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A constructive approach to coupled fixed point theorems in metric spaces

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ABSTRACT. In this paper we establish the existence and uniqueness of a coupled fixed point for operators $F : X \times X \to X$ satisfying a new type of contractive condition, which is weaker than all the corresponding ones studied in literature so far. We also provide constructive features to our coupled fixed point results by proving that the unique coupled fixed point of F can be approximated by means of two distinct iterative methods: a Picard type iterative method of the form $x_{n+1} = F(x_n, x_n),\ n \geq 0$, with $x_0 \in X$, as well as a two step iterative method of the form $y_{n+1} = F(y_{n-1}, y_n),\ n \geq 0$, with $y_0, y_1 \in X$. We also give appropriate error estimates for both iterative methods. Essentially we point out that all coupled fixed point theorems existing in literature, that establish the existence and uniqueness of a coupled fixed point with equal components, could be derived in a much more simpler manner.

1. INTRODUCTION

Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

There exists an extraordinary rich literature on the existence of coupled fixed points of mixed monotone, monotone and not monotone mappings that has developed in the last eight years, namely, after the publication of the seminal paper by Bhaskar Gnana and Lakshmikantham [19].

To give an idea on the extremely rapid development of this topic, let us search on Google Scholar for the current citations of that article: we find 486 papers that cite [19]. In SCOPUS database we find 233 papers that cite [19], while in ISI Web of Science database we find 227 papers, and all those papers were published in the last 7 years or so. Quite impressive !

In view of its extraordinary importance as a corner stone in fixed point theory, we state here the main results in [19]. To do so, we need the following notions.

Let$(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, (u, v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v.$$

We say that a mapping $F : X \times X \to X$ has the *mixed monotone property* if $F(x, y)$ is monotone nondecreasing in x and is monotone non increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

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and, respectively,

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

The next theorem is the main existence result in ([19], Theorem 2.1).

Theorem 1.1 (Bhaskar and Lakshmikantham [19]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \text{ for each } x \geq u, y \leq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F(\overline{x}, \overline{y}) \text{ and } \overline{y} = F(\overline{y}, \overline{x}).$$

In [19], Bhaskar and Lakshmikantham also established four other related results: Theorem 2.2, which shows that continuity of T is not necessary in the above result;Theorem 2.4, which establish conditions under which the coupled fixed point is actually unique, and Theorems 2.5 and 2.6 that give sufficient conditions for which the components of the coupled fixed point are identical, that is, one has $x = F(x, x)$. In this case, it is also said, see [38], etc. that x is a fixed point of F.

One of these Theorems reads as follows

Theorem 1.2. ([19], Theorem 2.6) *In addition to the hypothesis of Theorem 1, suppose that $x_0, y_0 \in X$ are comparable. Then for the coupled fixed point $(\overline{x}, \overline{y})$ we have $\overline{x} = \overline{y}$, that is, F has a fixed point:*

$$F(\overline{x}, \overline{x}) = \overline{x}.$$

In the same paper [19], Section 3, the authors applied Theorems 2.2 and 2.4 described above to the study of existence and uniqueness of a unique solution to a periodic boundary value problem.

Following the model established by [19], several authors devoted their efforts to extend, generalize or improve these results. Only a few of them also considered the application of the coupled fixed point theorems to the study of various nonlinear functional equations: 1) integral equations and systems of integral equations ([3], [4], [6], [17], [22], [23], [47], [48]); (periodic) two point boundary value problems ([11], [19], [21], [49]); nonlinear Hammerstein integral equations ([46]); nonlinear elliptic problems and delayed hematopoesis models ([50]); systems of differential and integral equations ([51]); nonlinear matrix and nonlinear quadratic equations ([2], [17]), initial value problems for ODE ([5]; [45]) etc.

But, if we look carefully to those applications, we can notice that in all cases, without any exception, it is not used an existence theorem but only the theorem which guarantees the existence and uniqueness of the coupled point and also the equality of its components.

