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Fixed point theorems for Zamfirescu mappings in metric spaces endowed with a graph

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ABSTRACT. Let (X, d) be a metric space endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X. We define the notion of G-Zamfirescu maps and obtain a fixed point theorem for such mappings. This extends and subsumes many recent results which were obtained for mappings on metric spaces endowed with a graph and for cyclic operators.

1. INTRODUCTION

We remind the reader few basic notions concerning graphs, the connectivity of graphs, G-contraction and Picard operator. Let (X, d) be a metric space and let G be a graph with no parallel edges. The set $V(G)$ denotes the vertices (or nodes) of the graph G and $E(G)$ denotes the set of its edges. In Jachymski's results ([13]), the graph G is a directed graph for which the set $V(G)$ coincides with X and the set $E(G)$ contains all loops, i.e. $E(G) \supseteq \Delta$, where Δ is the diagonal of the Cartesian product $X \times X$. Because G has no parallel edges, G can be identify with the pair $(V(G), E(G))$.

According to the basic definition ([14], Def.8.2.1.), a *path* of length N between two vertices x, y of a directed graph G is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. It is known that a graph G is *connected* if there is a path between any two vertices. If the undirected graph $\tilde{G}$, obtained from G by ignoring the direction of edges, is connected, then the graph G is said to be *weakly connected*.

One of the statement of an important theorem of Jachymski ([13], Th.3.2.) uses the equivalence class $[x]_G$ of the relation R defined on $V(G)$ by rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

If G is such that $E(G)$ is symmetric and x is a vertex in G, then $V(G_x) = [x]_G$ ([13]), where the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the *component* of G containing x. In this case, G_x is connected.

Recently, some results have appeared giving sufficient conditions for f to be a PO if (X, d) is endowed with a graph (following Petruşel and Rus [18], we say f is a *Picard operator* PO if f has a unique fixed point x^* and $\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$ and it is a *weakly Picard operator* WPO if the sequence $(f^n x)_{n\in\mathbb{N}}$ converges, for all $x \in X$ and the limit, which may depend on x, is a fixed point of T). The first result in this direction was given by J. Jachymski [13]. He also presented a new proof for the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0, 1]$. The idea concerning of the Bernstein operators comes from I. A. Rus [24]. See also [23].

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Definition 1.1 ([13], Def. 2.1)**.** We say that a mapping $f : X \to X$ is a Banach G-contraction or simply a G-contraction if f preserves edges of G, i.e.,

$$\forall x, y \in X\left((x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)\right) \tag{1.1}$$

and f decreases weights of edges of G in the following way:

$$\exists \alpha \in (0,1), \forall x, y \in X\left((x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leqslant \alpha d(x, y)\right). \tag{1.2}$$

Theorem 1.1 ([13], Th 3.2)**.** *Let (X, d) be complete, and let the triple (X, d, G) have the following property:*

(P:) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ *with* $(x_{k_n}, x) \in E(G)$ *for* $n \in \mathbb{N}$.

Let $f : X \to X$ be a Banach G-contraction, and $X_f = \{x \in X \,|(x, fx) \in E(G)\}$. Then the following statements hold:

1. card$Fix\, f = $ card$\{[x]_{\tilde{G}} \,|x \in X_f\}$.
2. Fix $f \neq \emptyset$ iff $X_f \neq \emptyset$.
3. *f has a unique fixed point if there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\tilde{G}}$.*
4. *For any $x \in X_f$, $f\,\big|_{[x]_{\tilde{G}}}$ is a PO.*
5. *If $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*
6. *If $X' := \cup\{[x]_{\tilde{G}} \,|x \in X_f\}$ then $f\,|_{X'}$ is a WPO.*
7. *If $f \subseteq E(G)$, then f is a WPO.*

In 2011, Nicolae, O'Regan and Petruşel [16] extended the notion of multi-valued contraction on a metric space with a graph in considering the fixed point theorem shown below.

Theorem 1.2 ([16])**.** *Let $F : X \to CB(X)$ be a multi-valued map with nonempty closed values. Assume that*

(1) there exists $\lambda \in (0,1)$ such that $D(F(x), F(y)) \leq \lambda d(x, y)$ for all $(x, y) \in E(G)$, where $D(A, B) = max\{sup_{x\in A} inf_{y\in B} d(x, y), sup_{y\in B} inf_{x\in A} d(x, y)\}$, $\forall A, B \in CB(X)$, is Pompeiu-Hausdorff metric;
(2) for each $(x, y) \in E(G)$, each $u \in F(x)$ and $v \in F(y)$ satisfying $d(u, v) \leq a \cdot d(x, y)$ for some $a \in (0,1)$, $(u, v) \in E(G)$ holds;
(3) X has the Property P.

