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Common fixed points of two finite families of nonexpansive mappings by iterations

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ABSTRACT. We study a Mann type iterative scheme for two finite families of nonexpansive mappings and establish $\triangle-$ convergence and strong convergence theorems. The obtained results are applicable in uniformly convex Banach spaces (linear domain) and CAT (0) spaces (nonlinear domain) simultaneously.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space. A mapping $W : X^2 \times I \to X$ is a convex structure [16] in X if

$$d\left(u, W\left(x, y, \alpha\right)\right) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$$

for all $u, x, y \in X$ and $\alpha \in I = [0, 1]$. A metric space X together with a convex structure W is known as convex metric space. A nonempty subset C of a convex metric space X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in I$.

If $X = \mathbb{R}$ (the set of reals), $W(x, y, \alpha) = \alpha x + (1 - \alpha)y$ and define $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for $x, y \in \mathbb{R}$, then X is a convex metric space. It is remarkable that every $CAT(0)$ space is a convex metric space (see [9] and references therein). To know more about $CAT\ (0)$ spaces and convex metric spaces, we refer the reader to [1, 6].

A convex metric space X is a (i) strictly convex if for any $x, y \in X$ and $\alpha \in I$, there exists a unique element $z \in X$ such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(z, y) = \alpha d(x, y)$ and (ii) uniformly convex [15] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d\left(W(x, y, \frac{1}{2}), u\right) \leq (1 - \delta)r$, whenever $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

The Ishikawa iterative scheme is a two-step iterative scheme and has been used to approximate common fixed points of mappings by a number of researchers (see, for example, [3, 4, 8, 7, 13]).

Denote by $J = \{1, 2, 3, ..., N\}$, the indexing set. To reduce computational cost of a two-step iterative scheme for two finite families $\{S_n : n \in J\}$ and $\{T_n : n \in J\}$ of nonexpansive mappings on a convex subset C of a Banach space, Khan et al. [10] introduced the following one-step implicit iterative scheme (see also [14])

(1.1) $$x_0 \in K,\ x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n S_n x_n,$$

where $S_n = S_{n(\text{mod} N)}$ and $T_n = T_{n(\text{mod} N)}, 0 \leq \alpha_n,\ \beta_n, \gamma_n \leq 1$ and satisfy $\alpha_n + \beta_n + \gamma_n = 1$. Explicit iterative scheme has less computational cost and is simpler than an implicit iterative scheme, therefore we suggest and analyze the following iterative scheme

(1.2) $$x_{n+1} = W\left(T_n x_n, W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)$$

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for two finite families $\{S_n : n \in J\}$ and $\{T_n : n \in J\}$ of nonexpansive mappings such that $S_n = S_{n(\text{mod}N)}, T_n = T_{n(\text{mod}N)}, 0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and $\alpha_n + \beta_n < 1$.

(1.2) is well-defined if X is strictly convex and is equivalent to an analogue of (1.1) in the Banach space setting.

Let $\{x_n\}$ be a bounded sequence in a metric space X. We define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$ for all $x \in X$. The asymptotic radius of $\{x_n\}$ with respect to $C \subseteq X$ is defined as

$$r(\{x_n\}) = \inf_{x\in C} r\left(x, \{x_n\}\right).$$

A point $y \in C$ is called the asymptotic center of $\{x_n\}$ with respect to $C \subseteq X$ if $r(y, \{x_n\}$ $\leq r(x, \{x_n\})$ for all $x \in C$. The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [4] that a bounded sequence has a unique asymptotic center with respect to closed convex subset in a complete and uniformly convex metric space.

A sequence $\{x_n\}$ in (X, d) is Fejér monotone with respect to a subset C of X if $d(x_{n+1}, x)$ $\leq d(x_n, x)$ for all $x \in C$. A selfmapping T on a subset C of X is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T if $Tx = x$.

A sequence $\{x_n\}$ in X (i) is an approximate common fixed point sequence for two finite families $\{S_n, n \in J\}$ and $\{T_n : n \in J\}$ of mappings if $\lim_{n\to\infty} d\left(x_n, S_l x_n\right) = 0 = \lim_{n\to\infty} d\left(x_n, T_l x_n\right)$, for all $l \in J$ (ii) $\triangle-$converges to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$[11]. In this case, we write x as $\triangle-$limit of $\{x_n\},\ i.e.,\ \triangle - \lim_n x_n = x$.

It has been shown that $\triangle-$convergence coincides with weak convergence in Banach spaces with Opial's property[12]. Moreover, the two concepts share many useful properties. Inspired and motivated by the work of Khan et al. [10], we approximate common fixed points of two finite families of nonexpansive mappings by the iterative scheme (1.2) in a convex metric space.

For the development of our main results, some key results are listed below.

Lemma 1.1. ([4]) *Let C be a nonempty closed convex subset of a uniformly convex metric space and $\{x_n\}$ a bounded sequence in C such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in C such that* $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$ *(a real number), then* $\lim_{m\to\infty} y_m = y$.

