CARPATHIAN J. MATH. **31** (2015), No. 3, 269 - 275

Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Systems of variational relations with lower semicontinuous set-valued mappings

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ABSTRACT. In this paper, we use fixed point techniques to establish existence criteria of the solution for a system of two variational relations with lower semicontinuous set-valued mappings.

1. INTRODUCTION

In [16], Luc proposed a general model for a large class of problems of optimization and nonlinear analysis, introducing the concept of variational relation problem. The standard relations of equality, inequality, inclusion or nonempty intersection are replaced, in Luc's paper, with an arbitrary relation between the elements of two or three given sets. This approach proved to be a powerful tool for studying a wide class of problems in diverse fields of pure and applied mathematics and for this reason Luc's problem, as well as other types of variational relation problems or systems of variational relation problems have been investigated in many recent papers. Thus, Luc's problem is studied in locally G-convex spaces in [10], on Hadamard manifolds in [22], or in absence of convexity in [17, 23]. Then, by using the Kakutani-Fan-Glicksberg fixed point theorem, Balaj and Lin obtain in [4] sufficient conditions for the existence of solution of problem (LVRP) when the variable z is missing. New types of variational relation problems are introduced in [5], [11], [24], [24].

Recently, Yang [25] studied the existence of solution for the following system of two variational relations:

$(SVRP_1)$ Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in S_1(\bar{x},\bar{y})$, $\bar{y} \in S_2(\bar{x},\bar{y})$ and

$$\begin{cases} R_1(\bar{x},\bar{y},z_1) & \text{holds for all } z_1 \in S_1(\bar{x},\bar{y}) \\ R_2(\bar{x},\bar{y},z_2) & \text{holds for all } z_2 \in S_2(\bar{x},\bar{y}), \end{cases}$$

when X and Y are compact convex sets in two vector normed spaces, $S_1 : X \times Y \rightrightarrows X$ and $S_2 : X \times Y \rightrightarrows Y$ are two continuous set-valued mappings with nonempty compact convex values and $R_1(x,y,z_1)$ and $R_2(x,y,z_2)$ are closed relations between elements $x \in X$, $y \in Y$ and $z_1 \in X$ and respectively, $x \in X$, $y \in Y$ and $z_2 \in Y$.

Earlier, Lin and Wang [14] established sufficient conditions for the existence of solution of $(SVRP_1)$ when X and Y are compact convex metrizable subsets of two locally convex Hausdorff topological vector spaces, and S_1 and S_2 are set-valued mappings only in the variable x ($S_1(x,y) = S_1(x), S_2(x,y) = S_2(x)$).

In concrete problems, relations R_1 and R_2 are determined by equalities and inequalities of real functions or by inclusions and intersections of multivalued mappings. Some typical examples of problems of type $(SVRP_1)$, encountered in the literature, are given bellow:

Received: 14.09.2014; In revised form: 09.01.2015; Accepted: 31.01.2015
2010 *Mathematics Subject Classification.* 49J40, 49J53.
Key words and phrases. *System of variational relations, set-valued mapping, fixed point.*

(a) Let Z be a topological vector space and $F_1 : X \times Y \times X \rightrightarrows Z$, $F_2 : X \times Y \times Y \rightrightarrows Z$, $C : X \rightrightarrows Z$ be set-valued mappings with nonempty values. Problems (I) and (II) below, called simultaneous generalized vector quasi-equilibrium problems, are studied in [2], [3], [15].