On the other hand, in almost all these papers, there is not paid attention to the constructive features of such a theorem, that is, there is no explicit mention neither on the method by which one could approximate that coupled fixed point, nor on the order of convergence and / or error estimates of the iterative processes involved.

Starting from these two important observations, our main aim in this paper is twofold: first, we want to show that the coupled fixed point theorems of the form given by Theorem 1.2 could be derived from a weaker contraction condition than (1.1) (and the corresponding conditions in many other related papers), and secondly, to provide constructive methods for approximating the coupled fixed points. For these methods, we give both *a priori* and *a posteriori* error estimates

In the present paper we shall be concerned only with the case of some coupled fixed point theorems in metric spaces only.

The study of coupled fixed point theorems of mixed monotone mappings in partially ordered metric spaces will be done in a forthcoming paper [15].

Preliminaries

In order to prove the main results in the present paper, we shall make use of the following two Lemmas, which are important tools by themselves and will also contribute to significantly simplify the proofs.

Lemma 1.1. *If* $\{x_n\}_{n\geq 0}$ *is a sequence of non negative real numbers satisfying*

$$x_{n+1} \leq \alpha_1 x_n + \alpha_2 x_{n-1}, n \geq 1, \tag{1.2}$$

where $\alpha_1,\ \alpha_2 \in (0,1)$ *are such that* $\alpha_1 + \alpha_2 \leq 1$, *then*

a) $\{x_n\}_{n\geq 0}$ *is convergent;*

b) There exist $L > 0$ *and* $\theta \in (0,1]$ *such that*

$$x_n \leq L \cdot \theta^n, \text{ for all } n \geq 1. \tag{1.3}$$

c) If $\alpha_1 + \alpha_2 < 1$ *then* $x_n \to 0$ *as* $n \to \infty$.

Proof. We consider the auxiliary sequence $\{y_n\}_{n\geq 0}$, defined by

$$y_{n+1} = x_{n+1} + (1-\alpha_1)x_n,\ n \geq 0,$$

and prove first that, if $\{x_n\}_{n\geq 0}$ satisfies (1.2), then $\{y_n\}_{n\geq 0}$ is convergent. Indeed, by the definition of y_n and (1.2) we have

$$\begin{aligned} y_{n+1} &= x_{n+1} + (1-\alpha_1)x_n \leq \alpha_1 x_n + \alpha_2 x_{n-1} + (1-\alpha_1)x_n \\ &= x_n + \alpha_2 x_{n-1} \leq x_n + (1-\alpha_1)x_{n-1} = y_n, \end{aligned}$$

which shows that $\{y_n\}_{n\geq 0}$ is a decreasing sequence of non negative real numbers, hence it is convergent. Denote

$$\lim_{n\to\infty} y_n = l,$$

and let $0 < \epsilon < 1$ be arbitrary. Then there exist a rank $N = N(\epsilon)$ such that for all $n \geq N$

$$l - \epsilon - (1-\alpha_1)x_n \leq x_{n+1} \leq l + \epsilon - (1-\alpha_1)x_n. \tag{1.4}$$

Let now $m \in \mathbb{N}$ and take $n = m + N$ in (1.4) to get

$$\begin{aligned} x_{N+m+1} &\leq l + \epsilon - (1-\alpha_1)x_{N+m} \\ &\leq l + \epsilon - (1-\alpha_1)\left[l - \epsilon - (1-\alpha_1)x_{N+m-1}\right] \\ &= l\left[1-(1-\alpha_1)\right] + \epsilon\left[1+(1-\alpha_1)\right] + (1-\alpha_1)^2 x_{N+m-1}. \end{aligned}$$

Inductively, we obtain

$$x_{N+m+1} \leq l\left[1-(1-\alpha_1)+(1-\alpha_1)^2+\ldots+(-1)^m(1-\alpha_1)^m\right]+$$

$$+\epsilon\left[1+(1-\alpha_1)+\cdots+(1-\alpha_1)^m\right]+(1-\alpha_1)^{m+1}x_N,\ m \in \mathbb{N}. \tag{1.5}$$

