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Fixed points for mappings defined on generalized gauge spaces

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ABSTRACT. In this article, the distinct classes of continuous pseudo-gauge structures and pseudometrics (perfect, quasi-perfect, sequentially complete) are defined and studied in depth. The conditions under which the set of fixed points of a given mapping of a space with concrete pseudo-gauge structure is non-empty are determined. Some examples are proposed.

1. INTRODUCTION

By a space we understand a completely regular topological Hausdorff space. We use the terminology from [14, 16, 20].

Let $\preceq$ be a pre-order on a set E, i.e. $\preceq$ is a binary relation on E such that $x \preceq x$ for each $x \in E$ and relations $x \preceq y$ and $y \preceq z$ imply $x \preceq z$. If $a \preceq b$ and $b \npreceq a$, then we put $a \prec b$. The pre-order $\preceq$ is an order if relations $x \preceq y$ and $y \preceq x$ imply $x = y$. A set $L \subset E$ is called *upper (lower) semi-bounded* if there exists $b \in E$ such that $x \preceq b$ ($b \succeq x$) for all $x \in L$.

A *supremum (infimum)* of a non-empty subset $L \subset E$ is an element $a = \vee L$ ($a = \wedge L$) satisfying the following conditions:

- $x \preceq a$ ($a \preceq x$) for each $x \in L$;

- if $b \in E$ and $x \preceq b$ ($b \preceq x$) for each $x \in L$, then $a \prec b$ ($b \prec a$).

If $a, b \in E$ are the supremums (infimums) of the set $L \subset E$, then simultaneous we have $a \preceq b$ and $b \preceq a$. A maximal (minimal) element need not be a supremum (infimum). The pre-ordered space E is *reticulated* if any non-empty upper (lower) semi-bounded subset L has a unique supremum $\vee L$ (infimum $\wedge L$). In this case from $a \preceq b$ and $b \preceq a$ it follows $a = b$.

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2. SPACES WITH PSEUDO-GAUGE STRUCTURES

A *Banach metric scale* is a non singleton partially ordered Banach space E such that:

- $x < y$ implies $x + z < y + z$;

- E is a reticulate lattice;

- for any non-empty lower semi-bounded chain L of E and $b = \wedge L$ there exists a sequence $A = \{x_n \in L : n \in \mathbb{N}\}$ such that $lim_{n\to\infty} x_n = b$;

- if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$.

Fix a Banach metric scale E.

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If L is a non-empty lower semi-bounded chain of E, $b = \wedge L$ and $A = \{x_n \in L : n \in \mathbb{N}\}$ is a sequence such that $lim_{n\to\infty} x_n = b$, then $b = \wedge A$.

A function $\rho : X \times X \longrightarrow \mathbb{E}$ is called an *E-pseudometric* or simply *pseudometric* on a space X if:

(P1) $\rho(x,x) = 0$, $\rho(x,y) = \rho(y,x)$ and $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$.

If $\rho(x,y) = 0$ if and only if $x = y$, then ρ is called an *E-metric* on the space X. General concepts of metric scales and of pseudometrics were examined by distinct authors (by instance, see [1, 18, 20]).

Let ρ be a pseudometric on a space X.

The pseudometric ρ is non-negative: $\rho(x,x) \leq \rho(x,y) + \rho(y,x) = \rho(x,y) + \rho(x,y)$ and $0 = \rho(x,x) \leq \rho(x,y) \leq \rho(x,y) + \rho(x,y)$.

If $\varepsilon > 0$ and $x \in X$, then the set $B(x,\rho,\varepsilon) = \{y \in X : \|\rho(x,y)\| < \varepsilon\}$ is called the *ε-ball* of the space X with center x and radius ε or, simply, the *ε-ball* about x. The pseudometric ρ generate on X the topology $T(\rho)$ with the open base $\{B(x,\rho,\varepsilon) : x \in X, \varepsilon > 0\}$.

If $x \in X$, then we put $\rho(x,H) = \wedge\{\rho(x,y) : y \in H\}$. If r is a positive number, then $\rho(x,H) \leq r$, if $B(x,\rho,r) \cap H \neq \emptyset$.

A pseudometric ρ is called a *continuous pseudometric* on a space X if:

(P2) the set $B(x,\rho,\varepsilon)$ is open in X for all $x \in X$ and $\varepsilon > 0$.

Proposition 2.1. *Let E be a Banach metric scale and ρ be an E-pseudometric on a space X. Then the function $d_\rho(x,y) = \|\rho(x,y)\|$ is a real-valued pseudometric on the space X with the following properties:*

1. The pseudometric ρ is continuous if and only if the pseudometric d_ρ is continuous.

2. The pseudometric ρ is a metric if and only if the pseudometric d_ρ is a metric.

3. The metric ρ is complete if and only if the metric d_ρ is complete.

Proof. Let $\varepsilon > 0$. The assertions of the Proposition follow from the equality $\{x \in X : \|\rho(x,y)\| < \varepsilon\} = \{x \in X : d_\rho(x,y) < \varepsilon\}$. □

Remark 2.1. Let E be a Banach metric scale and ρ be an E-pseudometric on a space X. Then the function $d_\rho(x,y) = \|\rho(x,y)\|$ is a real-valued pseudometric on the space X with the following properties:

1. The pseudometric ρ is continuous if and only if the pseudometric d_ρ is continuous.
2. The pseudometric ρ is a metric if and only if the pseudometric d_ρ is a metric.
3. The metric ρ is complete if and only if the metric d_ρ is complete.

A *pseudo-gauge E-structure*, or simple a *pseudo-gauge structure* on a space X is a non-empty family $\mathcal{G} = \{d_\alpha : X \times X \longrightarrow E : \alpha \in A\}$ of continuous pseudometrics.

The pseudo-gauge E-structure $\mathcal{G} = \{d_\alpha : \alpha \in A\}$ generate on X the topology $T(\mathcal{G}) = \vee\{T(d_\alpha) : \alpha \in A\}$.

If $T(\mathcal{G})$ is the topology of the space X, then $\mathcal{G}$ is called a *gauge E-structure* or a *gauge structure* (see [20] for $E = \mathbb{R}$).

Fix a space X and a pseudo-gauge E-structure $\mathcal{G} = \{d_\alpha : \alpha \in A\}$ on the space X.

We put $Z(x,\mathcal{G}) = \{y \in X : d_\alpha(x,y) = 0 \textit{ for each } \alpha \in A\}$ for any $x \in X$. A sequence $\{x_n : n \in \mathbb{N}\}$ is called $\mathcal{G}$-Cauchy if for each $\alpha \in A$ and each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $d_\alpha(x_n, x_k) < \varepsilon$ for all $n, k \geq m$.

We say that the pseudo-gauge E-structure $\mathcal{G}$:

- is sequentially quasi-complete if any $\mathcal{G}$-Cauchy sequence has an accumulation point in X;

- is sequentially complete if any $\mathcal{G}$-Cauchy sequence has an accumulation point in X and the set $Z(x,\mathcal{G})$ is compact for each point $x \in X$.

On a compact space any pseudo-gauge E-structure is complete. If $x \in X$ is an accumulation point of a $\mathcal{G}$-Cauchy sequence $\{x_n : n \in \mathbb{N}\}$, then $lim_{n\to\infty} d_\alpha(x, x_n) = 0$ for each $\alpha \in A$.