(I) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S_1(\bar{x}, \bar{y})$, $\bar{y} \in S_2(\bar{x}, \bar{y})$ and

$$\begin{cases} F_1(\bar{x}, \bar{y}, z_1) \subseteq C(\bar{x}) & \text{holds for all } z_1 \in S_1(\bar{x}, \bar{y}) \\ F_2(\bar{x}, \bar{y}, z_2) \subseteq C\bar{x}) & \text{holds for all } z_2 \in S_2(\bar{x}, \bar{y}). \end{cases}$$

(II) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S_1(\bar{x}, \bar{y})$, $\bar{y} \in S_2(\bar{x}, \bar{y})$ and

$$\begin{cases} F_1(\bar{x}, \bar{y}, z_1) \cap C(\bar{x}) \neq \emptyset & \text{for all } z_1 \in S_1(\bar{x}, \bar{y}) \\ F_2(\bar{x}, \bar{y}, z_2) \cap C(\bar{x}) \neq \emptyset & \text{for all } z_2 \in S_2(\bar{x}, \bar{y}). \end{cases}$$

(b) Let Z be a topological vector space, $f_1, f_2 : X \times Y \to Z$, $\theta : X \to Z$, $\eta : Y \to Z$ and $C : Z \to Z$ be a set-valued mapping such that for every $z \in Z$, $C(z)$ is a closed convex cone with nonempty interior. The generalized symmetric quasi-equilibrium problem consists in finding $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S_1(\bar{x}, \bar{y})$, $\bar{y} \in S_2(\bar{x}, \bar{y})$ and

$$\begin{cases} f_1(z_1, \bar{y}) - f_1(\bar{x}, \bar{y}) \notin -\operatorname{int} C(\eta(\bar{y})) & \text{for all } z_1 \in S_1(\bar{x}, \bar{y}) \\ f_2(\bar{x}, z_2) - f_2(\bar{x}, \bar{y}) \notin -\operatorname{int} C(\theta(\bar{x})) & \text{for all } z_2 \in S_2(\bar{x}, \bar{y}), \end{cases}$$

and it has been considered in [8], [9], [18].

In all the results established in the papers mentioned above, the set-valued mappings S_1 and S_2 are either upper semicontinuous, or they have open fibers. To the best of our knowledge there is no general existence result for problem $(SVRP_1)$, when the set-valued mappings S_1 and S_2 are assumed lower semicontinuous (not necessarily with open fibers). This might have been caused by the fact the lower semicontinuity is more difficult to study than upper semicontinuity. As we shall see, a major problem consists in the fact that the intersection of lower semicontinuous mappings need not be lower semicontinuous.

Our task in this paper is to prove several existence results for a problem more general than $(SVRP_1)$, which do not involve the upper semicontinuity of S_1 and S_2, but only the lower semicontinuity.

For a subset A of a topological vector space, the standard notations $\text{cl}\, A$ and $\text{co}\, A$ designate respectively the closure and the convex hull of A.

Let us fix the data of this problem: For each index $i \in \{1, 2\}$, X_i is a convex, metrizable subset of a locally convex Hausdorff topological vector space, Z_i is a topological space, $S_i : X_1 \times X_2 \rightrightarrows X_i$, $P_i : X_1 \times X_2 \rightrightarrows Z_i$ are set-valued mappings with nonempty values. Furthermore, $R_1(x_1, x_2, z_1)$ is a relation between elements $x_1 \in X_1, x_2 \in X_2, z_1 \in Z_1$ and $R_2(x_2, x_1, z_2)$ is a relation linking $x_2 \in X_2, x_1 \in X_1, z_2 \in Z_2$. Consider the the set-valued mappings $G_1 : X_2 \times Z_1 \rightrightarrows X$ and $G_2 : X_1 \times Z_2 \rightrightarrows Y$ defined by

$$\begin{cases} G_1(x_2, z_1) = \{x_1 \in X_1 : R_1(x_1, x_2, z_1) \text{ holds }\} & \text{for all } (x_2, z_1) \in X_2 \times Z_1, \\ G_2(x_1, z_2) = \{x_2 \in X_2 : R_2(x_2, x_1, z_2) \text{ holds }\} & \text{for all } (x_1, z_2) \in X_1 \times Z_2. \end{cases} \tag{1.1}$$