Similarly, one obtains

$$\begin{aligned} x_{N+m+1} &\geq l - \epsilon - (1-\alpha_1)x_{N+m} \\ &\geq l - \epsilon - (1-\alpha_1)\left[l + \epsilon - (1-\alpha_1)x_{N+m-1}\right] \\ &= l\left[1-(1-\alpha_1)\right] - \epsilon\left[1+(1-\alpha_1)\right] + (1-\alpha_1)^2 x_{N+m-1}, \end{aligned}$$

and hence, inductively, we obtain

$$\begin{aligned} x_{N+m+1} &\geq l\left[1-(1-\alpha_1)+(1-\alpha_1)^2+\ldots+(-1)^m(1-\alpha_1)^m\right] \\ &-\epsilon\left[1+(1-\alpha_1)+\cdots+(1-\alpha_1)^m\right]+(1-\alpha_1)^{m+1}x_N,\ m\in\mathbb{N}. \end{aligned} \tag{1.6}$$

Therefore, by (1.5) and (1.6) we get the following double inequality

$$\begin{aligned} l\cdot a_m - \epsilon\cdot b_m + (1-\alpha_1)^{m+1}x_N &\leq x_{N+m+1} \\ &\leq l\cdot a_m + \epsilon\cdot b_m + (1-\alpha_1)^{m+1}x_N, \end{aligned} \tag{1.7}$$

where

$$a_m = \frac{1-(-1)^{m+1}(1-\alpha_1)^{m+1}}{2-\alpha_1} \to \frac{1}{2-\alpha_1} \text{ as } n\to\infty,$$

and

$$b_m = \frac{1-(1-\alpha_1)^{m+1}}{\alpha_1} \to \frac{1}{\alpha_1} \text{ as } n\to\infty,$$

since $1-\alpha_1<1$.

Now we can choose $m_0=m_0(\epsilon)$ such that for $m\geq m_0$ one has

$$\left|a_m - \frac{1}{2-\alpha_1}\right| < \frac{\epsilon}{3l},\quad \left|b_m - \frac{1}{\alpha_1}\right| < \frac{1}{3},$$

and

$$\left|(1-\alpha_1)^{m+1}x_N\right| < \frac{\epsilon}{3}.$$

Let us denote $n_0=\max\{N,m_0\}$. Then, in view of the previous three inequalities, for all $m\geq n_0-N$, by (1.7) we have

$$\left|x_{m+N+1} - \frac{1}{2-\alpha_1}\right| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{\alpha_1}+\frac{\epsilon}{3} = \epsilon + \frac{\epsilon}{\alpha_1} := \epsilon_1,$$

which shows that

$$\lim_{n\to\infty} x_n = \frac{1}{2-\alpha_1},$$

and this proves part a) of the Lemma.

b) Let $f(t)=t^2-\alpha_1 t-\alpha_2$, $t\in\mathbb{R}$. Since $f(0)=-\alpha_2<0$, $f(1)=1-\alpha_1-\alpha_2\geq 0$ and f is continuous, it follows that there exists $\theta\in(0,1]$ such that $f(\theta)=0$, that is,

$$\theta^2 = \alpha_1\theta + \alpha_2. \tag{1.8}$$

Assume $x_0>0$ or $x_1>0$ (otherwise, $\{x_n\}$ would be the trivial null sequence) and denote

$$L = \max\{x_0, \frac{x_1}{\theta}\} > 0.$$

Then (1.3) is clearly satisfied for $n=0$ and $n=1$.

We proceed by induction. Suppose that (1.3) is true for $n=k$ and $n=k+1$ and then prove that it is also true for $n=k+2$.

Indeed, by (1.8), (1.2), and the induction hypothesis we get

$$x_{k+2} \leq \alpha_1 x_{k+1} + \alpha_2 x_k \leq \alpha_1 L\theta^{k+1} + \alpha_2 L\theta^k = L\theta^k(\alpha_1\theta+\alpha_2) = L\theta^{k+2},$$

as required.

c) If $\alpha_1 + \alpha_2 < 1$, then by the arguments above, we have $\theta \in (0,1)$ and hence, in view of (1.3), the conclusion follows. □

Remark 1.1.

