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Fixed point theorems for cyclic non-self single-valued almost contractions

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ABSTRACT. Let X be a Banach space, A and B two non-empty closed subsets of X and let $T : A \cup B \to X$ be an operator. We define the notion of cyclic non-self almost contraction and we give a corresponding fixed point theorem.

1. INTRODUCTION

Most of the results in metric fixed point theory deal with sigle-valued self mapping $T : X \to X$ and multi-valued self mappings $T : X \to \mathcal{P}(X)$ satisfying a certain contraction type condition, where X is a set endowed with a certain metric structure. These results are mainly generalizations of Banach contraction mapping principle, which can be briefly stated as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq a \cdot d(x, y), \forall x, y \in X, \tag{1.1}$$

where $a \in [0, 1)$ is a constant. Then T has a unique fixed point in X, say x^* and the sequence $\{T^n x_0\}$ converges to x^* for all $x_0 \in X$.

Notice that any contraction is continuous on X and $T(X) \subset X$. These two conditions makes this important result not applicable to most of the nonlinear problems where the associated operator T is actually a non-self operator. In 1968 R. Kannan in [7] extended Banach's Contraction principle to mappings that don't need to be continuous by considering instead of (1.1) the next condition: there exists $a \in [0, 0.5)$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \forall x, y \in X.$$

Following the Kannan's theorem a lot of papers were developed to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of the self operator T. For some other fixed point results see, for example [5, 17, 18] and references therein.

One of them, known as *weak contraction, almost contraction* or *Berinde operator* was introduced by Berinde in [2, 3, 4, 5] and is a simple but more general contraction condition that includes most of the conditions in Rhoades' classification [16]. The corresponding fixed point theorems is stated as follows.

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Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $l \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \textit{ for all } x, y \in X.$$

(1) $Fix(T) = \{x \in X \,:\, Tx = x\} \neq \emptyset$;
(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}$, $x_n = Tx_{n-1}, n \geq 1$ converges to some $x^* \in Fix(T)$;
(3) the following estimate holds

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1}), n \geq 0, i \geq 1. \tag{1.2}$$

This result, established mainly in [4], has some important features that differentiates it from other similar results in literature: 1) the fixed points set of almost contractions is not a singleton, in general; 2) the fixed points of almost contractions can be obtained by means of Picard iteration, like in the case of Banach contractions and, 3) the error estimate is of the same form as in the case of the usual contraction mapping principle.

Note that the estimate (1.2) includes the *a priori* estimate

$$d(x_i, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_0, x_1), i \geq 1,$$

obtained from (1.2) by taking $n := 1$, as well as the *a posteriori* estimate

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta}\, d(x_n, x_{n-1}), n \geq 0,$$

obtained from (1.2) by taking $i := 1$.

On the other hand, the contraction mapping principle has been extended by W.A. Kirk, P.S. Srinivasan and P. Veeramani [8] by considering a cyclical contractive condition as given by the next theorem.

Theorem 1.3. *Let A and B be two nonempty closed subsets of a complete metric space, and suppose $T : A \cup B \to A \cup B$ satisfies the following conditions:*

$$T(A) \subseteq B \textit{ and } T(B) \subset A$$

and there exist a constant $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq a \cdot d(x, y), \textit{ for } x \in A, y \in B. \tag{1.3}$$

Then T has a unique fixed point in $A \cap B$.

Inspired by the results in [8] other fixed point theorems were obtained see for example [9, 10, 11, 12, 13, 14, 15] and references therein. All the above presented results have been established for *self* mappings.

Very recently, V. Berinde and M. Păcurar [6] established a fixed point theorem for single-valued non self almost contractions. In order to state their main result, we need some notions and notations.

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx$ $(0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, Tx) = d(x, y) + d(y, Tx),\ y \in \partial K, \tag{1.4}$$

where we denoted $d(x, y) = \|x - y\|$.

In general, the set Y of points y satisfying condition (1.4) above may contain more than one element. The following concept is essential for stating Theorem 1.4.

Definition 1.1. Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (1.4). If, for any such elements x, we have

$$d(y, Ty) \leq d(x, Tx), \tag{1.5}$$

for all corresponding $y \in Y$, then we say that T has property (M).