There exist a space $X/\mathcal{G}$ and a gauge E-structure $\bar{\mathcal{G}} = \{\bar{d}_\alpha : \alpha \in A\}$ and a surjection $\pi_{\mathcal{G}} : X \longrightarrow X/\mathcal{G}$ such that $d(x, y) = \bar{d}_\alpha(\pi_{\mathcal{G}}(x), \pi_{\mathcal{G}}(y))$ for all $x, y \in X$ and $\alpha \in A$.

On $(X/\mathcal{G}, \bar{\mathcal{G}})$ we consider the topology $T(\bar{\mathcal{G}})$. The pseudometrics d_α are continuous if and only if the mapping $\pi_{\mathcal{G}}$ is continuous.

The pseudo-gauge E-structure $\mathcal{G}$ is sequentially quasi-complete if and only if the gauge E-structure $\bar{\mathcal{G}}$ is sequentially complete.

The pseudo-gauge E-structure $\mathcal{G}$ is called a *quasi-perfect* pseudo-gauge E-structure on a space X if it satisfies the following conditions:

(P3) the set $Z(x, \mathcal{G})$ is countably compact for each point $x \in X$;

(P4) If $x \in X$, the set U is open in X and $Z(x, \mathcal{G}) \subseteq U$, then there exist $\varepsilon > 0$ and a finite subset $B \subseteq A$ such that $\cap\{B(x, d_\beta, \varepsilon) : \beta \in B\}$.

The pseudo-gauge E-structure $\mathcal{G}$ is called a *perfect* pseudo-gauge E-structure on a space X if it is quasi-perfect satisfy the following condition:

(P5) the set $Z(x, \mathcal{G})$ is compact for each point $x \in X$.

A mapping $g : X \longrightarrow Y$ of space X into a space Y is called closed if the set $g(H)$ is closed in Y for each closed subset H of the space X. The mapping g is a *perfect mapping* if it is continuous, closed and the fibers $g^{-1}(y)$, $y \in Y$, are compact. The mapping g is a *quasi-perfect mapping* if it is continuous, closed and the fibers $g^{-1}(y)$, $y \in Y$, are countably compact.

The pseudo-gauge E-structure $\mathcal{G}$ is *quasi-perfect* if and only if the mapping $\pi_{\mathcal{G}} : X \longrightarrow X/\mathcal{G}$ is quasi-perfect. The pseudo-gauge E-structure $\mathcal{G}$ is *perfect* if and only if the mapping $\pi_\rho : X \longrightarrow X/\mathcal{G}$ is perfect.

The pseudo-gauge E-structure $\mathcal{G}$ is a gauge E-structure if and only if it is *quasi-perfect* and for any two distinct points $x, y \in X$ there exists $\alpha \in A$ such that $d_\alpha(x, y) \neq 0$ (i.e. the mapping $\pi_{\mathcal{G}}$ is one-to-one and, therefore, is a homeomorphism).

On a compact space any pseudo-gauge structure is perfect and sequentially complete.

Let $\mathcal{G} = \{d_\alpha : Y \times Y \to \mathbb{R} : \alpha \in A\}$ be a pseudo-gauge structure on a countably compact space X and the quotient space $X/\mathcal{G}$ is metrizable. Then the pseudo-gauge structure $\mathcal{G}$ is quasi-perfect and is sequentially quasi-complete. Moreover, if the space X is not compact, then the pseudo-gauge structure $\mathcal{G}$ is not perfect and is not sequentially complete.

Example 2.1. Let Y be an infinite compact space and $\mathcal{G} = \{d_\alpha : Y \times Y \to \mathbb{R} : \alpha \in A\}$ be a family of continuous pseudometrics on Y and for any two distinct points $x, y \in Y$ there exists $\alpha \in A$ such that $d_\alpha(x, y) \neq 0$. Then $(Y, \mathcal{G})$ is a compact sequentially complete gauge space. Let X be the set Y with the discrete topology. Then $\mathcal{G}$ is a pseudo-gauge structure on X, $X/\mathcal{G} = Y$, the mapping $\pi_{\mathcal{G}}$ is one-to-one and non-perfect. In particular, $\mathcal{G}$ is not a quasi-perfect pseudo-gauge structure on X.

Moreover:

- if in Y there exists an infinite convergent sequence, then the pseudo-gauge structure $\mathcal{G}$ is not sequentially complete on X;

- if Y is a space without infinite convergent sequences, then the pseudo-gauge structure $\mathcal{G}$ is sequentially complete on X.

Example 2.2. Let Y be an infinite compact space and $\mathcal{G}^* = \{\rho_\alpha : Y \times Y \to \mathbb{R} : \alpha \in A\}$ be a family of continuous pseudometrics on Y and for any two distinct points $x, y \in Y$ there exists $\alpha \in A$ such that $\rho_\alpha(x, y) \neq 0$. Then $(Y, \mathcal{G}^*)$ is a compact gauge space. The gauge structure $\mathcal{G}^*$ is sequentially complete. Fix an infinite space Z. We put $X = Y \times Z$. Let $\pi_Y : X \longrightarrow Y$ be the projection $\pi_Y(y, z) = y$ for any $(y, z) \in Y \times Z = X$. The mapping π is open. If the space Y is first-countable and the space Z is countably compact, then the mapping π_Y is quasi-perfect. If the space Z is compact, then the mapping π_Y is perfect. For any $\alpha \in A$ and all $u = (y_1, z_1) \in X$, $v = (y_2, z_2) \in X$ we put $d_\alpha(u, v) = \rho_\alpha(y_1, y_2)$. Then $\mathcal{G} = \{d_\alpha : X \times X \to \mathbb{R} : \alpha \in A\}$ is a quasi-gauge structure on X with the following properties:

(1) the quasi-gauge structure $\mathcal{G}$ is sequentially quasi-complete if and only if the space Z is countably compact;

(2) the quasi-gauge structure $\mathcal{G}$ is quasi-perfect if and only if the space Z is countably compact and the mapping π is closed;

(3) the quasi-gauge structure $\mathcal{G}$ is perfect if and only if the space Z is compact;

(4) $\pi_{\mathcal{G}} = \pi_Y$.

Consider some particular cases:

Case 1. Let Z be the subspace of rational numbers of the unity segment $Y = [0, 1]$, $\rho(u, v) = |u - v|$ for all $u, v \in Y$, $\mathcal{G}^* = \{\rho\}$. Thus the gauge space $(Y, \mathcal{G}^*)$ is a metric compact space. We put $x_n = (2^{-n} \cdot 2^{1/2}, 2^{-n})$. Then $\{x_n : n \in \mathbb{N}\}$ is a non-convergent Cauchy sequence of the pseudo-gauge space $(X, \mathcal{G})$.

Case 2. Let Ω be the first uncountable ordinal number and Y be the space of all ordinal numbers $\nu \leq \Omega$ in the topology induced on Y by the natural linear order on Y. The space Y is compact and not first-countable (in point Ω. The space $Z = Y \setminus \{\Omega\}$ is first-countable and countably compact. The set $F = \{(\nu, \nu) : \nu \in Z\}$ is closed in $X = Y \times Z$ and the set $\pi_Y(F)$ is not closed in Y. In this case the quasi-gauge structure $\mathcal{G}$ is sequentially quasi-complete and not quasi-perfect.