• Part a) in Lemma 1.1 is Theorem 2.2 in [7] (since the original paper is written in Romanian, we included here the original proof given by Bărbosu [7], for the convenience of the reader). The general result for the case of a subconvex combination of k terms is given in [43], see also [8]. The general Lemma should be useful for proving similar tripled, quadruple etc. fixed point results;

• From Lemma 1.1 part b) we obtain a result from [38], in the case $k = 2$. The proof given here for part b) is also adapted after [38], where the general case of k terms is established.

The next Lemma is due to Ostrowski ([31]) and can be also found in an extended form in [9]:

Lemma 1.2. *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ be two sequences of positive real numbers and $q \in (0,1)$ such that*

i) $a_{n+1} \leq q\, a_n + b_n, n \geq n_0$; ii) $b_n \to 0$ as $n \to \infty$.

Then

$$\lim_{n\to\infty} a_n = 0.$$

2. Coupled fixed point theorems in metric spaces

The first main result of this paper is stated in the next theorem.

Theorem 2.3. *Let (X,d) be a complete metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, z \in X$:*

$$d\left(F\left(x,y\right), F\left(y,z\right)\right) \leq kd\left(x,y\right) + ld\left(y,z\right), \tag{2.9}$$

where $k, l \in (0,1)$ with $k + l < 1$. Then

1) F has a unique coupled fixed point with identical components, i.e., there exists $\overline{x} \in X$, such that $F(\overline{x}, \overline{x}) = \overline{x}$;

2) The sequence $\{x_n\}_{n\geq 0}$, defined by

$$x_{n+1} = F(x_n, x_n),\ n \geq 0, \tag{2.10}$$

converges to $\overline{x}$, for any initial value $x_0 \in X$;

3) The sequence $\{y_n\}_{n\geq 0}$, defined by

$$y_{n+1} = F(y_{n-1}, y_n),\ n \geq 0 \tag{2.11}$$

converges to $\overline{x}$, for any initial values $y_0, y_1 \in X$;

4) The following estimates hold

$$d(x_{n+i-1}, \overline{x}) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1}), \quad n = 0,1,2,\ldots;\ i = 1,2,\ldots \tag{2.12}$$

$$d(y_n, \overline{x}) \leq \theta^n \max\left\{y_0, \frac{y_1}{\theta}\right\}, \quad n = 1,2,\ldots, \tag{2.13}$$

for some $\theta \in (0,1)$ and $\delta = k + l < 1$.

Proof. We first prove claims 1) and 2). Let $T : X \to X$ be defined by $T(x) = F(x, x)$, for any $x \in X$. Then

$$d(T(x), T(y)) = d(F(x, x), F(y, y)) \leq d(F(x, x), F(x, y)) + d(F(x, y), F(y, y))$$

and by using the contractive condition (2.9), one obtains

$$d(T(x), T(y)) \leq (k + l)d(x, y), \forall x, y \in X,$$

which, by assumption $k+l < 1$, shows that T is a contraction in the complete metric space (X, d).

So, by the contraction mapping principle, claims 1), 2) and the estimate (2.12) easily follow, see for example [9] for this form of the the error estimate.

3) We first prove that the sequence $\{d(y_n, y_{n+1})\}_{n\geq 0}$ converges to zero. Indeed, by condition (2.9) we get

$$d(y_n, y_{n+1}) = d(F(y_{n-2}, y_{n-1}), F(y_{n-1}, y_n))$$

$$\leq k \cdot d(y_{n-2}, y_{n-1}) + l \cdot d(y_{n-1}, y_n),\ n \geq 2,$$

and the claim easily follows by Lemma 1 with $\alpha_1 = k$, $\alpha_2 = l$ and $x_n = d(y_{n-1}, y_n)$. In order to prove that $\{y_n\}_{n\geq 0}$ converges to $\overline{x}$, the unique fixed point of T (and unique coupled fixed point of F), we denote