Theorem 1.4. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self almost contraction, that is, a mapping for which there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \textit{for all} \;\; x, y \in K. \tag{1.6}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{1.7}$$

then T has a fixed point in K.

A similar result, but for multi-valued self almost contractions has been obtained in [1]. Starting from this background, our aim in this paper is to obtain fixed point for single-valued self almost contractions, as well as for single-valued non-self almost contractions, thus extending or generalising most of the related results in literature.

2. Fixed point theorems for cyclic almost contractions

We extend Theorem 1.2 by considering cyclical assumptions.

Theorem 2.5. *Let (X, d) be a complete metric space, A and B two nonempty closed subsets of X, and $T : A \cup B \to A \cup B$ satisfies the following conditions:*

$$T(A) \subseteq B \textit{ and } T(B) \subset A \tag{2.8}$$

and there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \forall x \in A, y \in B \text{ or } x \in B, y \in A. \tag{2.9}$$

Then T has a unique fixed point in $A \cap B$.

Proof. It readily follows that for any $x \in A \cup B$, we have

$$d(Tx, T^2x) \leq \delta d(x, Tx)$$

and this implies that the sequence $\{x_n\}$, $x_n = Tx_{n-1}, n \geq 1$, is a Cauchy sequence. Consequently $\{x_n\}$ converges to some point $x^* \in X$. However in view of (2.8) an infinite number of terms of the sequence $\{x_n\}$ lie in A and an infinite number of terms lie in B. Therefore $x^* \in A \cap B$ so $A \cap B \neq \emptyset$. Hence the restriction of T to $A \cap B$ is an almost contraction and applying Theorem 1.2 we get the desired conclusion. □

Remark 2.1. We should note that, since the almost contraction condition is not symmetric, instead of requiring that it should be satisfied $\forall x \in A, y \in B$, we must require that it is satisfied $\forall x \in A, y \in B$ or $x \in B, y \in A$.

Note also that a similar estimate to (1.2),

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), n \geq 0, i \geq 1, \tag{2.10}$$

may be obtained for the cyclical almost contractions in Theorem 2.5, too.

Now, we illustrate Theorem 2.5 by the following example.

Example 2.1. Let $X = [0, 1]$ with the usual norm and $T : X \to X$, $Tx = \frac{2}{3}$ if $x < 1$ and $T1 = 0$.

We consider $A = \left[0, \frac{2}{3}\right]$ and $B = \left[\frac{1}{3}, 1\right]$ and prove that all assumptions of Theorem 2.5 are satisfied. Indeed, T is a cyclic self almost contraction, that is, there exist $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$|Tx - Ty| \leq \delta|x - y| + L|y - Tx|, \forall x \in A, y \in B \text{ or } x \in B, y \in A. \tag{2.11}$$

Case 1. If $x \in \left[0, \frac{2}{3}\right]$, $y \in \left[\frac{1}{3}, 1\right)$ then $Ty = \frac{2}{3}$, $Tx = \frac{2}{3}$, and (2.11) obviously holds.
Case 2. If $x \in \left[0, \frac{2}{3}\right]$, $y = 1$ then $Tx = \frac{2}{3}$, $Ty = 0$ and (2.11) becomes

$$\frac{2}{3} \leq \delta|x - 1| + L\left|1 - \frac{2}{3}\right|,$$

which is always true for any $\delta \in (0, 1)$ if we take $L \geq 2$.
Case 3. If $x = 1$, $y \in \left[0, \frac{2}{3}\right]$, then $Tx = 0$, $Ty = \frac{2}{3}$, and (2.11) becomes

$$\frac{2}{3} \leq \delta|1 - y| + L|y|,$$

which is true for all $y \in \left[0, \frac{2}{3}\right]$ if we take $\delta = L = \frac{2}{3}$, in view of the fact that

$$1 = |1| = |1 - y + y| \leq |1 - y| + |y|, \forall y \in \mathbb{R}.$$

Therefore, for $L = 2$, and $\delta = \frac{2}{3}$, T satisfies (2.11).