We associate to R_1 and R_2 two new relations $\overline{R}_1$, and respectively $\overline{R}_2$, defined as follows:

$$\overline{R}_1(x_1, x_2, z_1) \text{ holds iff } x_1 \in \text{ cl}\, G_1(x_2, z_1),$$

$$\overline{R}_2(x_2, x_1, z_2) \text{ holds iff } x_2 \in \text{ cl}\, G_2(x_1, z_2).$$

The problem considered in this paper is:

$(SVRP)$ Find $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ such that $\bar{x}_1 \in S_1(\bar{x}_1, \bar{x}_2)$, $\bar{x}_2 \in S_2(\bar{x}_1, \bar{x}_2)$ and

$$\begin{cases} \overline{R}_1(\bar{x}_1, \bar{x}_2, z_1) & \text{holds for all } z_1 \in P_1(\bar{x}_1, \bar{x}_2) \\ \overline{R}_2(\bar{x}_2, \bar{x}_1, z_2) & \text{holds for all } z_2 \in P_2(\bar{x}_1, \bar{x}_2). \end{cases}$$

2. Preliminaries

We recall here some definitions and known results concerning set-valued mappings. If X and Y are topological spaces, a set-valued mapping $T : X \rightrightarrows Y$ is said to be: (i) lower semicontinuous if for every open subset B of Y the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ is open; (ii) upper semicontinuous if for every open subset B of Y the set $\{x \in X : T(x) \subseteq B\}$ is open; (iii) compact if $\text{cl}\,(T(X))$ is a compact set in Y. It is often convenient to characterize the upper semicontinuity in terms of nets, as in the following lemma:

Lemma 2.1. *A set-valued $T : X \rightrightarrows Y$ with compact values is upper semicontinuous if and only if for every net $\{x_t\}$ in X converging to $x \in X$ and for any net $\{y_t\}$, with $y_t \in T(x_t)$, there exist $y \in T(x)$ and a subnet $\{y_{t_i}\}$ of $\{y_t\}$ converging to y. Moreover, when X and Y are metrizable the above statement remains true if the net $\{x_t\}$ is replaced by a sequence $\{x_n\}$.*

The following lemmas will be used in the next section.

Lemma 2.2. [21] *Let X be a nonempty compact, convex, metrizable subset of a locally convex Hausdorff topological vector space and $S : X \rightrightarrows X$ be a compact, lower semicontinuous set-valued mapping with nonempty, closed and convex values. Then S has a fixed point.*

Lemma 2.3. [20] *Let X and Y be topological spaces and D a closed subset of X. Suppose that $F_1 : D \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$ are lower semicontinuous mappings such that $F_1(x) \subseteq F_2(x)$ for all $x \in D$. Then the mapping $F : X \rightrightarrows Y$ defined by*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in D \\ F_2(x) & \text{if } x \in X \setminus D. \end{cases}$$

is also lower semicontinuous.

3. Main results

Theorem 3.1. *Let X_1, X_2 be convex, metrizable sets, each in a locally convex Hausdorff topological vector space, Z_1, Z_2 be topological spaces, $S_1 : X_1 \times X_2 \rightrightarrows X_1$, $S_2 : X_1 \times X_2 \rightrightarrows X_2$ $P_1 : X_1 \times X_2 \rightrightarrows Z_1$, $P_2 : X_1 \times X_2 \rightrightarrows Z_2$ be set-valued mappings with nonempty values and $R_1(x_1, x_2, z_1)$, $R_2(x_2, x_1, z_2)$ be relations between elements $x_i \in X_i, z_i \in Z_i$ (i = 1, 2). Assume that for each index $i \in \{1, 2\}$ the following conditions hold:*

(i) *S_i is compact, lower semicontinuous with closed and convex values;*
(ii) *the set $D_i = \{(x_1, x_2) \in X_1 \times X_2 : x_i \in S_i(x_1, x_2)\}$ is closed;*
(iii) *for each $(x_1, x_2) \in D_{3-i}$ there exists $u_i \in S_i(x_1, x_2)$ such that $\overline{R}_i(u_i, x_{3-i}, z_i)$ holds for all $z_i \in P_i(x_1, x_2)$;*
(iv) *relation $\overline{R}_i$ is convex in the first variable;*
(v) *the set-valued mapping $T_i : X_1 \times X_2 \rightrightarrows X_i$ defined by*

$$T_i(x_1, x_2) = \{u_i \in S_i(x_1, x_2) : \overline{R}_1(u_i, x_{3-i}, z_i) \text{ holds for all } z_i \in P_i(x_1, x_2)\}$$

is lower semicontinuous.

