{l} K_{r_i})$, $r_i \geq 2$, $l \geq 2$. In this case G has $t = l + 1$ true twin equivalence classes, $n = 1 + \sum_{i=1}^{l} r_i$ and $dim(G) = \sum_{i=1}^{l}(r_i - 1) = n - t$.

Corollary 2.2. *Let H be a graph of order n_2. Let G be a nontrivial connected graph of order n_1, having t_1 true twin equivalence classes. Then $dim(G \boxtimes H) \geq n_2(n_1 - t_1)$.*

Theorem 2.2 and Corollary 2.2 lead to the following result.

Theorem 2.4. *Let H be a self k-resolved graph of order n_2 and let G be a nontrivial connected graph of order n_1, having t_1 true twin equivalence classes and diameter $D(G) < k$. If $dim(G) = n_1 - t_1$, then $dim(G \boxtimes H) = n_2(n_1 - t_1)$.*

Lemma 2.1. *A nontrivial connected graph is self 2-resolved if and only if it does not have true twin vertices.*

Proof. Necessity. Let G be a 2-resolved graph. Let x and y be two adjacent vertices in G. Without loss of generality, we take $w \in V(G)$ such that $2 \leq k = d_G(x,w)$ and $y \in I[x,w]$. So, there exists a shortest path $x, y, u_2, ..., u_{k-1}, w$ from x to w and, as a consequence, $u_2 \in N_G[y]$ and $u_2 \notin N_G[x]$. Therefore, G does not have true twin vertices.

Sufficiency. If for every $u, v \in V(G)$, $N_G[u] \neq N_G[v]$, then for each pair of adjacent vertices x and y, there exists $w \in V(G) - \{x,y\}$ such that ($d_G(x,w) = 2$ and $y \in I[x,w]$) or ($d_G(y,w) = 2$ and $x \in I[y,w]$). On the other hand, if $d_G(u,v) \geq 2$, then for $w = u$ we have $d_G(v,w) \geq 2$ and $u \in I[v,w]$. Therefore, G is self 2-resolved. □

By Lemma 2.1 we deduce the following consequence of Theorem 2.4.

Corollary 2.3. *Let H be a connected graph of order $n_2 \geq 3$. If H does not have true twin vertices and $n_1 \geq 2$, then $dim(K_{n_1} \boxtimes H) = n_2(n_1 - 1)$.*

The following remark emphasizes some particular cases of the above result.

Remark 2.2. Let $n_1 \geq 2$ be an integer.

- For any tree T of order $n_2 \geq 3$, $dim(K_{n_1} \boxtimes T) = n_2(n_1 - 1)$.
- For any $n_2 \geq 4$, $dim(K_{n_1} \boxtimes C_{n_2}) = n_2(n_1 - 1)$.
- For any hypercube $Q_r = \underbrace{K_2 \square \cdots \square K_2}_{r}$, $r \geq 2$, $dim(K_{n_1} \boxtimes Q_r) = 2^r(n_1 - 1)$.

- For any integers $m, n \geq 2$, $dim(K_{n_1} \boxtimes (P_n \square P_m)) = n \cdot m \cdot (n_1 - 1)$.

Now we proceed to study the strong product of path graphs.

Theorem 2.5. *For any integers n_1 and n_2 such that $2 \leq n_1 < n_2$,*

$$\left\lfloor \frac{n_1 + n_2 - 2}{n_1 - 1} \right\rfloor \leq dim(P_{n_1} \boxtimes P_{n_2}) \leq \left\lceil \frac{n_1 + n_2 - 2}{n_1 - 1} \right\rceil.$$

Proof. Let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$ be the set of vertices of P_{n_1} and P_{n_2}, respectively. With the above notation we suppose that two consecutive vertices of V_i are adjacent, $i \in \{1, 2\}$.

Let $\alpha = \left\lceil \frac{n_2 - 1}{n_1 - 1} \right\rceil - 1$. We define the set S of cardinality $\left\lceil \frac{n_1 + n_2 - 2}{n_1 - 1} \right\rceil$ as follows:

$$S = \{(u_1, v_1), (u_{n_1}, v_{n_1}), (u_1, v_{2(n_1-1)+1}), (u_{n_1}, v_{3(n_1-1)+1}), ..., (u_1, v_{\alpha(n_1-1)+1}), (u_{n_1}, v_{n_2})\}$$

if $\left\lceil \frac{n_2 - 1}{n_1 - 1} \right\rceil$ is odd, and

$$S = \{(u_1, v_1), (u_{n_1}, v_{n_1}), (u_1, v_{2(n_1-1)+1}), (u_{n_1}, v_{3(n_1-1)+1}), ..., (u_{n_1}, v_{\alpha(n_1-1)+1}), (u_1, v_{n_2})\}$$

if $\left\lceil \frac{n_2 - 1}{n_1 - 1} \right\rceil$ is even.