If there exists $x_0, x_1 \in X$ such that $x_1 \in [x_0]^1_G \cap F(x_0)$, then F has a fixed point.

An existence theorem of a fixed point for set valued mappings (not necessarily unique) in metric spaces endowed with a graph was given by Beg, Butt and Radojević.

Theorem 1.3 ([1], Th 3.1)**.** *Let (X, d) be a complete metric space and suppose that the triple (X, d, G) has the Property P. Let $F : X \to CB(X)$ be a G-contraction and*

$$X_F := \{x \in X : (x, u) \in E(G) \;\; for \;\; some \;\; u \in F(x)\}.$$

Then the following statements hold:

(1) For any $x \in X_F$, $F|_{[x]_{\tilde{G}}}$ has a fixed point.
(2) If $X_F \neq \emptyset$ and G is weakly connected, then F has a fixed point in X.
(3) If $X' := \cup\{[x]_{\tilde{G}} : x \in X_F\}$ then $F|_{X'}$ has a fixed point.
(4) If $F \subseteq E(G)$ then F has a fixed point.
(5) $Fix\, F \neq \emptyset$ if and only if $X_F \neq \emptyset$.

We recall that:

Definition 1.2 ([27]). Let (X,d) be a metric space. $T : X \to X$ is called a Zamfirescu operator if there exist the real numbers α, β and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$ and $0 \leq \gamma < \frac{1}{2}$, such that, for each $x, y \in X$, at least one of the following is true:

(z_1) $d(Tx,Ty) \leqslant \alpha \cdot d(x,y)$;
(z_2) $d(Tx,Ty) \leqslant \beta [d(x,Tx) + d(y,Ty)]$;
(z_3) $d(Tx,Ty) \leqslant \gamma [d(x,Ty) + d(y,Tx)]$.

Zamfirescu [27] proved that if X is complete, then every Zamfirescu mapping has a unique fixed point. The aim of this paper is to study the existence of fixed points for Zamfirescu mappings in metric spaces endowed with a graph G by introducing the concept of G-Zamfirescu mappings. Several theorems concerning the existence and uniqueness of the fixed point for contractive mappings in metric spaces endowed with a graph have been considered recently in [1, 7, 8, 10, 13, 16].

2. Main results

Throughout this section we assume that (X,d) is a metric space, and G is a graph such that $V(G) = X$, $E(G) \supseteq \Delta$ and the graph G has no parallel edges. The set of all fixed points of a mapping T is denoted by $FixT$.

Following the idea of Jachymski [13], we will define a new mapping, G-Zamfirescu mapping:

Definition 2.3. Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be a G-Zamfirescu mapping if:

1. $\forall x, y \in X \; ((x,y) \in E(G) \Rightarrow (Tx,Ty) \in E(G))$.
2. there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$ and $0 \leq c < \frac{1}{2}$, such that, for each $(x,y) \in E(G)$, at least one of the following is true:
   (z_1) $d(Tx,Ty) \leqslant a \cdot d(x,y)$;
   (z_2) $d(Tx,Ty) \leqslant b [d(x,Tx) + d(y,Ty)]$;
   (z_3) $d(Tx,Ty) \leqslant c [d(x,Ty) + d(y,Tx)]$.

Remark 2.1. If the mapping T satisfies the condition (z_1), $\forall (x,y) \in E(G)$ then T is a Banach $G-$contraction (see [13], Definition 2.1) and if the mapping T satisfies the condition (z_2), $\forall (x,y) \in E(G)$ then T is a $G-$Kannan mapping (see [8], Definition 4).

Remark 2.2. If T is a G-Zamfirescu mapping, then T is both a G^{-1}-Zamfirescu mapping and a $\tilde{G}$-Zamfirescu mapping.