Then, there exists $(\overline{x}_1, \overline{x}_2) \in X_1 \times X_2$ such that $\overline{x}_1 \in S_1(\overline{x}_1, \overline{x}_2)$, $\overline{x}_2 \in S_2(\overline{x}_1, \overline{x}_2)$ and

$$\begin{cases} \overline{R}_1(\overline{x}_1, \overline{x}_2, z_1) & \text{holds for all } z_1 \in P_1(\overline{x}_1, \overline{x}_2), \\ \overline{R}_2(\overline{x}_2, \overline{x}_1, z_2) & \text{holds for all } z_2 \in P_2(\overline{x}_1, \overline{x}_2). \end{cases}$$

Proof. Define the set-valued mappings $\widetilde{S}_1 : X_1 \times X_2 \rightrightarrows X_1$, $\widetilde{S}_2 : X_1 \times X_2 \rightrightarrows X_2$ by

$$\widetilde{S}_i(x_1, x_2) = \begin{cases} T_i(x_1, x_2) & \text{if } (x_1, x_2) \in D_{3-i} \\ S_i(x_1, x_2) & \text{if } (x_1, x_2) \in X_1 \times X_2 \setminus D_{3-i}. \end{cases}$$

From (iii) and (iv) it follows readily that the values of T_1 and T_2 are nonempty and convex. By Lemma 2.3, $\widetilde{S}_1$, $\widetilde{S}_2$ are lower semicontinuous and, clearly, their values are nonempty, closed and convex. By Lemma 2.2, applied to the mapping $\widetilde{S} = \widetilde{S}_1 \times \widetilde{S}_2$, there exists $(\overline{x}_1, \overline{x}_2) \in X_1 \times X_2$ such that $\overline{x}_1 \in \widetilde{S}_1(\overline{x}_1, \overline{x}_2)$, $\overline{x}_2 \in \widetilde{S}_2(\overline{x}_1, \overline{x}_2)$. Since $\widetilde{S}_i(\overline{x}_1, \overline{x}_2) \subseteq S_i(\overline{x}_1, \overline{x}_2)$, we have $(\overline{x}_1, \overline{x}_2) \in D_1 \cap D_2$. From $\overline{x}_i \in T_i(\overline{x}_1, \overline{x}_2)$, it follows that $\overline{R}_i(\overline{x}_i, \overline{x}_{3-i}, z_i)$ holds for all $z_i \in P_i(\overline{x}_1, \overline{x}_2)$. □

In bringing the conclusion of Theorem 3.1 down to more details, let us examine conditions (i)- (iv). Conditions (i) and (ii) are standard. Condition (iii) can be replaced by another one, of KKM type, as stated in the next proposition. Recall that a set-valued mapping Q from a nonempty set Y to a convex subset X of a vector space is said to be generalized KKM if for every finite set $\{y_0, y_1, ..., y_n\} \subseteq Y$, there exists $\{x_0, x_1, \ldots, x_n\} \subseteq X$ such that for each index set $I \subseteq \{0, \ldots, n\}$ one has $\text{co}\{x_i : i \in I\} \subseteq \bigcup_{i \in I} Q(y_i)$.