We will show that S is a metric generator for $P_{n_1} \boxtimes P_{n_2}$.

Let $(u_i, v_j), (u_k, v_l)$ be two different vertices of $P_{n_1} \boxtimes P_{n_2}$. We differentiate two cases.

Case 1. $j = l$. We suppose, without loss of generality, that $i < k$. If $j \in \{1, ..., n_1\}$ and $d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_{n_1}, v_{n_1})) = d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_j), (u_{n_1}, v_{n_1}))$, then from $\max\{n_1 - i, n_1 - j\} = \max\{n_1 - k, n_1 - j\}$ we have $n_1 - j \geq n_1 - i > n_1 - k$. Hence, $j < k$ and, as a consequence,

$$\begin{aligned} d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_1, v_1)) &= \max\{i - 1, j - 1\} \\ &< k - 1 \\ &= \max\{k - 1, j - 1\} \\ &= d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_j), (u_1, v_1)). \end{aligned}$$

Thus, if $j \in \{1, ..., n_1\}$, then we deduce $r((u_i, v_j)|S) \neq r((u_k, v_j)|S)$.

An analogous procedure can be used to show that for $j \in \{t(n_1 - 1) + 1, ..., (t + 1)(n_1 - 1) + 1\}$, where $t \in \{1, .., \alpha - 1\}$, and for $j \in \{\alpha(n_1 - 1) + 1, ..., n_2\}$, it follows $r((u_i, v_j)|S) \neq r((u_k, v_j)|S)$.

Case 2. $j \neq l$. We suppose, without loss of generality, that $j < l$ and we differentiate two subcases.

Subcase 2.1. $l < n_1$. Since $(u_1, v_1), (u_{n_1}, v_{n_1}) \in S$, we only must consider the case when $d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_1, v_1)) = d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_l), (u_1, v_1))$ and $d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_{n_1}, v_{n_1})) = d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_l), (u_{n_1}, v_{n_1}))$. In such a situation, since $j < l$, we have $k < i$. Hence,

$$\begin{aligned} d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_1, v_{n_2})) &= \max\{i - 1, n_2 - j\} \\ &> \max\{k - 1, n_2 - l\} \\ &= d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_l), (u_1, v_{n_2})). \end{aligned}$$

So, if $(u_1, v_{n_2}) \in S$, then $r((u_i, v_j)|S) \neq r((u_k, v_l)|S)$. Moreover, if $(u_1, v_{n_2}) \notin S$, then $(u_1, v_{2n_1-1}) \in S$. Hence, from

$$\begin{aligned} d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_1, v_{2n_1-1})) &= \max\{i-1, 2n_1-1-j\} \\ &= 2n_1 - 1 - j \\ &> 2n_1 - 1 - l \\ &= \max\{k-1, 2n_1-1-l\} \\ &= d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_l), (u_1, v_{2n_1-1})), \end{aligned}$$

we have $r((u_i, v_j)|S) \neq r((u_k, v_l)|S)$.

Subcase 2.2. $l \geq n_1$. Since $j < l$ and $i, k \leq n_1$, we have that $i, k \leq l$. So,

$$\begin{aligned} d_{P_{n_1} \boxtimes P_{n_2}}((u_k, v_l), (u_1, v_1)) &= \max\{k-1, l-1\} \\ &= l - 1 \\ &> \max\{i-1, j-1\} \\ &= d_{P_{n_1} \boxtimes P_{n_2}}((u_i, v_j), (u_1, v_1)). \end{aligned}$$

Thus, in this case $r((u_i, v_j)|S) \neq r((u_k, v_l)|S)$ as well.

We conclude that S is a metric generator for $P_{n_1} \boxtimes P_{n_2}$ and, as a consequence, the upper bound follows.