Case 3. Let Z be the space from the Case 2, Y and $\mathcal{G}^*$ be as in the Case 1. In this case the quasi-gauge structure $\mathcal{G}$ is sequentially quasi-complete, quasi-perfect and not perfect.

Example 2.3. Let τ be an uncountable cardinal number $D = \{0, 1\}$ be a discrete space, $B = \{b_n : n \in \mathbb{N}\}$ be an infinite convergent sequence of the space D^τ to the point $b \in D^\tau \setminus L$. Let $X = D^\tau \setminus \{b\}$. The space X is not countably compact and B be a discrete closed subset of the space X. Fix a pseudo-gauge E-structure $\mathcal{G} = \{d_\alpha : X \times X \longrightarrow E : \alpha \in A\}$ on a space X.

The space D^τ is the Stone-Čech compactification βX of the space X and the space $X/\{d\}$ is a compact metrizable space for each continuous E-pseudometric d on X. Thus for each continuous E-pseudometric d on X there exists a continuous E-pseudometric $e(d)$ on $\beta X = D^\tau$ such that $e(d)(x, y) = d(x, y)$ for all $x, y \in X$. If the cardinality $|A| < \tau$, then the set $F(A) = \{x \in X : e(d)(b, x) = 0\}$ is of cardinality τ and non-empty.

Therefore, the pseudo-gauge space $(X, \mathcal{G})$ has the following properties:

(1) $B = \{b_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the pseudo-gauge space $(X, \mathcal{G})$;

(2) the pseudo-gauge space $(X, \mathcal{G})$ is not quasi-complete;

(3) if $|A| < \tau$, then the gauge space $(X/\mathcal{G}, \bar{\mathcal{G}})$ is complete;

(4) if $\mathcal{G}$ is a gauge structure on X, then $|A| \geq \tau$ and the gauge space $(X, \mathcal{G}) = (X/\mathcal{G}, \bar{\mathcal{G}})$ is not quasi-complete.

3. Some general reduction principles

Fix a Banach metric scale E and a pseudo-gauge structure $\mathcal{G}$ on a space X.

If $f : X \longrightarrow Y$ is a set-valued mapping of a space X into a space Y, then $f(x)$ is a non-empty closed subset of Y for any $x \in X$.

For any set-valued mapping $f : X \longrightarrow X$ denote by $Fix(f) = \{x \in X : x \in f(x)\}$ the set of fixed points of the mapping f, by $Fix_{\mathcal{G}}(f) = \{x \in X : d(x, y) = 0 \textit{ for some } y \in f(x)\}$ is the set of $\mathcal{G}$-fixed points of the mapping f and by $Fix_{v\mathcal{G}}(f) = \{x \in X : d(x, f(x)) = 0\}$ is the set of virtual $\mathcal{G}$-fixed points of the mapping f. Obviously, $Fix(f) \subseteq Fix_{\mathcal{G}}(f) \subseteq Fix_{v\mathcal{G}}(f)$. In general, the sets $Fix(f)$, $Fix_G(f)$, $Fix_{v\mathcal{G}}(f)$ are distinct. If $E = \mathbb{R}$, then $Fix_{\mathcal{G}}(f) = Fix_{v\mathcal{G}}(f)$.

Example 3.4. Let $E = \mathbb{R} \times \mathbb{R}$, $X = \{(0,0),(0,1),(1,0)\}$ be a subspace of E, $d((x, y),(u, v)) = (|x - u|, |y - v|)$ for each pair of points $(x, y),(u, v) \in X$ is the E-metric on X, $\mathbb{G} = \{d\}$ is a gauge structure on X. Consider the set-valued mapping $f : X \longrightarrow X$, where $f(0, 1) = (1, 0)$, $f(0, 1) = (0, 0)$, and $f(0, 0) = \{(0, 1),(1, 0)\}$. Then $Fix(f) = \emptyset$, $Fix_G(f) = \emptyset$, $Fix_{v\mathcal{G}}(f) = \{(0, 0)\}$.

Assume that $n \geq 0$, $f^{(0)}(x) = x$ for each $x \in X$, $f^{(1)} = f$ and $f^{(n+1)}(x) = f(f^{(n)}(x))$ for each $x \in X$.

In the applications of pseudometric spaces (X, d), the complementary assumptions compensate the situation $d(x, y) = 0$ for some distinct points $x, y \in X$. We mention that in some cases the study of problems on spaces with pseudometrics can be reduced to the metric spaces.

Theorem 3.1. *Let E be a Banach metric scale, $\mathcal{G} = \{d_\alpha : X \times X \longrightarrow E : \alpha \in A\}$ be a pseudo-gauge E-structure on a space X and $g : X \longrightarrow X$ be a mapping. Assume that for any two points $x, y \in X$ for which $g(x) \neq g(y)$ there exists $\alpha = a(x, y) \in A$ such that $d_\alpha(x, y) \neq 0$. Then there exists a mapping $f : X/\mathcal{G} \longrightarrow X/\mathcal{G}$ such that:*

1. $f(\pi_{\mathcal{G}}(x)) = \pi_{\mathcal{G}}(g(x))$ for each $x \in X$.

2. If $b \in X$ and $\pi_{\mathcal{G}}(b)$ is a fixed point of the mapping f, then $g(b)$ is a fixed point of the mapping g.

3. If $a, b \in X$, $\{g^n(a) : n \in \mathbb{N}\}$ is a Cauchy sequence and $lim_{n\to\infty} d_\alpha(b, g^n(a)) = 0$ for any $\alpha \in A$, then $g(b)$ is a fixed point of the mapping g.

4. Assume that $\mathcal{P}$ are properties of mappings of pseudo-gauge E-structures and of E and for any mapping of a E-gauge space with properties $\mathcal{P}$ the set of fixed points is non-empty. If the pseudo-gauge E-structure $\mathcal{G}$ on the space X, mapping g and E has the properties $\mathcal{P}$, then the set of $\mathcal{G}$-fixed points of the mapping g is non-empty.

Proof. If $x, y \in X$ and $d_\alpha(x, y) = 0$ for all $\alpha \in A$, then $g(x) = g(y)$.

There exists a subset Y of X such that for each $x \in X$ there exists a unique point $y(x) \in Y$ such that $\rho(x, y(x)) = 0$. Then $\pi_\rho(Y) = X/\mathcal{G}$ and $\pi_{\mathcal{G}}|Y$ is a one-to-one mapping of Y onto $X/\mathcal{G}$.

For each $y \in Y$ there exists a unique point $h(y) = y(g(y)) \in Y$ such that $d_\alpha(g(y), h(y)) = 0$ for all $\alpha \in A$. Now, for each $x \in X$ we put $f(\pi_{\mathcal{G}}(x)) = \pi_{\mathcal{G}}(h(y(x)))$. If $x, x' \in X$ and $d_\alpha(x, x') = 0$ for all $\alpha \in A$, then $y(x) = y(x')$, $g(x) = g(x')$ and $f(\pi_{\mathcal{G}} x)) = f(\pi_{\mathcal{G}}(x'))$. Thus the mapping f is correct defined and $f(\pi_{\mathcal{G}}(x)) = \pi_{\mathcal{G}}(g(x))$ for each $x \in X$.