Example 2.1. Any Zamfirescu mapping is a G_0-Zamfirescu mapping, where the graph G_0 is defined by

$$V(G_0) = X \text{ and } E(G_0) = X \times X.$$

Example 2.2. Let $X = \{0,1,2,3\}$ be endowed with the Euclidean metric $d(x,y) = |x - y|$. The mapping $T : X \to X$, $Tx = 0$, $for\ x \in \{0,1\}$ $and\ Tx = 1$, $for\ x \in \{2,3\}$ is a G-Zamfirescu mapping satisfying (z_1) from Definition 2.3 with the constant $a = \frac{2}{3}$, where

$$V(G) = X \text{ and } E(G) = \{(0,1);(0,2);(2,3);(0,0);(1,1));(2,2);(3,3)\},$$

but is not a Zamfirescu mapping because

- $d(T1,T2) = 1$ and $d(1,2) = 1$ so (z_1) from Definition 1.2 is false;
- $d(T1,T2) = 1$ and $d(1,T1) + d(2,T2) = 2$ so (z_2) from Definition 1.2 is false;
- $d(T1,T2) = 1$ and $d(1,T2) + d(2,T1) = 2$ so (z_3) from Definition 1.2 is false.

We prove some lemmas first. The first one is derived from elementary calculus, so we skip the proof of that.

Lemma 2.1. *Let $\alpha \in [0,1)$ and let $\{x_n\}$ converge to zero. Then the sequence $\{y_n\}$, defined by*

$$y_n = \sum_{i=1}^{n} \alpha^{n-i} x_i$$

converges to zero.

Lemma 2.2. *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Zamfirescu mapping. If $x, y \in X$ satisfy the condition $(x,y) \in E\left(\tilde{G}\right)$ then we have*

$$d\left(Tx,Ty\right) \leqslant \alpha d\left(x,y\right) + 2\alpha d\left(x,Tx\right), \tag{2.3}$$

and

$$d\left(Tx,Ty\right) \leqslant \alpha d\left(x,y\right) + 2\alpha d\left(x,Ty\right), \tag{2.4}$$

where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

Proof. Since T is a G-Zamfirescu mapping, by Remark 2.2, T is a $\tilde{G}$ Zamfirescu mapping. But $(x,y) \in E\left(\tilde{G}\right)$ so at least one of the conditions $(z_1), (z_2), (z_3)$ is satisfied.

If the pair (x,y) satisfies (z_1) we get $d\left(Tx,Ty\right) \leqslant ad\left(x,y\right)$ so both conditions (2.3) and (2.4) are satisfied.

If the pair (x,y) satisfies (z_2) then

$$\begin{aligned} d\left(Tx,Ty\right) &\leqslant b\left[d\left(x,Tx\right) + d\left(y,Ty\right)\right] \\ &\leqslant b\left[d\left(x,Tx\right) + d\left(y,x\right) + d\left(x,Tx\right) + d\left(Tx,Ty\right)\right] \end{aligned}$$

which implies that

$$d\left(Tx,Ty\right) \leqslant \frac{b}{1-b} d\left(x,y\right) + \frac{2b}{1-b} d\left(x,Tx\right)$$

so the condition (2.3) is true. In the same manner we can prove that the condition (2.4) is true.

If the pair (x,y) satisfies (z_3) the proof is identical as above. □

Lemma 2.3. *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Zamfirescu mapping. If $x \in X$ satisfies the condition $(x,Tx) \in E\left(\tilde{G}\right)$ then we have*

$$d\left(T^n x, T^{n+1} x\right) \leqslant \alpha^n d\left(x,Tx\right) \tag{2.5}$$

for all $n \in \mathbb{N}^*$, where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

Proof. Because T is a G-Zamfirescu mapping, using Remark 2.2, T is a $\tilde{G}$ Zamfirescu mapping.

Let $(x,Tx) \in E\left(\tilde{G}\right)$. An easy induction shows that $\left(T^n x, T^{n+1} x\right) \in E\left(\tilde{G}\right)$ for all $n \in \mathbb{N}$. Then for all $n > 0$, by (2.4), we get

$$d\left(T^{n+1}x, T^n x\right) \leqslant \alpha d\left(T^n x, T^{n-1} x\right) + 2\alpha d\left(T^n x, T^n x\right) = \alpha d\left(T^n x, T^{n-1} x\right)$$

which implies

$$d\left(T^n x, T^{n+1} x\right) \leqslant \alpha d\left(T^{n-1} x, T^n x\right)$$

where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, so we get

$$d\left(T^n x, T^{n+1}x\right) \leqslant \alpha^n d\left(x, Tx\right), \ \forall n \in \mathbb{N}^*.$$

