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On a generalized coupled fixed point theorem in $\mathcal{C}[0,1]$ and its application to a class of coupled systems of functional-integral equations

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ABSTRACT. In this paper, we present a result about the existence of a generalized coupled fixed point in the space $\mathcal{C}[0,1]$. Moreover, as an application of the result, we study the problem of existence and uniqueness of solution in $\mathcal{C}[0,1]$ for a general system of nonlinear functional-integral equations with maximum.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is a very important tool in nonlinear analysis. Its significance lies in the vast applicability in a great number of branches of mathematics and other sciences. The Banach contraction mapping principle is the pivotal result in this theory. Generalizations of the above principle has been an important branch of research. Particularly, one of these generalization uses the comparison functions. These functions are defined as functions $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying that φ is increasing and $\varphi^n(t) \to 0$ for $t > 0$, where φ^n denotes the n-iteration of φ.
The above mentioned result is the following [2, 5].

Theorem 1.1. *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \phi(d(x, y)), \quad \textit{for any } x, y \in X,$$

where ϕ is a comparison function. Then T has a unique fixed point.

In this paper, we will work with the space $\mathcal{C}[0,1]$ which denotes the set of all real functions defined and continuous on the interval $[0,1]$. In $\mathcal{C}[0,1]$, we consider the usual distance defined by

$$d(x, y) = \sup\{|x(t) - y(t)|\colon t \in [0,1]\} \quad \text{for any } x, y \in \mathcal{C}[0,1].$$

For $x \in \mathcal{C}[0,1]$, we define the function Hx on $[0,1]$ by

$$(Hx)(t) = \max_{0 \leq \tau \leq t} |x(\tau)| \quad \text{for } t \in I.$$

In [3] it is proved the following result.

Lemma 1.1. *Suppose that $x, y \in \mathcal{C}[0,1]$ then*

(i) $Hx \in \mathcal{C}[0,1]$.
(ii) $d(Hx, Hy) \leq d(x, y)$.

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The aim purpose of this paper is to present a result about of the existence and uniqueness of a generalized coupled fixed point in $\mathcal{C}[0,1]$ and, as application, to study the existence and uniqueness of solutions in $\mathcal{C}[0,1]$ of the following coupled system of functional-integral equations with maximum

$$\begin{cases} x(t) = f(t, x(t), y(t), \int_0^t g(s, x(s), y(s))\mathrm{d}s) \\ y(t) = f(t, (Hx)(t), (Hy)(t), \int_0^t g(s, (Hx)(s), (Hy)(s))\mathrm{d}s) \end{cases} \tag{1.1}$$

with $t \in [0,1]$.
The main ingredient used in the paper is Theorem 1.1.

2. Generalized coupled fixed point theorem

We start this section with the following definition.

Definition 2.1. An element $(x,y) \in \mathcal{C}[0,1] \times \mathcal{C}[0,1]$ is called a generalized coupled fixed point of a mapping $G\colon \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ if $G(x,y) = x$ and $G(Hx, Hy) = y$.

The next theorem is the main result of the paper and it gives us a sufficient condition for the existence and uniqueness of a generalized coupled fixed point.

Theorem 2.2. *Suppose that* $G\colon \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ *is a mapping satisfying*

$$d(G(x,y), G(u,v)) \le \phi(\max(d(x,u), d(y,v)))\,, \quad \textit{for any } x,y,u,v \in \mathcal{C}[0,1]\,, \tag{2.2}$$

where ϕ *is a comparison function.*
Then G has a unique generalized coupled fixed point.