Lemma 1.2. ([5]) *Let X be a uniformly convex metric spac with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{w_n\}$ and $\{z_n\}$ are sequences in X such that* $\limsup_{n\longrightarrow\infty} d(w_n, x) \leq r,\ \limsup_{n\longrightarrow\infty} d(z_n, x) \leq r$ *and* $\lim_{n\longrightarrow\infty} d(W(w_n, z_n, a_n), x) = r$ *for some* $r \geq 0$, *then* $\lim_{n\to\infty} d(w_n, z_n) = 0$.

Lemma 1.3. ([2]) *Let C be a nonempty closed subset of a complete metric space X and $\{x_n\}$ be Fejér monotone sequence with respect to C. Then $\{x_n\}$ converges to some $x \in C$ if and only if* $\lim_{n\to\infty} d(x_n, C) = 0$,*where* $d(x_n, C) = \inf_{y\in C} d(x_n, y)$.

From now onwards, for finite families $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$ of nonexpansive mappings on C, we set $F = \cap_{i\in J} \left(F(T_i) \cap F(S_i)\right) \neq \emptyset$.

2. Main results

First of all, we prove a pair of lemmas which play key role in the proof of our main theorems.

Lemma 2.4. *Let C be a closed and convex subset of a convex metric space X and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive mappings on C such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (1.2), we have that*
(i) $\{x_n\}$ is Fejér monotone with respect to F
(ii) $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$ and
(iii) $\lim_{n\to\infty} d(x_n, F)$ exists.

Proof. For $p \in F$, it follows from (1.2) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_n x_n, W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_n x_n, p) + (1-\alpha_n) d\left(W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d(T_n x_n, p) + \beta_n d(S_n x_n, p) + (1-\alpha_n-\beta_n) d(x_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n) d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

This proves that (i) *$\{x_n\}$ is Fejér monotone with respect to F and* (ii) *$\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.* From $\inf_{p\in F} d(x_{n+1}, p) \leq \inf_{p\in F} d(x_n, p)$, it follows that (iii) $\lim_{n\to\infty} d(x_n, F)$ exists. □

Lemma 2.5. *Let C be a closed and convex subset of a uniformly convex metric space X.Then the sequence $\{x_n\}$ in (1.2) is an approximate common fixed point sequence for the families $\{S_n, n \in J\}$ and $\{T_n : n \in J\}$ of nonexpansive mappings on C.*

Proof. To establish the result, we have to show that

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n), \text{ for all } l \in J.$$

From Lemma 2.4, we have that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.
Assume that $\lim_{n\to\infty} d(x_n, p) = c$. The result is trivial if $c = 0$. If $c > 0$, then $\lim_{n\to\infty} d(x_n, p) = c$ can be written as

$$\lim_{n\to\infty} d\left(W\left(T_n x_n, W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c \tag{2.3}$$

Since T_n is nonexpansive, we have $d(T_n x_n, p) \leq d(x_n, p)$ for each $p \in F$. Taking $\limsup$ on both sides, we obtain $\limsup_{n\to\infty} d(T_n x_n, p) \leq c$.

Note that

$$\begin{aligned} d\left(W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(S_n x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} d\left(W\left(S_n x_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{2.4}$$

Taking $x=p, r=c, a_n=\alpha_n, w_n=T_nx_n, z_n=W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)$ in Lemma 1.2 and using (2.3) and (2.4), we get

$$\lim_{n\to\infty} d\left(T_nx_n, W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right)=0. \tag{2.5}$$

Observe that

$$\begin{aligned} d(x_{n+1}, T_nx_n) &= d\left(W\left(T_nx_n, W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), T_nx_n\right) \\ &\leq (1-\alpha_n)d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), T_nx_n\right) \\ &\leq (1-a)d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), T_nx_n\right). \end{aligned}$$

Taking $\limsup$ on both sides in the above inequality and using (2.5), we have

$$\lim_{n\to\infty} d(x_{n+1}, T_nx_n)=0. \tag{2.6}$$

Moreover by triangular inequality,

$$\begin{aligned} d(x_{n+1}, p) &= d\left(x_{n+1}, T_nx_n\right)+d\left(T_nx_n, W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\quad +d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \end{aligned}$$

Taking $\liminf$ on both sides of the above estimate and then utilizing (2.5)-(2.6), we have

$$c \leq \liminf_{n\to\infty} d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{2.7}$$

Combining (2.4) and (2.7), we get

$$\lim_{n\to\infty} d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)=c. \tag{2.8}$$

By Lemma 1.2 (with $x=p, r=c, a_n=\frac{\beta_n}{1-\alpha_n}, w_n=S_nx_n, z_n=x_n$) and (2.8), we get

$$\lim_{n\to\infty} d(x_n, S_nx_n)=0. \tag{2.9}$$

Note that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d\left(W\left(T_nx_n, W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_n\right) \\ &\leq \alpha_n d(T_nx_n, x_n)+(1-\alpha_n)\, d\left(W\left(S_nx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\ &\leq \alpha_n d\left(T_nx_n, x_n\right)+\beta_n d\left(S_nx_n, x_n\right) \\ &\leq \alpha_n\left\{d\left(x_{n+1}, T_nx_n\right)+d\left(x_{n+1}, x_n\right)\right\}+\beta_n d\left(S_nx_n, x_n\right). \end{aligned}$$