□

Lemma 2.4. *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a G-Zamfirescu operator. If there exist $x, y \in X$ such that $(x,y) \in E\left(\tilde{G}\right)$ then*

$$d\left(T^n x, T^n y\right) \leq \alpha^n d\left(x, y\right) + 2\alpha \sum_{i=1}^{n} \alpha^{n-i} d\left(T^{i-1}x, T^i x\right)$$

for all $n \in \mathbb{N}^*$, where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

Proof. Since T is a G-Zamfirescu mapping, by Remark 2.2, T is a $\tilde{G}$ Zamfirescu mapping. Let $x, y \in X$ such that $(x,y) \in E\left(\tilde{G}\right)$. An easy induction shows that $(T^n x, T^n y) \in E\left(\tilde{G}\right)$ for all $n \in \mathbb{N}$.

Then by Lemma 2.2 we have

$$d\left(T^n x, T^n y\right) \leqslant \alpha d\left(T^{n-1}x, T^{n-1}y\right) + 2\alpha d\left(T^{n-1}x, T^n x\right). \tag{2.6}$$

Using the relation (2.6) and elementary calculus we get

$$d\left(T^n x, T^n y\right) \leq \alpha^n d\left(x, y\right) + 2\alpha \sum_{i=1}^{n} \alpha^{n-i} d\left(T^{i-1}x, T^i x\right) \tag{2.7}$$

for all $n \in \mathbb{N}$, where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$. □

The main result of this paper is given in the following theorem.

Theorem 2.4. *Let (X,d) be a complete metric space endowed with a graph G, $T : X \to X$ be a G-Zamfirescu operator. We suppose that:*

(i.) G is weakly connected;

(ii.) $X_T = \left\{x \in X \,\middle|\, (x, Tx) \in E(\tilde{G})\right\} \neq \emptyset$;

(iii.) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(\tilde{G})$ for $n \in \mathbb{N}$.

Then T is a PO.

Proof. By (ii.) the set $X_T \neq \emptyset$. This means that there exists at least one $x \in X$ such that $(x, Tx) \in E(\tilde{G})$. Because $a < 1$, $b, c < \frac{1}{2}$ then $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} \in [0,1)$. From Lemma 2.3 we have that the sequence $\{T^n x\}_{n\geqslant 0}$ is Cauchy. Since (X,d) is complete, the sequence $\{T^n x\}_{n\geqslant 0}$ converges to $x^* \in X$. By (iii.) there is a subsequence $\{T^{k_n}x\}_{n\geqslant 0}$ such that $\left(T^{k_n}x, x^*\right) \in E(\tilde{G})$, $\forall n \in \mathbb{N}$. Using the definition of the G-Zamfirescu operator we have that $\left(T^{k_n+1}x, Tx^*\right) \in E\left(G\right)$, for all $n \in \mathbb{N}$. From Lemma 2.2 we obtain

$$d\left(T^{k_n+1}x, Tx^*\right) \leqslant \alpha d\left(T^{k_n}x, x^*\right) + 2\alpha d\left(T^{k_n}x, T^{k_n+1}x\right). \tag{2.8}$$

Letting $n \to \infty$ in the relation (2.8), we get

$$d\left(x^*, Tx^*\right) \leqslant 0$$

which implies that $d\left(x^*, Tx^*\right) = 0$, so $Tx^* = x^*$ and $x^* \in FixT$.

Let $x \in X_T$ and $y \in X$. Since G is a weakly connected graph, there exists a path $(x_i)_{i=0}^{N}$ in X from $x \in X_T$ to y i.e. $x_0 = x, x_N = y$, and $(x_{i-1}, x_i) \in E(G)$ for all $i = \overline{1, N}$. We have proved that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to $x^* \in FixT$. But $(x, x_1) \in E(G)$, so from Lemma 2.3 we get

$$d(T^n x, T^n x_1) \le \alpha^n d(x, x_1) + 2\alpha \sum_{i=1}^{n} \alpha^{n-i} d\left(T^{i-1}x, T^i x\right)$$

for all $n \in \mathbb{N}^*$, where $\alpha = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} < 1$.