Proof. Consider the cartesian product $\mathcal{C}[0,1] \times \mathcal{C}[0,1]$ endowed with the distance

$$\tilde{d}((x_1,y_1),(x_2,y_2)) = \max(d(x_1,x_2), d(y_1,y_2))\,.$$

It is easily seen that $(\mathcal{C}[0,1] \times \mathcal{C}[0,1], \tilde{d})$ is a complete metric space.
Now, we consider the mapping $\tilde{G}\colon \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathcal{C}[0,1] \times \mathcal{C}[0,1]$ defined by

$$\tilde{G}(x,y) = (G(x,y), G(Hx,Hy)) \quad \text{for any } x,y \in \mathcal{C}[0,1]\,.$$

Now, we check that $\tilde{G}$ satisfies assumptions of Theorem 1.1.
In fact, taking into account (2.2), for any $x,y,u,v \in \mathcal{C}[0,1]$ we have

$$\begin{aligned} \tilde{d}(\tilde{G}(x,y), \tilde{G}(u,v)) &= \tilde{d}(G(x,y), G(Hx,Hy), G(u,v), G(Hu,Hv)) \\ &= \max[d(G(x,y),G(u,v)), d(G(Hx,Hy), G(Hu,Hv))] \\ &\le \max[\phi(\max(d(x,u),d(y,v))), \phi(\max(d(Hx,Hu), d(Hy,Hv)))]. \end{aligned}$$

By Lemma 1.1, we have

$$\begin{aligned} d(Hx,Hu) &\le d(x,y) \\ d(Hy,Hv) &\le d(y,v)\,. \end{aligned}$$

Consequently, since ϕ is increasing, it follows that

$$\begin{aligned} \tilde{d}(\tilde{G}(x,y), \tilde{G}(u,v)) &\le \phi(\max(d(x,u), d(y,v))) \\ &= \phi(\tilde{d}((x,y),(u,v)))\,. \end{aligned}$$

Therefore, by Theorem 1.1, there exists a unique $(x_0, y_0) \in \mathcal{C}[0,1] \times \mathcal{C}[0,1]$ such that

$$\tilde{G}(x_0, y_0) = (x_0, y_0)\,.$$

This means, by the definition of $\tilde{G}$, that

$$\begin{aligned} G(x_0, y_0) &= x_0 \\ G(Hx_0, Hy_0) &= y_0\,, \end{aligned}$$

or, equivalently, that (x_0, y_0) is a generalized coupled fixed point of G.
This complete the proof. □

3. Application and example

Now, we present an application of Theorem 2.2, to the study of the existence and uniqueness of solutions in $\mathcal{C}[0,1]$ of Problem (1.1).
Problem (1.1) will be studied under the following assumptions:

H1 $f \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

H2 f satisfies

$$|f(t,x,y,z) - f(t,u,v,w) \leq \phi(\max(|x-u, |y-v|, |z-w|))$$

for any $t \in [0,1]$ and $x, y, z, u, v, w \in \mathbb{R}$, where ϕ is a comparison function.

H3 g satisfies

$$|g(t,x,y) - g(t,u,v) \leq \max(|x-u, |y-v|)$$

for any $t \in [0,1]$ and $x, y, u, v \in \mathbb{R}$.

Theorem 3.3. *Under assumptions H1-H3, Problem* (1.1) *has a unique solution in* $\mathcal{C}[0,1] \times \mathcal{C}[0,1]$.

Proof. Consider the mapping G defined on $\mathcal{C}[0,1] \times \mathcal{C}[0,1]$ by

$$G(x,y)(t) = f(t, x(t), y(t), \int_0^t g(s, x(s), y(s)\mathrm{d}s)) \quad \text{for } x, y \in \mathcal{C}[0,1] \text{ and } t \in [0,1]\,.$$

In virtue of H1, it is clear that G applies $\mathcal{C}[0,1] \times \mathcal{C}[0,1]$ into $\mathcal{C}[0,1]$.
Now, we check that G satisfies assumptions of Theorem 2.2.
In fact, taking into account H1-H3, for $x, y, u, v \in \mathcal{C}[0,1]$, we get

$$\begin{aligned} &d(G(x,y), G(u,v)) = \sup\{|G(x,y)(t) - G(u,v)(t)| \colon t \in [0,1]\} \\ &= \sup\{|f(t, x(t), y(t), \int_0^t g(s, x(s), y(s)\mathrm{d}s)) - f(t, u(t), v(t), \int_0^t g(s, u(s), v(s)\mathrm{d}s))|\} \\ &= \sup\{\phi(\max(|x(t) - u(t)|, |y(t) - v(t)|, |\int_0^t g(s, x(s), y(s)\mathrm{d}s)) - \int_0^t g(s, u(s), v(s)\mathrm{d}s))|\}\,. \end{aligned}$$