We will show that $dim(P_{n_1} \boxtimes P_{n_2}) \geq \left\lfloor \frac{n_1+n_2-2}{n_1-1} \right\rfloor$ by contradiction. Let $n_2 - 1 = x(n_1-1) + y$, where $n_1 - 1 > y \geq 0$. Now we suppose that there exists a metric generator for $P_{n_1} \boxtimes P_{n_2}$, say S', of cardinality x. Note that a vertex $(u_r, v_t) \in S'$ distinguishes two vertices $(u_1, v_j), (u_2, v_j)$ if and only if $|t-j| < r-1$. Analogously, a vertex $(u_r, v_t) \in S'$ distinguishes two vertices $(u_{n_1-1}, v_{j'}), (u_{n_1}, v_{j'})$ if and only if $|t-j'| < n_1-r$. Hence, a vertex $(u_r, v_t) \in S'$ distinguishes, at most, $2n_1-3$ pairs of vertices of the form $(u_1, v_j), (u_2, v_j)$ or $(u_{n_1-1}, v_{j'}), (u_{n_1}, v_{j'})$. Thus, if S' is a metric generator, then $2n_2 - x \leq (2n_1-3)x$ and, as a consequence, $n_2 - 1 \leq x(n_1-1) - 1$, a contradiction. □

Conjecture 2.3. *For any integers n_1 and n_2 such that $2 \leq n_1 < n_2$, $dim(P_{n_1} \boxtimes P_{n_2}) = \left\lceil \frac{n_1+n_2-2}{n_1-1} \right\rceil$.*

Theorem 2.6. *For any integer $n \geq 2$, $dim(P_n \boxtimes P_n) = 3$.*

Proof. Let $V = \{v_1, v_2, ..., v_n\}$ be the set of vertices of P_n. Now, with the above notation, we suppose that two consecutive vertices of V are adjacent. We will show that $S' = \{(u_1, v_1), (u_n, v_1), (u_n, v_n)\}$ is a metric generator for $P_n \boxtimes P_n$. Let $(u_i, v_j), (u_k, v_l)$ be two different vertices of $P_n \boxtimes P_n$. We only must consider the case when $d_{P_n \boxtimes P_n}((u_i, v_j), (u_1, v_1)) = d_{P_n \boxtimes P_n}((u_k, v_l), (u_1, v_1))$ and $d_{P_n \boxtimes P_n}((u_i, v_j), (u_n, v_n)) = d_{P_n \boxtimes P_n}((u_k, v_l), (u_n, v_n))$. In such a case, if $j < l$, then $k < i$ and, as a consequence,

$$\begin{aligned} d_{P_n \boxtimes P_n}((u_i, v_j), (u_n, v_1)) &= \max\{n-i, j-1\} \\ &< \max\{n-k, l-1\} \\ &= d_{P_n \boxtimes P_n}((u_k, v_l), (u_n, v_1)). \end{aligned}$$

Analogously, if $j > l$, then we have $d_{P_n \boxtimes P_n}((u_i, v_j), (u_n, v_1)) > d_{P_n \boxtimes P_n}((u_k, v_l), (u_n, v_1))$. We conclude that S' is a metric generator for $P_n \boxtimes P_n$ and, as a consequence, $dim(P_n \boxtimes P_n) \leq 3$. In order to show that $dim(P_n \boxtimes P_n) \geq 3$, we suppose that there exists a metric generator for $P_n \boxtimes P_n$ of cardinality two. Since $(0, 0)$ is not a possible distance vector, and the diameter of $P_n \boxtimes P_n$ is $n-1$, there are $n^2 - 1$ possible distance vectors, but the order of $P_n \boxtimes P_n$ is n^2, a contradiction. So, $dim(P_n \boxtimes P_n) \geq 3$ and, as a consequence, $dim(P_n \boxtimes P_n) = 3$. □

The following claim will be useful in the proof of Theorem 2.7.

Claim 2.4. *Let C be a cycle graph. If x, y, u and v are vertices of C such that $u \neq v$, $x \neq y$, x, y are adjacent and $d_C(u, x) = d_C(v, x)$, then $d_C(u, y) \neq d_C(v, y)$.*

Theorem 2.7. *For any integers n_1 and n_2 such that $\frac{n_1 - 1}{2} \geq \left\lfloor \frac{n_2}{2} \right\rfloor \geq 2$, $dim(P_{n_1} \boxtimes C_{n_2}) \leq n_1$.*

Proof. Let $V_1 = \{u_0, u_1, ..., u_{n_1-1}\}$ and $V_2 = \{v_0, v_1, ..., v_{n_2-1}\}$ be the set of vertices of P_{n_1} and C_{n_2}, respectively. Here we suppose that v_0 and v_{n_2-1} are adjacent vertices in C_{n_2} and, with the above notation, two consecutive vertices of V_i are adjacent, $i \in \{1, 2\}$. Let S be the set of vertices of $P_{n_1} \boxtimes C_{n_2}$ of the form (u_i, v_i), where the subscript of the second component is taken modulo n_2. We will show that S is a metric generator for $P_{n_1} \boxtimes C_{n_2}$. To begin with, we consider two different vertices (u_i, v_j) and (u_k, v_l) of $P_{n_1} \boxtimes C_{n_2}$.