Proposition 3.1. *For $i \in \{1, 2\}$, condition (iii) in Theorem 3.1 is fulfilled if*

(iii') *for every $(x_1, x_2) \in D_{3-i}$ and any finite subset $\{z_i^0, z_i^1, \ldots, z_i^n\}$ of $P_i(x_1, x_2)$ there exists a corresponding subset $\{u_i^0, u_i^1, \ldots, u_i^n\}$ of $S_i(x_1, x_2)$ such that for any nonempty set $J \subseteq \{0, \ldots, n\}$ and every $u_i \in co\{u_i^j : j \in J\}$ one can find some index $j \in J$ such that $\overline{R}_i(u_i, x_{3-i}, z_i^j)$ holds.*

Proof. Let i be a fixed index and (x_1, x_2) be an arbitrary element of D_{3-i}. Define $Q : P_i(x_1, x_2) \rightrightarrows S_i(x_1, x_2)$ by

$$Q(z_i) = \{u_i \in S_i(x_1, x_2) : \overline{R}(u_i, x_{3-i}, z_i) \text{ holds}\}.$$

By (iii'), it follows readily that Q is a generalized KKM map. The values of Q are compact since they are closed subsets of the compact $\text{cl}(S(X))$. By [6, Lemma 2.1], there is $u_i \in \bigcap_{z_i \in P_i(x_1, x_2)} Q(z_i)$. Then u_i is a point in $S_i(x_1, x_2)$ that satisfies (iii). □

To progress further, we seek now conditions that guarantee the lower semicontinuity of the set-valued mappings T_i. To this aim, for each $i \in \{1, 2\}$ we introduce the set-valued mappings $H_i, K_i : D_{3-i} \rightrightarrows X_i$ defined by,

$$H_i(x_1, x_2) = \{u_i \in X_i : R_i(u_i, x_{3-i}, z_i) \text{ holds for all } z_i \in P_1(x_1, x_2)\},$$

$$K_i(x_1, x_2) = S_i(x_1, x_2) \cap H_i(x_1, x_2).$$

Lemma 3.4. *Let E be a topological vector space. If $\{A_i\}_{i \in I}$ is a family of open convex subsets of E and B is a closed convex set in E such that $B \cap \bigcap_{i \in I} A_i \neq \emptyset$, then*

$$cl\big(B \cap \bigcap_{i \in I} A_i\big) = B \cap \bigcap_{i \in I} cl\ A_i.$$

Proof. Since $B \cap \bigcap_{i \in I} A_i$ is contained in the closed set $B \cap \bigcap_{i \in I} \text{cl } A_i$, $\text{cl}\big(B \cap \bigcap_{i \in I} A_i\big) \subseteq B \cap \bigcap_{i \in I} \text{cl } A_i$.

We demonstrate the opposite inclusion. Fix any $y \in B \cap \bigcap_{i \in I} A_i$. Given any $x \in B \cap \bigcap \bigcap_{i \in I}$ cl A_i, the point $z_\lambda = \lambda x + (1-\lambda)y$ belongs to B, as well as, to every set A_i for $0 \le \lambda < 1$, hence $z_\lambda \in B \cap \bigcap_{i \in I} A_i$. Since x is the limit of these points z_λ as $\lambda \uparrow 1$, it follows that $x \in \text{cl}\big(B \cap \bigcap_{i \in I} A_i\big)$. □

Proposition 3.2. *For $(x_1, x_2) \in X_1 \times X_2$ assume that:*

(i) $K_1(x_1, x_2) \neq \emptyset$;

(ii) R_1 *is open and convex in the first variable.*

Then, cl $K_1(x_1, x_2) = T_1(x_1, x_2)$.

Proof. If G_1 is the set valued-mapping defined in (1.1), observe that $K_1(x_1, x_2) = S_1(x_{1,2}) \cap \bigcap_{z_1 \in P_1(x_1,x_2)} G_1(x_2, z_1)$. By (ii) the sets $G_1(x_2, z_1)$ are open and convex. In view of Lemma 1 we have

$$\text{cl } H_1(x_1, x_2) = S_1(x_1, x_2) \cap \bigcap_{z_1 \in P_1(x_1,x_2)} \text{cl } G_1(x_2, z_1) =$$

$$S_1(x_1, x_2) \cap \bigcap_{z_1 \in P_1(x_1,x_2)} \{u_1 \in X_1 : \overline{R}_1(u_1, x_2, z_1) \text{ holds}\} = T_1(x_1, x_2).$$