Assume that $b \in X$ and $\pi_{\mathcal{G}}(b)$ is a fixed point of the mapping f, i.e. $f(\pi_\rho(b)) = \pi_\rho(b)$. Suppose that $g(g(b)) \neq g(b)$. Then $d_\alpha(b, g(b)) \neq 0$ for some $\alpha \in A$ and $\pi_{\mathcal{G}}(b) \neq \pi_{\mathcal{G}}(g(b)) = f(\pi_{\mathcal{G}}(b)) = \pi_{\mathcal{G}}(b)$, a contradiction. The assertion 2 is proved. The assertion 4 follows from the assertion 2.

Fix $a \in X$. Let $b \in X$ and $lim_{n\to\infty} d_\alpha(b, g^n(a)) = 0$ for each $\alpha \in A$. Then $lim_{n\to\infty} d_\alpha(g(b), g^n(a)) = 0$ and $d_\alpha(b, g(b)) = 0$ for each $\alpha \in A$. Thus b is a $\mathcal{G}$-fixed point of g and $g(g(b)) = g(b)$. The assertion 3 is proved. The proof is complete. □

Theorem 3.2. *Let $\mathcal{G}$ be a pseudo-gauge E-structure on a space X and $f : X \longrightarrow X$ be a mapping.*

1. Assume $x = y$ if $d(f(x), f(y)) = 0$ for each $d \in \mathcal{G}$. Then $Fix_G(f)$ = $Fix(f)$. In particular, from $Fix_G(f) \neq \emptyset$ it follows $Fix(f) \neq \emptyset$.

2. Assume that f is a set-valued mapping and for any $x \in X$ with $x \notin f(x)$ there exists $d \in \mathcal{G}$ such that $d(x, f(x)) > 0$. Then $Fix(f)$ = $Fix_\mathcal{G}(f)$. In particular, from $Fix_\mathcal{G}(f) \neq \emptyset$ it follows $Fix(f) \neq \emptyset$.

3. Assume that $n \geq 0$ and for any $x \in X$ with $f^{(n+1)}(x) \neq f^{(n)}(x)$ there exists $d \in \mathcal{G}$ such that $d(f^{(n)}(x)), f^{(n+1)}(x)) > 0$. Then $f^{(n)}(b) \in Fix(f)$ for each $b \in Fix_\mathcal{G}(f)$. In particular, from $Fix_\mathcal{G}(f) \neq \emptyset$ it follows $Fix(f) \neq \emptyset$.

Proof. Assertion 1 immediately follows from Theorem 3.1. Assertion 2 is obvious. We mention, that for single-valued mappings Assertion 2 is the Assertion 3 for $n = 0$.

Let $n \geq 0$. Suppose that for any $x \in X$ with $f^{(n+1)}(x) \neq f^{(n)}(x)$ there exists $d \in \mathcal{G}$ such that $d(f^{(n)}(x), f^{(n+1)}(x)) > 0$. Fix $b \in Fix_\mathcal{G}(f)$. Let $c = f^{(n)}(b)$. Then $c \in Fix_\mathcal{G}(f)$ and $d(b, c) = 0$ for each $d \in \mathcal{G}$. Assume that $f(c) \neq c$. Then $c = f^{(n)}(x) \neq f^{(n+1)}(x) = f(c)$ and there exists $d \in \mathcal{G}$ such that $d(f(f^{(n)}(x)), f^{(n+1)}(x)) > 0$, i.e. $d(c, f(c)) > 0$, a contradiction. Assertion 3 is proved. The proof is complete. □

Corollary 3.1. *Assume that $\mathcal{P}$ are properties of mappings of pseudo-gauge E-structures and of Banach m-scales E and for any mapping of a E-gauge space with properties $\mathcal{P}$ the set of fixed points is non-empty. Let E be a Banach m-scale, $\mathcal{G}$ = $\{d_\alpha : X \times X \longrightarrow E : \alpha \in A\}$ be a pseudo-gauge E-structure on a space X and $g : X \longrightarrow X$ be a mapping with properties $\mathcal{P}$. If for each $x \in X$ the subspace $Z(x, \rho)$ is a fixed point space, then $Fix(g)$ = $\{x \in X : g(x) = x\}$ is a non-empty set.*

Example 3.5. Let $\mathbb{R}$ be the space of reals with the distance $\rho(x, y)$ = $|x - y|$. Fix a number $k \in \mathbb{R} \setminus \{0, 1\}$. Consider the mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where $\varphi(x) = kx$ for each $x \in X$. Let $\mathbb{D}$ = $\{-1, 1\}$ be a discrete space. Let X = $\mathbb{R} \times \mathbb{D}$ and $\psi(x, i)$ = $(\varphi(x), -i)$ for each point $(x, i) \in \mathbb{R} \times \mathbb{D}$ = X. On X consider the continuous pseudometric $d((x, i), (y, j)) = \rho(x, y)$ for all points $(x, i), (y, j) \in \mathbb{R} \times \mathbb{D}$ = X. Then $\mathbb{G} = \{d\}$ is a perfect pseudo-gauge structure on X, $d(\psi(x, i), \psi(y, j))$ = $|k| \cdot d((x, i), (y, j))$ for all points $(x, i), (y, j) \in X$, $Fix(\psi) = \emptyset$ and $Fix_\mathcal{G}(\psi) = \{(0, -1), (0, 1)\}$.

Example 3.6. Let $\mathbb{R}$ be the space of reals with the distance $\rho(x, y)$ = $|x - y|$. Fix a number $k \in \mathbb{R} \setminus \{0, 1\}$. Consider the mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where $\varphi(x) = kx$ for each $x \in X$. Let $\mathbb{I}$ = $[0, 1]$ be the unite interval as a subspace of the space of reals $\mathbb{R}$. Let X = $\mathbb{R} \times \mathbb{I}$ and n be a natural number. We put $\psi(x, t)$ = $(\varphi(x), max\{0, 2^{-1}t - 2^{-n-1})$ for each point $(x, t) \in \mathbb{R} \times \mathbb{I}$ = X. On X consider the continuous pseudometric $d((x, t), (y, t')) = \rho(x, y)$ for all points $(x, t), (y, t') \in X$. Then $\mathbb{G} = \{d\}$ is a perfect pseudo-gauge structure on X, $d(\psi(x, t), \psi(y, t'))$ = $|k| \cdot d((x, t), (y, t'))$ for all points $(x, t), (y, t') \in X$, $Fix(\psi) = \{(0, 0)\}$ and $Fix_\mathcal{G}(\psi) = \{(0, t) : t(0, 1) \in \mathbb{I}\}$. By construction, if $\psi^{(n)}(x, t) \neq \psi^{(n+1)}(x, t)$, then $d(\psi^{(n)}(x, t), \psi^{(n+1)}(x, t)) > 0$. Hence $\psi^n(Fix_\mathcal{G}(\psi))$ = $Fix(\psi)$.