$$\Delta_n = d(y_n, \overline{x}),\ n \geq 0.$$

We have

$$d(y_n, \overline{x}) = d(F(y_{n-1}, y_n), F(\overline{x}, \overline{x})) \leq d(F(y_{n-1}, y_n), F(y_n, \overline{x}))$$

$$+d(F(y_n, \overline{x}), F(\overline{x}, \overline{x}))$$

and so by the contractive condition (2.9) this yields

$$\Delta_{n+1} \leq k \cdot d(y_n, \overline{x}) + l \cdot d(y_n, \overline{x}) + kd(y_{n-1}, y_n),\ n \geq 1,$$

that is,

$$\Delta_{n+1} \leq (k + l) \cdot \Delta_n + kd(y_{n-1}, y_n),\ n \geq 1,$$

Now apply Lemma 2 with $a_n = \Delta_n$, $b_n = kd(y_{n-1}, y_n)$ and $q = k + l$, to get the desired conclusion.

4) The estimate (2.13) follows by using Lemma 1 b).

□

Remark 2.2. Theorem 2.3 corresponds to a coupled fixed point theorem established in [41] (Theorem 2.2), where only conclusion 1) is obtained by using a more restrictive contractive condition than our condition (2.15). This condition reads as follows: for all $x, y, u, v \in X$ we have

$$d\left(F\left(x, y\right), F\left(u, v\right)\right) \leq kd\left(x, u\right) + ld\left(y, v\right), \tag{2.14}$$

where $k, l \in (0, 1)$ with $k + l < 1$.

It is apparent that (2.14) is more restrictive than (2.9), since (2.9) is obtained by (2.14) by taking $y = u$ and then denoting $v := z$.

The next theorem considers, instead of (2.9), an independent condition of Kannan type [25] and corresponds to Theorem 2.5 and Corollary 2.7 in [41].

Theorem 2.4. *Let (X, d) be a complete metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, z \in X$:*

$$d\left(F\left(x,y\right),F\left(y,z\right)\right) \leq kd\left(F(x,y),x\right) + ld\left(F(y,z),y\right), \tag{2.15}$$

where $k, l \in (0,1)$ with $k + l < 1$. Then

1) F has a unique coupled fixed point with identical components, i.e., there exists $\overline{x} \in X$, such that $F(\overline{x}, \overline{x}) = \overline{x}$;

2) The sequence $\{x_n\}_{n\geq 0}$, defined by

$$x_{n+1} = F(x_{n-1}, x_n),\ n \geq 0 \tag{2.16}$$

converges to $\overline{x}$, for any initial values $x_0, x_1 \in X$;

3) The following estimate holds, for some $\theta \in (0, 1)$:

$$d(y_n, \overline{x}) \leq \theta^n \max\left\{y_0, \frac{y_1}{\theta}\right\}, \quad n = 1, 2, \dots. \tag{2.17}$$

Proof. Let $x_0, x_1 \in X$ and $x_{n+1} = F(x_{n-1}, x_n)$, $n \geq 0$. Then, in view of (2.15), for $n \geq 2$ we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n, x_{n-1}), F(x_{n-1}, x_{n-2})) \\ &\leq kd(F(x_n, x_{n-1}), x_n) + ld(F(x_{n-1}, x_{n-2}), x_{n-1}) \\ &= kd(x_{n+1}, x_n) + ld(x_n, x_{n-1}) \end{aligned}$$

which yields

$$d(x_{n+1}, x_n) \leq \frac{l}{1-k} d(x_n, x_{n-1}),\ n \geq 1.$$

By induction, from the previous inequality we obtain

$$d(x_{n+1}, x_n) \leq \theta^n d(x_1, x_0),\ n \geq 0, \tag{2.18}$$

where $\theta = \dfrac{l}{1-k} < 1$. Now by (2.18), in a straightforward manner, we conclude that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. As (X, d) is complete, it follows that $\{x_n\}_{n\geq 0}$ is convergent. Let

$$x^* = \lim_{n\to\infty} x_n. \tag{2.19}$$

In order to prove that (x^*, x^*) is a (coupled) fixed point of F, let us estimate the distance $d(F(x^*, x^*), x^*)$. We have

$$\begin{aligned} d(F(x^*, x^*), x^*) \leq d(F(x^*, x^*), x_{n+1}) + d(x_{n+1}, x^*) = d(x_{n+1}, x^*) \\ +d(F(x_n, x_{n-1}), F(x^*, x^*)). \end{aligned}$$