Then by virtue of Theorem 2.5, T has a unique fixed point, $\frac{2}{3} \in \left[\frac{1}{3}, \frac{2}{3}\right] = A \cap B$.

Note that T in this example does not satisfy condition (1.3) in Theorem 1.3 and hence we cannot obtain the above conclusion by means of Theorem 1.3. Indeed, let us assume that (1.3) holds, that is,

$$|Tx - Ty| \leq \delta|x - y|, \forall x \in A, y \in B \text{ or } x \in B, y \in A,$$

for some $\delta \in (0, 1)$ and take $x = 1$ and $y = \frac{2}{3}$ in (1.3) to get

$$\frac{2}{3} \leq \delta\frac{1}{3} < \frac{1}{3},$$

a contradiction.

This shows that Theorem 2.5 properly extends Theorem 1.3.

3. Fixed point theorems for cyclic non-self almost contractions

Let X be a Banach space, A and B two nonempty closed subsets of X.

Definition 3.2. Let X be a Banach space, A and B two nonempty closed subsets of X. We say that $T : A \cup B \to X$ is a *cyclic non-self operator* if

(i) for any $x \in A$ we have $[x, Tx] \cap \partial B \neq \emptyset$;
(ii) for any $x \in B$ we have $[x, Tx] \cap \partial A \neq \emptyset$.

If $x \in A$ such that $Tx \notin B$ and $[x, Tx] \cap \partial B \neq \emptyset$ then we can always choose an $y \in \partial B$ such that $y = (1-\lambda)x + \lambda Tx$, with $\lambda \in (0,1)$ which actually expresses the fact that:

$$\|x - Tx\| = \|x - y\| + \|y - Ty\|, y \in \partial B. \tag{3.12}$$

Similarly, if $x \in B$ such that $Tx \notin A$ and $[x, Tx] \cap \partial A \neq \emptyset$ then we can always choose an $y \in \partial A$ such that $y = (1-\lambda)x + \lambda Tx$, with $\lambda \in (0,1)$, which actually expresses the fact that:

$$\|x - Tx\| = \|x - y\| + \|y - Ty\|, y \in \partial A. \tag{3.13}$$

We adapt now Definition 1.1 to the case of cyclic non-self mappings.

Definition 3.3. Let X be a Banach space, A and B two nonempty closed sursets of X and $T : A \cup B \to X$ is a cyclic non-self operator. Let $x \in A$ with $Tx \notin B$ and let $y \in \partial B$ be the corresponding elements given by (3.12). Also, let $u \in B$ with $Tu \notin A$ and let $v \in \partial A$ be the corresponding elements given by (3.13). If, for any such elements x, u, we have

$$\|y - Ty\| \leq \|x - Tx\|$$

and

$$\|v - Tv\| \leq \|u - Tu\|,$$

for all corresponding y, v, then we say that T has *cyclic property (M)*.

Definition 3.4. Let X be a Banach space, A and B two nonempty closed sursets of X. We say that $T : A \cup B \to X$ is a *cyclic non-self almost contraction* if

(i) T is a cyclic non-self operator;
(ii) for any $x \in A$ and $y \in B$ we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\| \tag{3.14}$$

for some constants $\delta \in (0,1)$ and $L \geq 0$.

Theorem 3.6. *Let X be a Banach space, A and B two nonempty closed subsets of X, and $T : A \cup B \to X$ a cyclic non-self almost contraction. If T has cyclic property (M) and satisfies Rothe's boundary condition:*

$$T(\partial A) \subset B \text{ and } T(\partial B) \subset A. \tag{3.15}$$

Then T has a fixed point in $A \cap B$.