Example 3.7. Let $\mathbb{R}$ be the space of reals with the distance $\rho(x, y)$ = $|x - y|$. Fix a number $k \in \mathbb{R} \setminus \{0, 1\}$. Consider the mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where $\varphi(x) = kx$ for each $x \in X$. Let $\mathbb{I}$ = $[0, 1]$ be the unite interval as a subspace of the space of reals $\mathbb{R}$. Let X = $\mathbb{R} \times \mathbb{I}$ and n be a natural number. We put $\psi(x, t)$ = $(\varphi(x), max\{0, 2^{-1}t - 2^{-n-1})$ for each point $(x,t) \in \mathbb{R} \times \mathbb{I} = X$. On X consider the continuous pseudometric $d((x,t),(y,t')) = \rho(x,y)$ for all points $(x,t),(y,t') \in X$. Then $\mathbb{G} = \{d\}$ is a perfect pseudo-gauge structure on X, $d(\psi(x,t),\psi(y,t')) = |k| \cdot d((x,t),(y,t'))$ for all points $(x,t),(y,t') \in X$, $Fix(\psi) = \{(0,0)\}$ and $Fix_{\mathcal{G}}(\psi) = \{(0,t) : t \in \mathbb{I}\}$. By construction, if $\psi^{(n)}(x,t) \neq \psi^{(n+1)}(x,t)$, then $d(\psi^{(n)}(x,t),\psi^{(n+1)}(x,t)) > 0$. Hence $\psi^n(Fix_{\mathcal{G}}(\psi)) = Fix(\psi)$.

Example 3.8. Let $\mathbb{R}$ be the space of reals with the distance $\rho(x,y) = |x-y|$. Fix a number $k \in \mathbb{R} \setminus \{0,1\}$. Consider the mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where $\varphi(x) = kx$ for each $x \in X$. Let $X = \mathbb{R} \times \mathbb{I}$. We put $\psi(x,t) = (\varphi(x), max\{0, 2^{-1}t)$ for each point $(x,t) \in \mathbb{R} \times \mathbb{I} = X$. On X consider the continuous pseudometric $d((x,t),(y,t')) = \rho(x,y)$ for all points $(x,t),(y,t') \in X$. Then $\mathbb{G} = \{d\}$ is a perfect pseudo-gauge structure on X, $d(\psi(x,t),\psi(y,t')) = |k| \cdot d((x,t),(y,t'))$ for all points $(x,t),(y,t') \in X$, $Fix(\psi) = \{(0,0)\}$ and $Fix_{\mathcal{G}}(\psi) = \{(0,t) : t \in \mathbb{I}\}$. By construction, $\psi^n(Fix_{\mathcal{G}}(\psi)) \neq Fix(\psi)$ for each $n \in \mathbb{N}$.

Example 3.9. Let $\mathbb{R}$ be the space of reals with the distance $\rho(x,y) = |x-y|$. Fix a number $k \in \mathbb{R} \setminus \{0,1\}$. Consider the mapping $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where $\varphi(x) = kx$ for each $x \in X$. Let $X = \mathbb{R} \times \mathbb{R}$. We put $\psi(x,t) = (\varphi(x), max\{0, 2^{-1}t - 2^{-n-1})$ for each point $(x,t) \in \mathbb{R} \times \mathbb{R} = X$. On X consider the continuous pseudometric $d((x,t),(y,t')) = \rho(x,y)$ for all points $(x,t),(y,t') \in X$. Then $\mathbb{G} = \{d\}$ is a pseudo-gauge structure on X, $d(\psi(x,t),\psi(y,t')) = |k| \cdot d((x,t),(y,t'))$ for all points $(x,t),(y,t') \in X$, $Fix(\psi) = \{(0,0)\}$ and $Fix_{\mathcal{G}}(\psi) = \{(0,t) : t \in \mathbb{R}\}$. By construction, $\psi^n(Fix_{\mathcal{G}}(\psi)) \neq Fix(\psi)$ for each $n \in \mathbb{N}$. Moreover, for each $(x,t) \in Fix_{\mathcal{G}}(\psi)$ there exists n such that $\psi(x,t) \in Fix(\psi)$. We mention that the pseudo-gauge structure $\mathbb{G} = \{d\}$ is not perfect.

4. Some applications of the reduction principles

Consider a Banach metric scale E and a pseudo-gauge structure $\mathcal{G}$ on a space X.

The results of above sections may be applied to the spaces with pseudo-gauge structures. For that are important the conditions which guarantee non-empties of the set $Fix_{\mathcal{G}}(f)$. A mapping $g : X \longrightarrow Y$ is called an *upper semicontinuous* mapping, if for any open subset V of Y the set $g^{\ominus 1}(V) = \{x \in X : g(x) \subseteq V\}$ is open in X.

Lemma 4.1. *Let $\mathbb{G}$ be a sequentially quasi-complete pseudo-gauge structure on a space X and $g : X \longrightarrow X$ be an upper semicontinuous compact-valued mapping. Then:*

1. The set $Fix(g)$ is closed in X.

2. The set $Fix_{\mathcal{G}}(g)$ is closed in X.

3. If $x \in Fix_{\mathcal{G}}(g)$, then $Z(x,\mathcal{G}) \subseteq Fix_{\mathcal{G}}(g)$.

Proof. Assume that $x \notin g(x)$. There exist two open subsets V and W of X such that $x \in V$, $g(x) \subseteq W$ and $V \cap W = \emptyset$. The set $U = V \cap g^{\ominus 1}(W)$ is open in X, $x \in U$ and $U \cap g(U) = \emptyset$. Assertion 1 is proved for arbitrary upper semicontinuous set-valued mappings. Assume that $x_0 \notin Fix_{\mathcal{G}}(g)$. Then for each $x \in g(x_0)$ there exist $r(x) > 0$ and $d_x \in \mathcal{G}$ such that $\|d_x(x_0,x)\| \geq 3r(x)$. Since $g(x_0)$ is a compact space and $g(x_0) \subseteq \cup\{B(x,d_x,r(x)) : x \in g(x_0)\}$, then there exists a non-empty finite subset L of $g(x_0)$ such that $g(x_0) \subseteq \cup\{B(x,d_x,r(x)) : x \in L\}$. Let $W = \cup\{B(x,d_x,r(x)) : x \in L\}$, $V_1 = \cap\{B(x_0,d_x,r(x)) : x \in L\}$ and $V = V_1 \cap g^{\ominus 1}(W)$. The set V is open in X, $x_0 \in Z(x_0,\mathcal{G}) \subseteq V$ and $V \cap Fix_{\mathcal{G}}(g) = \emptyset$. Assertions 2 and 3 are proved. □

A sequence $\{x_n : n \in \mathbb{N}\}$ is a Picard sequence of the set-valued mapping $g : X \longrightarrow X$ if $x_{n+1} \in g(x_n)$ for each $n \in \mathbb{N}$. If $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space $(X,\mathcal{G})$, then we say that $\{x_n : n \in \mathbb{N}\}$ is a $\mathcal{G}$-Picard sequence of the mapping g.

Lemma 4.2. *Let $\mathcal{G}$ be a sequentially quasi-complete pseudo-gauge structure on a space X, $g : X \longrightarrow X$ be an upper semicontinuous set-valued mapping and $\{x_n : n \in \mathbb{N}\}$ be a Picard sequence of the mapping g. Then:*

1.If $x \in X$ and $lim_{n\to\infty} d(x, x_n) = 0$ for each $d \in \mathcal{G}$, then $x \in Z(x, \mathcal{G}) \subseteq Fix_{\mathcal{G}}(g)$ and $\{x_n : n \in \mathbb{N}\}$ is a $\mathcal{G}$-Picard sequence of the mapping g.