□

As the closure of a lower semicontinuos set-valued mapping is lower semicontinuous [1, Lemma 17.22], in view of Proposition 3.2, T_1 will be lower semicontinuous on D_2 whenever K_1 will be so. Since $K_1(x_1, x_2) = S_1(x_1, x_2) \cap H_1(x_1, x_2)$ it would be desirable to find conditions such that the intersection of two set-valued mappings should be lower semicontinuous. In this direction, we observe that in general the intersection of two lower semicontinuous set-valued mappings is not lower semicontinuous, even if they have convex values and their intersection is nonempty (see [7], [13]). Thus, from the lower semicontinuity of the set-valued mappings S_1 and H_1, the lower semicontinuity of K_1, in general, does not follow. However, if we impose additional requirements on the set-valued mappings S_1 and P_1 and on the relation R_1, such an implication becomes true. Below we give an auxiliary result needed in what follows.

Lemma 3.5. [19] *If X and Y are Hausdorff topological spaces, $S : X \rightrightarrows Y$ is lower semicontinuous, $H : X \rightrightarrows Y$ has open graph and for all $x \in X$, $S(x) \cap H(x) \neq \emptyset$, then the intersection set-valued mapping $S \cap H$ is lower semicontinuous.*

Proposition 3.3. *If the set-valued mapping P_1 is upper semicontinuous and relation R_1 is open, then H_1 has open graph.*

Proof. We have to show that the set $\big(\text{Gr } H_1\big)^c := D_2 \times X_1 \setminus \text{Gr } H_1$ is closed in $D_2 \times X_1$. To this end, let $\{((x_1^n, x_2^n), u_1^n)\}$ be a sequence in $\big(\text{Gr } H_1\big)^c$ converging to $((x_1, x_2), u_1) \in D_2 \times X_1$. Then, for each n, there exists $z_1^n \in P_1(x_1^n, x_2^n)$ such that $R_1^c(u_1^n, x_2^n, z_1^n)$ holds. As P_1 is upper semicontinuous with compact values there exist $z_1 \in P(x_1, x_2)$ and a subnet $\{z_1^{n_k}\}$ of $\{z_1^n\}$ converging to z_1. Since relation R_1^c is closed, $R_1(u_1, x_2, z_1)$ holds, hence $((x_1, x_2), u_1) \in \big(\text{Gr } H_1\big)^c$. □

Combining Theorem 3.1 and Proposition 3.3 we get:

Theorem 3.2. *Assume that for each $i \in \{1,2\}$, the data of problem $(SVRP)$ satisfy the following conditions:*

(i) *S_i is compact, lower semicontinuous with closed and convex values;*
(ii) *P_i is upper semicontinuous with compact values;*
(iii) *the set $D_i = \{(x_1, x_2) \in X_1 \times X_2 : x_i \in S_i(x_1, x_2)\}$ is closed;*
(iv) *for each $(x_1, x_2) \in D_{3-i}$, there exists $u_i \in S_i(x_1, x_2)$ such that $R_i(u_i, x_{3-i}, z_i)$ holds for all $z_i \in P_i(x_1, x_2)$.*
(iv) *R_i is convex in the first variable and open.*

Then problem $(SVRP)$ has solution.

Proof. Let i be any of the indices 1, 2. By Proposition 3.3, the set-valued mapping H_i has open graph. By (iv), $S_i(x_1, x_2) \cap H(x_1, x_2) \neq \emptyset$ for all $(x_1, x_2) \in D_{3-i}$. Then, in view of Lemma 3.5, K_i is lower semicontinuous and by Proposition 3.2, T_i is also lower semicontinuous. Note that (iv) implies the condition similarly noted in Theorem 3.1 and that the convexity of R_i in the first variable is inherited by relation $\overline{R}_i$. The conclusion follows now from Theorem 3.1. □

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