Proof. If $T(A) \subset B$ and $T(B) \subset A$, that is $T(A \cup B) \subset A \cup B$, then T is in fact a cyclic self almost contraction and the conclusion follows by Theorem 2.5. Therefore, we consider the case $T(A \cup B) \subset X$. Let $x_0 \in \partial A$. By (3.15) we know that $Tx_0 \in B$. Denote $x_1 = Tx_0$. Now, if $Tx_1 \in A$, set $x_2 = Tx_1$. If $Tx_1 \notin A$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which belongs to ∂A, that is

$$x_2 = (1-\lambda)x_1 + \lambda Tx_1,$$

with $\lambda \in (0,1)$. Continuing in this way we obtain a sequence $\{x_n\}$ whose terms are satisfying one of the following properties:

(i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in A \cup B$;
(ii) $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial A \cup \partial B$, with $\lambda \in (0,1)$, if $Tx_{n-1} \notin A \cup B$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} \, : \, x_k = Tx_k - 1\}$$

and

$$Q = \{x_k \in \{x_n\} \, : \, x_k \neq Tx_k - 1\}.$$

Note that $\{x_n\} \subset A \cup B$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, by virtue of (3.15), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in P).

The strategy is to prove that $\{x_n\}$ is a Cauchy sequence. To prove this, we must consider three different cases: Case I. $x_n, x_{n-1} \in P$; Case II. $x_n \in P, x_{n+1} \in Q$; Case III. $x_n \in Q, x_n + 1 \in P$. Without losing generality suppose that $x_n \in A$ and $x_{n-1}, x_{n+1} \in B$.

Case I. $x_n, x_{n-1} \in P$.

In this case we have $x_n = Tx_{n-1}$ and $x_{n+1} = Tx_n$. By (3.14) we obtain

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\| \leq \delta \|x_n - x_{n-1}\| + L\|x_n - Tx_n - 1\|.$$

Since $x_n = Tx_{n-1}$ we obtain

$$\|x_{n+1} - x_n\| \leq \delta \|x_n - x_{n-1}\|. \tag{3.16}$$

Case II. $x_n \in P, x_{n+1} \in Q$.

In this case $x_n = Tx_{n-1}$ and $x_{n+1} \neq Tx_n$. By (3.12) we can choose an $x_{n+1} \in \partial B$, situated on the segment $[x_n, Tx_n]$ such that

$$\|x_x - Tx_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|.$$

Accordingly with

$$\|x_n - x_{n+1}\| \leq \|x_n - Tx_n\| = \|Tx_{n-1} - Tx_n\|$$

and (3.14) we get

$$\|x_n - x_{n+1}\| \leq \|Tx_{n-1} - Tx_n\| \leq \delta \|x_n - x_{n-1}\| + L\|x_n, Tx_{n-1}\|,$$

and since $x_n = Tx_{n-1}$ we obtain again inequality (3.16).

Case III. $x_n \in Q, x_{n+1} \in P$.

Since $x_{n+1} \in P$ we also have that $x_{n-1} \in P$ and hence $x_{n+1} = Tx_n, \; x_{n-1} = Tx_{n-2}$. Because T has property (M) it follows that

$$\|x_n - x_{n+1}\| = \|x_n - Tx_n\| \leq \|x_{n-1} - Tx_{n-1}\|.$$

Now, by (3.14) we obtain

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq \|x_{n-1} - Tx_{n-1}\| = \|Tx_{n-2} - Tx_{n-1}\| \\ &= \delta \|x_{n-2} - x_{n-1}\| + L\|x_{n-1} - Tx_{n-2}\| \\ &= \delta \|x_{n-2} - x_{n-1}\|. \end{aligned}$$

Hence

$$\|x_n - x_{n+1}\| \leq \delta \|x_{n-2} - x_{n-1}\|. \tag{3.17}$$

Summarizing all tree cases and using (3.16) and (3.17) we obtain the following inequality:

$$\|x_n - x_{n+1}\| \leq \delta \max\{\|x_{n-1} - x_n\|, \|x_{n-2} - x_{n-1}\|\}.$$

and this, by induction implies that $\{x_n\}$ is a Cauchy sequence. Consequently $\{x_n\}$ converges to some point $x^* \in X$. However in view that T is a cyclic non-self operator an infinite number of terms of the sequence $\{x_n\}$ lie in A and an infinite number of terms lie in B. Therefore $x^* \in A \cap B$ so $A \cap B \neq \emptyset$. Therefore the restriction of T to $A \cap B$ is an almost constraction.

□

Example 3.2. Let $X = \mathbb{R}$ and $K = [0,1]$ with the usual norm and $T : K \to X$ given by $Tx = \dfrac{2}{3}$ if $x \in [0,1) \setminus \left\{\dfrac{1}{2}\right\}$, $T\dfrac{1}{2} = -1$ and $T1 = 0$.