2. If $\{x_n : n \in \mathbb{N}\}$ is a $\mathcal{G}$-Picard sequence of the mapping g and x is an accumulation point of the sequence $\{x_n : n \in \mathbb{N}\}$, then $lim_{n\to\infty} d(x, x_n) = 0$ for each $d \in \mathcal{G}$ and $x \in Fix_{\mathcal{G}}(g)$.

Proof. Let $b \in X$ and $lim_{n\to\infty} d(b, x_n) = 0$ for each $d \in \mathcal{G}$. Then $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space $(X, \mathcal{G})$ and a $\mathcal{G}$-Picard sequence of the mapping g. The sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point x_0 and $b \in Z(x_0, \mathcal{G})$. By virtue of Lemma 4.1, we can assume that $b = x_0$ is an accumulation point of the sequence $\{x_n : n \in \mathbb{N}\}$. Since the mapping g is upper semicontinuous and $\{x_n : n \in \mathbb{N}\} \subseteq g(\{x_n : n \in \mathbb{N}\})$, the sequence $\{x_n : n \in \mathbb{N}\}$ has an accumulation point $c \in g(b)$. Then $d(b, c) = 0$ for all $d \in \mathcal{G}$ and $b \in Fix_{\mathcal{G}}(g)$. Assertion 1 is proved. Assertion 2 follows from Assertion 1. □

A set-valued mapping $g : X \longrightarrow X$ is called *$\mathcal{G}$-contractive* if there exists a family of non-negative numbers $\{k_d : d \in \mathbb{G}\}$ and for all $x, y \in X$ and $x' \in g(x)$ there exists $y' = c(g, x, y, x') \in g(y)$ for which $d(x', y') \leq k_d \cdot d(x, y)$ and $0 \leq k_d < 1$ for each $d \in \mathbb{G}$.

Lemma 4.3. *Let $\mathcal{G}$ be a pseudo-gauge structure on a space X and $g : X \longrightarrow X$ be a $\mathbb{G}$-contractive mapping. If $x_1 \in X$ is an arbitrary fixed point, $x_2 \in g(x_1)$ and $x_{n+2} = c(g, x_n, x_{n+1}, x_{n+1}) \in g(x_{n+1})$ for each $n \in \mathbb{N}$, then $T(g, x_1, x_2) = \{x_n : n \in \mathbb{N}\}$ is a $\mathcal{G}$-Picard sequence of the mapping g generated by x_1 and $x_2 \in g(x_1)$.*

Proof. Assume that the points $x_1 \in X$ and $x_2 \in g(x_1)$ are fixed and $x_{n+2} = c(g, x_n, x_{n+1}, x_{n+1}) \in g(x_{n+1})$ for each $n \in \mathbb{N}$. Let $b_d = d(x_1, x_2)$. Then $d(x_{n+1}, x_{n+m}) \leq (k_d^n : (1 - k_d)) \cdot b_d$. The proof is complete. □

Theorem 4.3. *Let $\mathcal{G}$ be a sequentially complete perfect pseudo-gauge structure on a space X and $g : X \longrightarrow X$ be a $\mathcal{G}$-contractive upper semicontinuous set-valued mapping. Then for each points $x_1 \in X$ and $x_2 \in g(x_1)$ the Picard orbit $T(g, x_1, x_2) = \{x_n : n \in \mathbb{N}\}$ is a $\mathcal{G}$-Picard sequence of the pseudo-gauge space $(X, \mathbb{G})$ with accumulation points in X. Moreover, if $z \in X$ is an accumulation point of the sequence $T(g, x_1, x_2)$, then $z \in Fix_{\mathcal{G}}(g)$ and $lim_{n\to\infty} d(z, x_n) = 0$ for each $d \in \mathcal{G}\}$.*

Proof. Follows from Lemmas 4.2 and 4.3. □

Theorem 4.4. *Let $\mathcal{G}$ be a sequentially quasi-complete pseudo-gauge structure on a space X and $g : X \longrightarrow X$ be a $\mathcal{G}$-contractive mapping. Then:*

1. For each point $x \in X$ the Picard orbit $T(g, x) = \{x_n : n \in \mathbb{N}\}$, where $x_1 = x$ and $x_{n+1} = g(x_n)$ for each $n \in \mathbb{N}$, is a Cauchy sequence of the pseudo-gauge space $(X, \mathcal{G})$ with accumulation points in X. Moreover, if $z \in X$ is an accumulation point of the sequence $T(g, x)$, then $z \in Fix_{\mathcal{G}}(g)$ and $lim_{n\to\infty} d(z, x_n) = 0$ for each $d \in \mathcal{G}$.

2. If $x, y \in Fix_{\mathcal{G}}(g)$, then $d(x, y) = 0$ for each $d \in \mathcal{G}$ and $Fix_{\mathcal{G}}(g) = \{z \in X : d(x, z) = 0$ for each $d \in \mathcal{G}\}$.

3. $Fix_{\mathcal{G}}(g)$ is a non-empty closed countably compact subspace of the space X.

Proof. Assertion 1 follows from Theorem 4.3.

If $x, y \in Fix_{\mathcal{G}}(g)$, then $d(x, g(x)) = 0$, $d(y, g(y)) = 0$, $d(x, y) \leq d(x, g(x)) + d(g(x), g(y)) + d(g(y), y) = d(g(x), g(y)) \leq k_d \cdot d(x, y)$, i.e. $d(x, y) = 0$ for each $d \in \mathcal{G}\}$. If $d \in \mathcal{G}\}$, $z \in X$ and $d(x, z) > 0$, then $d(x, g(z)) = d(g(x), g(z)) < d(x, z)$. Thus $z \neq g(z)$, $d(g(z), z) > 0$ and

$z \notin Fix_{\mathcal{G}}(g)$. If $d(x,z) = 0$, then $d(x, g(z)) = 0$ too. Hence $Fix_{\mathcal{G}}(g) = \{z \in X : d(x,z) = 0$ *for each* $d \in \mathcal{G}\}$. Assertion 2 is proved.

If $\{x_n \in Fix_{\mathcal{G}}(g) : n \in \mathbb{N}\}$, then, by virtue of Assertion 2, we have $d(x_n, x_m) = 0$ for all $n \in \mathbb{N}$ and each $d \in \mathcal{G}$. Hence, $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence and has an accumulation point in X. Since the pseudometrics $d \in \mathcal{G}$ are continuous, the set $Fix_{\mathcal{G}}(g)$ is closed in X. Assertion 3 is proved □

Some special cases of Theorem 4.4 were examined in [3, 12, 13, 17, 20]. We mention, that Assertions 2 and 3 of Theorem 4.4 are not true for set-valued mappings.

Example 4.10. Let $\mathcal{G}$ be a sequential complete gauge structure on a space X with an infinite closed subset F. Consider the set-valued mapping $g : X \longrightarrow X$, where $g(x) = F$ for each $x \in X$. The mapping g is upper semicontinuous (and lower semicontinuous too). The mapping g is compact-valued if and only if the set F is compact. If $x, y \in X$ and $x' \in g(x)$, then we put $c(g, x, y, x') = x'$. Then g is a $\mathcal{G}$-contractive mapping with $k_d = 0$ for any $d \in \mathcal{G}$. We have $Fix(g) = Fix_{\mathcal{G}} = F$.