By using the contractive condition (2.15), we have

$$\begin{aligned} d(F(x_n, x_{n-1}), F(x^*, x^*)) \leq d(F(x_n, x_{n-1}), F(x_{n-1}, x^*)) \\ +d(F(x_{n-1}, x^*), F(x^*, x^*)) \leq kd(F(x_n, x_{n-1}), x_n) + ld(F(x_{n-1}, x^*), x_{n-1}) \\ +kd(F(x_{n-1}, x^*), x_{n-1}) + ld(F(x^*, x^*), x^*) \\ = kd(x_{n+1}, x_n) + (k + l)d(F(x_{n-1}, x^*), x_{n-1}) + ld(F(x^*, x^*), x^*). \end{aligned}$$

This shows that

$$d(F(x^*, x^*), x^*) \leq \frac{k}{1-l} d(x_{n+1}, x_n) + \frac{k+l}{1-l} d(F(x_{n-1}, x^*), x_{n-1}). \tag{2.20}$$

Let us now evaluate $d(F(x_n, x^*), x_n)$. We have

$$d(F(x_n, x^*), x_n) = d(F(x_n, x^*), F(x_{n-1}, x_{n-2}))$$

$$\leq d(F(x_n, x^*), F(x^*, x_{n-1})) + d(F(x^*, x_{n-1}), F(x_{n-1}, x_{n-2}))$$
$$\leq kd(F(x_n, x^*), x_n) + ld(F(x^*, x_{n-1}), x^*) + kd(F(x^*, x_{n-1}), x^*)$$
$$+ld(F(x_{n-1}, x_{n-2}), x_{n-1}).$$

This gives

$$d(F(x_n, x^*), x_n) \leq \frac{k+l}{1-k} d(F(x^*, x_{n-1}), x^*) + \frac{l}{1-k} d(x_n, x_{n-1}). \tag{2.21}$$

Further,

$$d(F(x^*, x_{n-1}), x^*) \leq d(F(x^*, x_{n-1}), x_n) + d(x_n, x^*)$$
$$= d(F(x^*, x_{n-1}), F(x_{n-1}, x_{n-2})) + d(x_n, x^*)$$
$$\leq kd(F(x^*, x_{n-1}), x^*) + ld(F(x_{n-1}, x_{n-2}), x_{n-1}) + d(x_n, x^*)$$
$$= kd(F(x^*, x_{n-1}), x^*) + ld(x_n, x_{n-1}) + d(x_n, x^*)$$

and therefore

$$d(F(x^*, x_{n-1}), x^*) \leq \frac{l}{1-k} d(x_n, x_{n-1}) + \frac{1}{1-k} d(x_n, x^*).$$

Now, sumarizing all these computations, one obtains

$$d(F(x_{n-1}, x^*), x_{n-1}) \leq \frac{l(l+1)}{(1-k)^2} d(x_{n-1}, x_{n-2}) + \frac{k+l}{1-k} d(x_{n-1}, x^*). \tag{2.22}$$

and hence, by (2.20), (2.21) and (2.22), we have

$$d(F(x^*, x^*), x^*) \leq \frac{k}{1-l} d(x_{n+1}, x_n) + \frac{k+l}{1-l} d(F(x_{n-1}, x^*), x_{n-1})$$
$$\leq \frac{k}{1-l} d(x_{n+1}, x_n) + \frac{l(l+1)(k+l)}{(1-l)(1-k)^2} d(x_{n-1}, x_{n-2})$$
$$+\frac{(k+l)^2}{(1-l)(1-k)} d(x_{n-1}, x^*). \tag{2.23}$$

Now, by letting $n \to \infty$ in (2.23) and taking into account (2.18) and (2.19), we obtain $d(F(x^*, x^*), x^*) = 0$, which proves that

$$F(x^*, x^*) = x^*,$$

as claimed.