First we consider the case $i = k$ and we suppose, without loss of generality, that $j < l$. Now, if $d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_j), (u_i, v_i)) = d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_l), (u_i, v_i))$, then $d_{C_{n_2}}(v_j, v_i) = d_{C_{n_2}}(v_l, v_i)$. So, since $v_j \neq v_l$, for $i = 0$, Claim 2.4 leads to

$$\begin{aligned} d_{P_{n_1} \boxtimes C_{n_2}}((u_0, v_j), (u_1, v_1)) &= \max\{1, d_{C_{n_2}}(v_j, v_1)\} \\ &\neq \max\{1, d_{C_{n_2}}(v_l, v_1)\} \\ &= d_{P_{n_1} \boxtimes C_{n_2}}((u_0, v_l), (u_1, v_1)). \end{aligned}$$

Analogously, for $i \neq 0$, Claim 2.4 leads to

$$\begin{aligned} d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_j), (u_{i-1}, v_{i-1})) &= \max\{1, d_{C_{n_2}}(v_j, v_{i-1})\} \\ &\neq \max\{1, d_{C_{n_2}}(v_l, v_{i-1})\} \\ &= d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_l), (u_{i-1}, v_{i-1})). \end{aligned}$$

Hence, $r((u_i, v_j)|S) \neq r((u_i, v_l)|S)$.

Now we consider the case $i \neq k$. We suppose, without loss of generality, that $i < k$. If $k \leq \left\lfloor \frac{n_2}{2} \right\rfloor$, then $n_1 - 1 - i > \left\lfloor \frac{n_2}{2} \right\rfloor = D(C_{n_2})$. Thus,

$$\begin{aligned} d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_j), (u_{n_1-1}, v_{n_1-1})) &= \max\{d_{P_{n_1}}(u_i, u_{n_1-1}), d_{C_{n_2}}(v_j, v_{n_1-1})\} \\ &= \max\{n_1 - 1 - i, d_{C_{n_2}}(v_j, v_{n_1-1})\} \\ &> \max\{n_1 - 1 - k, d_{C_{n_2}}(v_l, v_{n_1-1})\} \\ &= d_{P_{n_1} \boxtimes C_{n_2}}((u_k, v_l), (u_{n_1-1}, v_{n_1-1})). \end{aligned}$$

Moreover, if $k > \left\lfloor \frac{n_2}{2} \right\rfloor$, then

$$\begin{aligned} d_{P_{n_1} \boxtimes C_{n_2}}((u_i, v_j), (u_0, v_0)) &= \max\{d_{P_{n_1}}(u_i, u_0), d_{C_{n_2}}(v_j, v_0)\} \\ &= \max\{i, d_{C_{n_2}}(v_j, v_0)\} \\ &< \max\{k, d_{C_{n_2}}(v_l, v_0)\} \\ &= d_{P_{n_1} \boxtimes C_{n_2}}((u_k, v_l), (u_0, v_0)). \end{aligned}$$

Hence, $r((u_i, v_j)|S) \neq r((u_k, v_l)|S)$. Therefore, the set S of cardinality n_1 is a metric generator for $P_{n_1} \boxtimes C_{n_2}$. □

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[1] DEPARTAMENT D'ENGINYERIA INFORMÀTICA I MATEMÀTIQUES
UNIVERSITAT ROVIRA I VIRGILI
AV. PAÏSOS CATALANS 26, 43007 TARRAGONA, SPAIN
E-mail address: juanalberto.rodriguez@urv.cat, dorota.kuziak@urv.cat

[2] DEPARTAMENTO DE MATEMÁTICAS, ESCUELA POLITÉCNICA SUPERIOR DE ALGECIRAS
UNIVERSIDAD DE CÁDIZ
AV. RAMÓN PUYOL S/N, 11202 ALGECIRAS, SPAIN
E-mail address: ismael.gonzalez@uca.es

[3] UNIVERSIDAD AUTÓNOMA DE GUERRERO
FACULTAD DE MATEMÁTICAS
CARLOS E. ADAME 5, COL. LA GARITA, ACAPULCO, GUERRERO, MÉXICO
E-mail address: josemariasigarretaalmira@hotmail.com