Using Lemma 2.1 we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha^{n-i} d\left(T^{i-1}x, T^i x\right) = 0$$

so $\lim\limits_{n\to\infty} d(T^n x, T^n y) = 0$, in conclusion the sequence $\{T^n x_1\}$ converges to x^*. Moreover the sequences $\{T^n x_i\}$ converge to x^* for all $i = \overline{1, N}$. In particular for $i = N$ the sequence $\{T^n x_N\}$ =$\{T^n y\}$ converges to x^*. Thus

$$\lim_{n\to\infty} T^n y = x^*$$

for all $y \in X$.

If there exists $y^* \in FixT$ then the sequences $\{T^n y^*\}$ converge to x^* which implies $y^* = x^*$ so T is a PO. □

The following example shows that the condition "G is weakly connected graph" is necessary for the G-Zamfirescu mapping to be a PO.

Example 2.3. Let $X := [0, 1]$ be endowed with the Euclidean metric d_E. We define the graph G by

$$E(G) = \{(x, y) \in (0, 1] \times (0, 1] \,|\, x \geqslant y\} \cup \{(0, 0)\}.$$

Set

$$Tx = \frac{x}{4} \text{ for } x \in (0, 1], \text{ and } T0 = \frac{1}{4}.$$

It is obvious that (X, d) is a complete metric space, G is not weakly connected because there is no path in $\tilde{G}$ from $x_0 = 0$ to $x_N = 1$ and T is a G-Zamfirescu mapping which satisfies (z_2) with $b = \frac{1}{3}$. Clearly, $T^n x \to 0$ for all $x \in X$, but T has no fixed points.

The next example underlines the importance of the nonempty property for the set X_T, property which ensures that the G-Zamfirescu mapping T is a PO.

Example 2.4. Let $X = \{3, 4, 5, ...\} = \mathbb{N}\setminus\{0, 1, 2\}$ be endowed with the Euclidean metric d_E. We define the graph G by

$$V(G) = X \text{ and } E(G) = \left\{\left(2^k n, 2^k (n+1)\right) : k \in \mathbb{N}, n \in \mathbb{N}\setminus\{0, 1, 2\}\right\} \cup \Delta.$$

Then (X, d) is a complete metric space and G is a weakly connected graph since for all $m, n \in \mathbb{N}$ with $m < n$ we have that the sequence $x_0 = m, x_1 = m + 1, ..., x_{n-m} = n$ is a path in G from m to n.

Set

$$Tx = 2x.$$

For all $(x, y) \in E(G)$ we have

$$(Tx, Ty) = (2x, 2y) \in E(G)$$

and the mapping T satisfies (z_2) with $\beta = \frac{1}{3}$ since for $(x,y) \in E(G)$ there is $k \in \mathbb{N}$ and $n \in \{3,4,5,...\}$ such that $(x,y) = \left(2^k n, 2^k (n+1)\right)$ and

$$d(Tx,Ty) = \left|2^{k+1}n - 2^{k+1}(n+1)\right| = 2^{k+1} = \frac{1}{3}2^k \cdot 6 < \frac{1}{3}2^k(2n+1)$$
$$= \frac{1}{3}\left(2^k n + 2^k(n+1)\right) = \frac{1}{3}\left(d(x,Tx) + d(y,Ty)\right).$$

If we suppose that there exists $x \in X$ such that $(x,Tx) \in E(\tilde{G})$, then there exist $k,\ n \in \mathbb{N}$ with $n \geq 3$ and $x = 2^k n, Tx = 2^k(n+1)$ which implies that $n = 1$. So the assumption is false, in conclusion $X_T = \emptyset$.

The property (ii.) from Theorem 2.4 is satisfied because every convergent sequence is a constant sequence. Clearly, $(T^n x)_{n\in\mathbb{N}}$ is not convergent for all $x \in X$.

Corollary 2.1. *Let (X,d) be a complete metric space endowed with a graph G, $T : X \to X$ be a Banach G-contraction. We suppose that:*

(i.) G is weakly connected;
(ii.) $X_T = \{x \in X \,|(x,Tx) \in E(G)\} \neq \emptyset$;
(iii.) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Then T is a PO.

Proof. If T is a Banach $G-$contraction with the constant $\alpha \in [0,1)$ then the operator T satisfies the condition (z_1) from Definition 2.3 for all $(x,y) \in E\left(\tilde{G}\right)$. T is a G-Zamfirescu operator and from Theorem 2.4 the mapping T is a PO. □

Corollary 2.2. *Let (X,d) be a complete metric space endowed with a graph G, $T : X \to X$ be a $G-$Kannan mapping (see* [8]*). We suppose that:*

(i.) G is weakly connected;
(ii.) $X_T = \{x \in X \,|(x,Tx) \in E(G)\} \neq \emptyset$;
(iii.) for any $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there is a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Then T is a PO.