Assume now that $\mathcal{G}$ is a pseudo-gauge structure on a space X. Let $\Phi = \cup\{Z(x, \mathcal{G}) : x \in F\}$. Obviously, $F \subseteq \Phi$. For some $\mathcal{G}$ we have $F \neq \Phi$. We have $Fix(g) = F$ and $Fix_{\mathcal{G}} = \Phi$.

A space X is called a *fixed point space* if for each continuous mapping we have $g(y) = y$ for some $y \in X$ (see [12, 13, 16]).

Any convex compact subset of a topological liner space is a fixed point space [8, 9, 16, 20].

Corollary 4.2. *Let $\mathcal{G}$ be a sequentially quasi-complete pseudo-gauge structure on a space X and $g : X \longrightarrow X$ be a $\mathcal{G}$-contractive mapping. If for each $x \in X$ the subspace $Z(x, \rho)$ is a fixed point space, then $Fix(g) = \{x \in X : g(x) = x\}$ is a non-empty set.*

Example 4.11. Let $\mathcal{G}_1$ be a sequential complete gauge structure on a space Y and Z be a compact fixed-point space. Consider the $X = Y \times \mathbb{Z}$. For each $\rho \in \mathcal{G}_1$ on X we consider the continuous pseudometric $d((x,z),(y,z')) = \rho(x,y)$ for all points $(x,z),(y,z') \in X$. Then $\mathcal{G} = \{d_\rho : \rho \in \mathcal{G}_1\}$ is a perfect pseudo-gauge structure on X.

Assume that $0 < k < 1$ and $g : X \longrightarrow X$ is a mapping such that $d(g(x,z), g(y,z')) \leq k \cdot d((x,z),(y,z'))$ for all points $(x,z),(y,z') \in X$. Then there exists a unique point $(x_0, z_0) \in X$ such that $Fix_{\mathcal{G}}(g) = \{(x,z) \in X : d((x,z),(x_0,z_0) = 0$ *for each* $d \in \mathcal{G}\} = \{x_0\} \times Z$. Then $f : Z \longrightarrow Z$, where $f(z) = z'$ if and only if $g(x_0, z) = g(x_0, z')$, is a continuous mapping of Z into Z. Since Z is a fixed-point space, there exists $z_1 \in Z$ such that $z_1 \in Fix(f)$. Then $(x_0, z_1) \in Fix(g)$ and $Fix(g) \neq \emptyset$.

Theorem 4.5. *Let E be a Banach metric scale and $\|x+y\| = \|x\| + \|y\|$ provided $0 \leq x$ and $0 \leq y$, $\mathcal{G}$ be a sequentially quasi-complete pseudo-gauge structure on a space X, $\{\varphi_d : X \longrightarrow E : d \in \mathcal{G}\}$ be functions, $0 \leq \varphi_d(x)$ for all $x \in X$ and $d \in \mathcal{G}$, and $g : X \longrightarrow X$ be a set-valued mapping with a closed graph $Gr(g) = \cup\{\{x\} \times g(x) : x \in X\}$ in $X \times X$. Assume that for each $x \in X$ there exists $s(x) \in g(x)$ such that $d(x, s(x)) \leq \varphi_d(x) - \varphi_d(s(x))$ for each $d \in \mathcal{G}$. Then:*

1. $Fix_{\mathcal{G}}(g)$ is a non-empty set of the space X.

2. If $x_0 \in X$ and $x_{n+1} = s(x_n)$, then $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space X and there exist two accumulation points b, c of $\{x_n : n \in \mathbb{N}\}$ such that $c \in g(b)$ and $b \in Fix_{\mathcal{G}}(g)$.

Proof. Note that the mapping g is called the Caristi operator on X (see [7]; [20], p. 75).

Fix $x_0 \in X$ and $x_{n+1} = s(x_n)$ for each $n \in \mathbb{N}$. Denote by $A(x_0)$ the set of all accumulation points of the sequence $\{x_n : n \in \mathbb{N}\}$ and $L(x_0) = \{x \in X : lim_{n\to\infty} d(x, x_n) = 0\}$. Obviously, $A(x_0) \subseteq L(x_0)$.

Fix $d \in \mathcal{G}$. Obviously, that $\Sigma\{d(x_i, x_{i+1} : 0 \leq i \leq n\} \leq f_d(x_0) - f_d(x_{n+1} \leq f(x_0)$ for each $n \in \mathbb{N}$. Hence the series $\Sigma\{d(x_i, x_{i+1} : 0 \leq i < \infty\}$ is convergent in E and $\Sigma\{d(x_i, x_{i+1} : 0 \leq i < \infty\} = \vee\{\Sigma\{d(x_i, x_{i+1} : 0 \leq i \leq n\} : n \in \mathbb{N}\} \leq f_d(x_0)$. In particular, the series $\Sigma\{\{d(x_i, x_{i+1})\| : 0 \leq i < \infty\}$ is convergent in E and $\Sigma\{\|d(x_i, x_{i+1})\| : 0 \leq i < \infty\} \leq \|f(x_0)\|$. Hence $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space $(X, \mathcal{G})$. Let $P = \{(x_n, x_{n+1} : n \in \mathbb{N}\} \subseteq X \times X$. By construction, $P \subseteq Gr(g)$.

Since $(X, \mathcal{G})$ is a sequentially complete pseudo-gauge space, there exists $b \in X$ such that b is an accumulation point of the sequence $\{x_n : n \in \mathbb{N}\}$. Moreover, $A(x_0)$ and $L(x_0)$ are non-empty compact subsets of the space X and $b \in A(x_0) \subseteq L(x_0)$.

Claim 1. Let V be an open subset of X and $A(x_0) \subseteq V$. Then the set $\{n : x_n \notin V\}$ is finite.

Follows from the following fact: any infinite subsequence of the sequence $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of the space $(X, \mathcal{G})$ and has accumulation points in $A(x_0)$.

Claim 2. $(A(x_0) \times A(x_0)) \cap cl_{X \times X} P \neq \emptyset$.

Assume that $(A(x_0) \times A(x_0)) \cap cl_{X \times X} P = \emptyset$. Since the set $A(x_0)$ is compact, then there exists an open subset V of X such that $A(x_0) \subseteq V$ and $(V \times V) \cap P = \emptyset$. By virtue of Claim 1, there exists $k \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq k$. Then $(x_n, x_{n+1}) \in (V \times V) \cap P$ for all $n \geq k$, a contradiction. Claim 2 is proved.

Without loss of generality, we can suppose that $(b, c) \in (A(x_0) \times A(x_0)) \cap cl_{X \times X} P$ for some $c \in A(x_0)$. Since the set $Gr(g)$ is closed in $X \times X$ and $P \subseteq Gr(g)$, we have $c \in g(b)$. By construction, $lim_{n \to \infty} d(b, x_n) = 0$ and $lim_{n \to \infty} d(c, x_n) = 0$ for each $d \in \mathcal{G}$. Hence $d(b, c) = 0$ for each $d \in \mathcal{G}$ and $b \in Fix_{\mathcal{G}}(g)$. □

Theorem 4.5 is true for regular Banach metric scales: every increasing sequence which is bounded from above is convergent (see [20], p. 80).