To prove the uniqueness of the fixed point x^*, let us assume there exists another fixed point $\overline{x}$ of F. Then

$$d(\overline{x}, x^*) = d(F(\overline{x}, \overline{x}), F(x^*, x^*)) \leq d(F(\overline{x}, \overline{x}), F(\overline{x}, x^*))$$
$$+d(F(\overline{x}, x^*), F(x^*, x^*)) \leq ld(F(\overline{x}, x^*), \overline{x}) + kd(F(\overline{x}, x^*), \overline{x})$$
$$= (k+l)d(F(\overline{x}, x^*), \overline{x}).$$

But

$$d(F(\overline{x}, x^*), \overline{x}) = d(\overline{x}, F(\overline{x}, x^*)) = d(F(\overline{x}, \overline{x}), F(\overline{x}, x^*))$$
$$\leq ld(F(\overline{x}, x^*), \overline{x})$$

and so

$$(1-l)d(F(\overline{x}, x^*), \overline{x}) \leq 0,$$

which leads to $d(F(\overline{x}, x^*), \overline{x}) = 0$ and therefore from

$$d(\overline{x}, x^*) \leq (k+l)d(F(\overline{x}, x^*), \overline{x}),$$

we conclude that $d(\overline{x}, x^*) = 0$, which ends the first part of the proof.

To obtain the error estimate (2.12), one uses the inequality (2.18). □

A similar result to that in Theorem 1.2 and corresponding to Theorem 2.6 and Corollary 2.8 in [41] and based on the following dual (and independent, see [40]) contractive condition

$$d\left(F\left(x,y\right),F\left(y,z\right)\right)\leq kd\left(F(x,y),x\right)+ld\left(F(y,z),y\right), \tag{2.24}$$

where $k,l\in(0,1)$ with $k+l<1$, can be similarly stated.

Note that Theorem 2.3 is actually a particular case of the more general result due to Presic [38] but adapted here to the existence of coupled fixed points.

3. Examples and open problems

Example 3.1. Let $X=\mathbb{R}$, $d\left(x,y\right)=|x-y|$ and $F:X\times X\to X$ be defined by

$$F\left(x,y\right)=\frac{x-3y}{5},\ (x,y)\in X^2.$$

Then F satisfies both condition

$$d\left(F\left(x,y\right),F\left(u,v\right)\right)\leq kd\left(x,u\right)+ld\left(y,v\right), \tag{3.25}$$

and

$$d\left(F\left(x,y\right),F\left(y,z\right)\right)\leq kd\left(x,y\right)+ld\left(y,z\right), \tag{3.26}$$

where $k,l\in(0,1)$ with $k+l<1$.

Problem 1. *Are conditions* (3.25) *and* (3.26) *in fact equivalent ?*

Example 3.2. Let $X=[0,1]$, $d\left(x,y\right)=|x-y|$ and $F:X\times X\to X$ be defined by

$$F\left(x,y\right)=\frac{2}{5},\ (x,y)\in\left[0,\frac{2}{3}\right)\times X$$

and

$$F\left(x,y\right)=\frac{1}{5},\ (x,y)\in\left[\frac{2}{3},1\right]\times X.$$

Then F satisfies condition

$$d\left(F\left(x,y\right),F\left(u,v\right)\right)\leq kd\left(F(x,y),x\right)+ld\left(F(u,v),u\right), \tag{3.27}$$

and

$$d\left(F\left(x,y\right),F\left(y,z\right)\right)\leq kd\left(F(x,y),x\right)+ld\left(F(y,z),y\right), \tag{3.28}$$

where $k,l\in(0,1)$ with $k+l<1$.

Problem 2. *Are conditions* (3.27) *and* (3.28) *in fact equivalent ?*

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