Proof. If T is a $G-$Kannan with the constant α then T is a G-Zamfirescu operator. From Theorem 2.4 we have that the mapping T is a PO. □

From Theorem 2.4, we obtain the following corollary concerning the fixed point of Zamfirescu operator in partially ordered metric spaces.

Corollary 2.3. *Let $(X, \leqslant)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. Let $T : X \to X$ be an increasing operator such that the following three assertions hold true:*

(i.) There exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$ and $0 \leq c < \frac{1}{2}$, such that, for each $x, y \in X$ with $x \leqslant y$, at least one of the following is true:
(z_1) $d(Tx,Ty) \leqslant ad(x,y)$;
(z_2) $d(Tx,Ty) \leqslant b[d(x,Tx) + d(y,Ty)]$;
(z_3) $d(Tx,Ty) \leqslant c[d(x,Ty) + d(y,Tx)]$.

(ii.) *For each* $x, y \in X$, *incomparable elements of* $(X, \leqslant)$, *there exists* $z \in X$ *such that* $x \leqslant z$ *and* $y \leqslant z$;

(iii.) *If an increasing sequence* (x_n) *converges to* x *in* X, *then* $x_n \leqslant x$ *for all* $n \in \mathbb{N}$.

Then T *is a PO.*

Proof. Consider the graph G with $V(G) = X$, and

$$E(G) = \{(x, y) \in X \times X \mid x \leqslant y\}.$$

Because the mapping T is an increasing one and (i.) holds true we get that the mapping T is a $G-$Zamfirescu mapping. By (ii.) G is a weakly connected graph and the condition (iii.) implies the condition (ii.) from Theorem 2.4. The conclusion follows now from Theorem 2.4. □

The next result shows that the fixed point theorem for cyclic Zamfirescu operators, proved in [19] by Petric and Zlatanov, is a consequence of Theorem 2.4.

Let $p \geqslant 2$ and $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space X. A mapping $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a *cyclical operator* if

$$T(A_i) \subseteq A_{i+1}, \quad \text{for all } i \in \{1, 2, ..., p\} \tag{2.9}$$

where $A_{p+1} := A_1$.

Theorem 2.5. *Let* $A_1, A_2, ..., A_p, A_{p+1} = A_1$ *be nonempty closed subsets of a complete metric space* (X, d) *and suppose that* $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ *is a cyclical operator, and there exist real numbers* $a \in [0, 1)$, $b \in \left[0, \frac{1}{2}\right)$ *and* $c \in \left[0, \frac{1}{2}\right)$ *such that for each pair* $(x, y) \in A_i \times A_{i+1}$, *for* $i \in \{1, 2, ..., p\}$, *at least one of the following is true:*

(z_1) $d(Tx, Ty) \leqslant a d(x, y)$;

(z_2) $d(Tx, Ty) \leqslant b[d(x, Tx) + d(y, Ty)]$;

(z_3) $d(Tx, Ty) \leqslant c[d(x, Ty) + d(y, Tx)]$.

Then T *is a PO.*

Proof. Let $Y = \cup_{i=1}^{p} A_i$ then (Y, d) is a complete metric space.

Let the graph G be such that $V(G) = Y$, and

$$E(G) = \{(x, y) \in Y \times Y : \exists\, i \in \{1, 2, ..., n\} \text{ such that } x \in A_i \text{ and } y \in A_{i+1}\}$$

$$\cup \{(x, x) : x \in Y\}.$$

Because T is a cyclic operator we get

$$(Tx, Ty) \in E(G), \text{ for all } (x, y) \in E(G)$$

and from hypothesis, the operator T is a G-Zamfirescu mapping and G is a weakly connected graph.

Now let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. If $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then there is $j \in \{1, 2, ..., n\}$ such that $x \in A_i$. However in view of (2.9), the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for all $i \in \{1, 2, ..., n\}$. The subsequence of the sequence $\{x_n\}$ formed by the terms which are in A_{j-1} satisfies the condition (ii.) from Theorem 2.4. In conclusion the operator T is a PO.

□

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