In similar way the generalized contractions of types of Ćirić - Reich - Rus ([20], p. 28), Krasnoselelskii - Zabrejko ([20], p. 29), Zamfirescu ([20], p. 29), Rus - Kasahara - Hicks - Rhoades ([20], p. 35), Niemytzky - Edelstein ([20], p. 38), Berinde - Choban [4, 5] are extended for set-valued mappings of pseudo-gauge spaces.

5. Spaces with finite pseudo-gauge structures

Let E be a Banach metric scale and m be a natural number. Consider the Banach metric scale E^m in which for all $(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m) \in E^m$ we have $\|(x_1, x_2, ..., x_m)\| = \|x_1\| + \|x_2\| + ... + \|x_m\|$ and $(x_1, x_2, ..., x_m) \leq (y_1, y_2, ..., y_m)$ if and only if $x_1 \leq y_1$, $x_2 \leq y_2$, ...,$x_1 \leq y_m$). The Banach metric scale E^m is regular if and only if E is a regular Banach metric scale.

Let $\mathcal{G} = \{d_1, d_2, ..., d_m\}$ be a finite pseudo-gauge structure on a space X. Then $d((x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)) = (d_1(x_1, y_1), d_2(x_2, y_2), ..., d_m(x_m, y_m)$ is a continuous E^m-pseudometric on the space d.

Hence the results from [10] are true for spaces with finite pseudo-gauge structures.

The matrix contractions may be applied to the finite pseudo-gauge spaces. Denote by M^*_m the family of all $m \times m$ matrices $S = (s_{ij})^m_m$ with the properties:

- $s_{ij} \in \mathbb{R}$ and $0 \leq s_{ij}$ for all $i, j \leq m$;
- $lim_{k \to \infty} S^k = \theta_m$, where θ_m is the zero $m \times m$ matrix.

If $S \in M^*_m$, then it is said to be that the matrix S is convergent to zero ([20], Section 6.0.3).

By virtue of Theorem 6.0.1 from [20], for any non-negative scalar $m \times m$ matrix S the following assertions are equivalent:

(i) S is a convergent to zero matrix;

(ii) $det(\varepsilon_m - S) \neq 0$ and $(\varepsilon_m - S)^{-1}$ has non-negative elements, where ε_m is the unit $m \times m$ matrix;

(iii) $det(\varepsilon_m - S) \neq 0$ and $(\varepsilon_m - S)^{-1} = \varepsilon_m + S + S^2 + ... + S^n + ...$

The matrix approach and reduction principles permit to extend the distinct fixed point theorems, proved for the $\mathbb{R}^m$-metrics (see [20, 19]).

A mapping $\varphi : X \longrightarrow X$ is a scalar S-contraction if there exists $S \in M_m^*$ such that $d(\varphi(x), \varphi(y)) \leq Sd(x, y)$ for all $x, y \in X$.

Theorem 5.6. *Let $(X, \mathcal{G})$ be a sequentially complete pseudo-gauge space and $\varphi : X \longrightarrow X$ be a scalar S-contraction. Then:*

1. There exists $x^* \in X$ such that $d(x^*, \varphi(x^*)) = 0$.

2. If $\{x_n : n \in \mathbb{N}\}$ and $lim_{n\to\infty} d(x_n, \varphi(x_n)) = 0$, then $lim_{n\to\infty} d(x_n, x^*) = 0$.

3. If $d(x, y) \neq 0$ provided $\varphi(x) \neq \varphi(y)$, then $Fix(\varphi) \neq \emptyset$.

4. Assume that $x = y$ if $d(\varphi(x), \varphi(y)) = 0$. Then $Fix(\varphi)$ is a singleton set.

Proof. The proofs of the assertions 1 and 2 are similar as in the case $E = \mathbb{R}$ (see [20], Theorems 6.1.1 and 6.1.2). The assertion 3 follows from the assertion 1 and Theorem 3.1. The assertion 4 follows from the assertion 1 and Theorem 3.2. □

6. Pseudometrics in ordered Banach algebras

In the present section the notion of the scalar S-contraction is extended for matrices with elements from a given ordered commutative Banach algebra (we use the notions from [2, 6, 15] about normed rings and ordered algebras).

A *Banach metric r-scale* is a non singleton partially ordered Banach space E such that:

- E is a Banach metric scale;

- E is a commutative ring with unity 1 such that $0 < 1$, $\parallel 1 \parallel = 1$, $\parallel xy \parallel \leq \parallel x \parallel \cdot \parallel y \parallel$ for all x, y from E, $xy \geq 0$ provided $x \geq 0$ and $y \geq 0$, $uv \leq uz$ provided $u \geq 0$ and $v \leq z$.

Fix a Banach metric r-scale E and a natural number m.

Let $M_m(E)$ be the family of all $m \times m$ matrices $S = (s_{ij})_m^m$ with the properties:

- $s_{ij} \in E$ and $0 \leq s_{ij}$ for all $i, j \leq m$;

- $\parallel detS \parallel + \parallel detS^2 \parallel + ... + \parallel detS^n \parallel + ... < \infty$;

- there exists the sum $|S| + |S^2| + ... + |S^n| + ...$, where $|A| = (\parallel a_{ij} \parallel)_m^m$ for each matrix $A = (a_{ij})_m^m$.

If $S \in M_m(E)$, then $lim_{k\to\infty} S^k = \theta_m$, where θ_m is the zero $m \times m$ matrix.

Let $\mathcal{G} = \{d_1, d_2, ..., d_m\}$ be a finite pseudo-gauge structure on a space X. Then $d((x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)) = (d_1(x_1, y_1), d_2(x_2, y_2), ..., d_m(x_m, y_m)$ is a continuous E^m-pseudometric on the space d.

The matrix approach and reduction principles permit to extend the distinct fixed point theorems, proved for the $\mathbb{R}^m$-metrics (see [20]).

A mapping $\varphi : X \longrightarrow X$ is an SE-contraction if there exists $S \in M_m(E)$ such that $d(\varphi(x), \varphi(y)) \leq Sd(x, y)$ for all $x, y \in X$.

Theorem 6.7. *Let $(X, \mathcal{G})$ be a sequentially complete pseudo-gauge space and $\varphi : X \longrightarrow X$ be an SE-contraction. Then:*

1.There exists $x^* \in X$ such that $d(x^*, \varphi(x^*)) = 0$.

2. If $\{x_n : n \in \mathbb{N}\}$ and $lim_{n\to\infty} d(x_n, \varphi(x_n)) = 0$, then $lim_{n\to\infty} d(x_n, x^*) = 0$.

3. If $d(x, y) \neq 0$ provided $\varphi(x) \neq \varphi(y)$, then $Fix(\varphi) \neq \emptyset$.

4. Assume that $x = y$ if $d(\varphi(x), \varphi(y)) = 0$. Then $Fix(\varphi)$ is a singleton set.

Proof. Let ε_m be the unit matrix. Then $(\varepsilon_m + S)^{-1} = \varepsilon_m + S + S^2 + ... + S^n + ...$ for any $S \in M_m(E)$. The proofs of the assertions 1 and 2 are similar as in the case $E = \mathbb{R}$ (see [20], Theorems 6.1.1 and 6.1.2). The assertion 3 follows from the assertion 1 and Theorem 3.1. The assertion 4 follows from the assertion 1 and Theorem 3.2. □

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