Re-arranging the terms in the above inequality, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{\alpha_n}{1-\alpha_n} d\left(x_{n+1}, T_nx_n\right)+\beta_n d\left(S_nx_n, x_n\right) \\ &\leq \frac{b}{1-b} d\left(x_{n+1}, T_nx_n\right)+bd\left(S_nx_n, x_n\right). \end{aligned}$$

Taking lim sup on both sides in the above inequality and then using (2.6) and (2.9), we have

(2.10) $$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

For each $l < N$, the inequality

$$d(x_n, x_{n+l}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + \ d(x_{n+l-1}, x_{n+l})$$

and (2.10) provide that

(2.11) $$\lim_{n\to\infty} d(x_n, x_{n+l}) = 0 \quad \text{for each } l < N.$$

Since $d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n)$, therefore it follows that

(2.12) $$\lim_{n\to\infty} d(x_n, T_n x_n) = 0.$$

Since

$$\begin{aligned} d(x_n, S_{n+l}x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, S_{n+l}x_{n+l}) + d(S_{n+l}x_{n+l}, S_{n+l}x_n) \\ &\leq 2d(x_n, x_{n+l}) + d(x_{n+l}, S_{n+l}x_{n+l}), \end{aligned}$$

therefore by lim sup on both sides in the above inequality and then using (2.9) and (2.11), we get that

$$\lim_{n\to\infty} d(x_n, S_{n+l}x_n) = 0 \quad \text{for each } l \in J.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0 \quad \text{for each } l \in J.$$

Since for each $l \in J$, the sequence $\{d(x_n, S_l x_n)\}$ is a subsequence of $\cup_{l=1}^{N}\{d(x_n, S_{n+l}x_n)\}$ and $\lim_{n\to\infty} d(x_n, S_{n+l}x_n) = 0$ for each $l \in J$, therefore

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 = \lim_{n\to\infty} d(x_n, T_l x_n) \text{ for each } l \in J.$$

□

The following is our $\triangle-$convergence result through the iterative scheme(1.2).

Theorem 2.1. *Let C be a closed and convex subset of a uniformly convex metric space X and let $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$ be two finite families of nonexpansive mappings of C such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ in (1.2), $\triangle-$converges to an element of F.*

Proof. It follows from Lemma 2.4 that $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 2.5, we have $\lim_{n\to\infty} d(u_n, T_l u_n) = 0 = \lim_{n\to\infty} d(u_n, S_l u_n)$ for each $l \in J$. We claim that u is the common fixed point of $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$. Now, we define a sequence $\{z_m\}$ in C by $z_m = T_m u$, where $T_m = T_{m(\text{mod}N)}$.
Observe that

$$\begin{aligned} d(z_m, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1}u_n) + ... + d(Tu_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Therefore, we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.1 that $T_m u = u$. Hence u is the common fixed point of $\{T_n : n \in J\}$. Similarly, we can show that u is the common fixed point of $\{S_i : i \in J\}$. Hence $u \in F$. Suppose $x \neq u$. Since $\lim_{n\to\infty} d(x_n, u)$ exists (by Lemma 2.4), so the uniqueness of asymptotic centre gives

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, u) \\
&= \limsup_{n\to\infty} d(u_n, u),
\end{aligned}$$

which is a contradiction. Hence $x = u$. Therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$, $\triangle-$converges to a common fixed point of $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$. □

We give a necessary and sufficient condition for strong convergence of the iterative scheme (1.2) in the following result.

Theorem 2.2. *Let C be a closed and convex subset of a complete convex metric space X and let $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$ be two finite families of nonexpansive mappings of C such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ in (1.2), converges strongly to $p \in F$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. It follows from Lemma 2.4 that $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n\to\infty} d(x_n, F)$ exists. Moreover, F is a closed subset of X. Hence, the result follows from Lemma 1.3. □

Recall that a mapping $T : C \to C$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Let f be a nondecreasing mapping on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$. Let $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$ be finite families of nonexpansive mappings on C with $F \neq \emptyset$. Then the two families are said to satisfy condition (A) if

$$d(x, Tx) \geq f(d(x, F)) \text{ or } d(x, Sx) \geq f(d(x, F)) \text{ for all } x \in C,$$

holds for at least one $T \in \{T_n : n \in J\}$ or one $S \in \{S_n : n \in J\}$.

The statement in the following remark follows easily from Lemma 2.5.

Remark 2.1. If C is a closed and convex subset of a complete and uniformly convex metric space X and $\{T_n : n \in J\}$ and $\{S_n : n \in J\}$ are two finite families of nonexpansive mappings on C such that $F \neq \emptyset$. Suppose that at least one $T \in \{T_n : n \in J\}$ or one $S \in \{S_n : n \in J\}$ satisfies condition (A) (or is semi-compact), then the sequence $\{x_n\}$ in (1.2), converges strongly to an element of F.

Remark 2.2. Our results hold in Banach spaces and $CAT(0)$ spaces simultaneously.

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