We prove that all assumptions of Theorem 3.6 are satisfied, if we consider $A = \left[0, \dfrac{2}{3}\right]$ and $B = \left[\dfrac{1}{3}, 1\right]$. Indeed, T is a cyclic non-self almost contraction, that is, there exist $\delta \in [0,1)$ and some $L \geq 0$ such that

$$|Tx - Ty| \leq \delta|x-y| + L|y - Tx|, \forall x \in A, y \in B \text{ or } x \in B, y \in A. \tag{3.18}$$

Case 1. If $x, y \in \left[0, \dfrac{2}{3}\right) \setminus \left\{\dfrac{1}{2}\right\}$, then $Tx = Ty$ and (3.18) obviously holds for any $\delta \in [0,1)$ and $L \geq 0$.

Case 2. If $x \in \left[0, \dfrac{2}{3}\right] \setminus \left\{\dfrac{1}{2}\right\}$ and $y = 1$ then $Tx = \dfrac{2}{3}$, $Ty = 0$ and (3.18) becomes

$$\frac{2}{3} \leq \delta|x-1| + L\Big|1 - \frac{2}{3}\Big|,$$

which is always true for any $\delta \in (0,1)$ if we take $L \geq 2$.

Case 3. If $x = 1$, $y \in \left[0, \dfrac{2}{3}\right] \setminus \left\{\dfrac{1}{2}\right\}$, then $Tx = 0$, $Ty = \dfrac{2}{3}$, and (3.18) becomes

$$\frac{2}{3} \leq \delta|1-y| + L|y|,$$

which is true for all $y \in \left[0, \dfrac{2}{3}\right]$ if we take $\delta = L = \dfrac{2}{3}$, in view of the fact that

$$1 = |1| = |1 - y + y| \leq |1-y| + |y|, \forall y \in \mathbb{R}.$$

Case 4. If $x \in \left[0, \dfrac{2}{3}\right] \setminus \left\{\dfrac{1}{2}\right\}$ and $y = \dfrac{1}{2}$, then $Tx = \dfrac{2}{3}$, $Ty = -1$ and (3.18) becomes

$$\Big|\frac{2}{3} - (-1)\Big| \leq \delta\Big|x - \frac{1}{2}\Big| + L\Big|\frac{1}{2} - \frac{2}{3}\Big|,$$

which is satisfied for any $\delta \in [0,1)$ if we take $L \geq 10$.

Case 5. If $x = 1$ and $y = \dfrac{1}{2}$, then $Tx = 0$, $Ty = -1$ and (3.18) becomes

$$\Big|0 - (-1)\Big| \leq \delta\Big|1 - \frac{1}{2}\Big| + L\Big|\frac{1}{2}\Big|,$$

which is satisfied for any $\delta \in [0,1)$ if we take $L \geq 2$.

Case 6. If $x = \dfrac{1}{2}$ and $y \in \left[0, \dfrac{2}{3}\right] \setminus \left\{\dfrac{1}{2}\right\}$, then $Tx = -1$, $Ty = \dfrac{2}{3}$ and (3.18) becomes

$$\Big|-1 - \frac{2}{3}\Big| \leq \delta\Big|\frac{1}{2} - y\Big| + L\Big|y + 1\Big|,$$

which is satisfied for any $\delta \in [0,1)$ if we take $L \geq \dfrac{5}{3}$, since $\Big|y+1\Big| \geq 1$.

Now, by summarising all six cases above, we conclude that (3.18) is satisfied for $\delta = \dfrac{2}{3}$ and $L = 10$. Property (M) also holds, since $d(0, T0) = \frac{2}{3} < \frac{3}{2} = d(\frac{1}{2}, T\frac{1}{2})$. Then, by virtue of Theorem 3.6, T has a unique fixed point, $\frac{2}{3} \in \left[\frac{1}{3}, \frac{2}{3}\right] = A \cap B$. Note that Theorem 2.5 cannot be applied to T in this example since T is a *non-self* almost contraction, while Theorem 2.5 applies to self almost contractions only.

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