Since ϕ is increasing, it follows that

$$\begin{aligned} &d(G(x,y),G(u,v)) = \sup\{|G(x,y)(t)-G(u,v)(t)|\colon t\in[0,1]\}\\ &\le \sup\{\phi(\max(d(x,u),d(y,v),\int_0^t |g(s,x(s),y(s))-g(s,u(s),v(s))\mathrm{d}s))|\}\\ &\le \sup\{\phi(\max(d(x,u),d(y,v),\int_0^t |\max(|x(s)-u(s)|,|y(s)-v(s)|)\mathrm{d}s))\}\\ &\le \sup\{\phi(\max(d(x,u),d(y,v),\max(d(x,u),d(y,v))\cdot t)\}\\ &\le \phi(\max(d(x,u),d(y,v))\,. \end{aligned}$$

Therefore, G satisfies assumptions of Theorem 2.2 and, consequently, G has a unique generalized coupled fixed point $(x_0,y_0)\in\mathcal{C}[0,1]\times\mathcal{C}[0,1]$. This means that

$$G(x_0,y_0)=x_0$$
$$G(Hx_0,Hy_0)=y_0\,,$$

or equivalently

$$\begin{cases} x_0(t)=f(t,x_0(t),y_0(t),\int_0^t g(s,x_0(s),y_0(s))\mathrm{d}s)\\ y_0(t)=f(t,(Hx_0)(t),(Hy_0)(t),\int_0^t g(s,(Hx_0)(s),(Hy_0)(s))\mathrm{d}s)\,. \end{cases}$$

This complete the proof. □

Before to present an example illustrating our results, we need the following lemmas which will be used later.

Lemma 3.2. *The inverse tangent function satisfies*

$$|\arctan x-\arctan y|\le|\arctan(|x-y|)\quad \textit{for any } x,y\in\mathbb{R}\,.$$

For a proof, wee Example 4.2 of [4] and to take into account the fact that $\arctan x$ is an odd function.
The next lemma is Lemma 2.1 of [1].

Lemma 3.3. *Let $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then these conditions are equivalent:*

(i) $\lim_{n\to\infty}\varphi^n(t)=0$ *for any* $t\ge 0$ *(φ^n denotes the n-iteration of φ).*
(ii) $\varphi(t)<t$ *for any* $t>0$.

Now, we are ready to present our example.

Example 3.1. Consider this coupled system of integral equation with maximum

$$(3.3)\qquad \begin{cases} x(t)=e^t+\frac{1}{3}(\arctan x(t)+\arctan y(t)+\arctan(\int_0^t \alpha(s+x(s)+y(s))\mathrm{d}s))\\ y(t)=e^t+\frac{1}{3}(\arctan(\sup\limits_{0\le\tau\le t}x(\tau))+\arctan(\sup\limits_{0\le\tau\le t}y(\tau))+\\ \qquad\qquad +\arctan(\int_0^t\alpha(s+\sup\limits_{0\le\tau\le s}x(\tau)+\sup\limits_{0\le\tau\le s}y(\tau))\,, \end{cases}$$

with $t \in [0,1]$ and $\alpha > 0$.
Notice that Problem 3.3 is a particular case of Problem 1.1 with

$$f(t,x,y,z) = e^t + \frac{1}{3}(\arctan x + \arctan y + \arctan z) \quad \text{and} \quad g(t,x,y) = \alpha(t+x+y).$$

It is clear that f and g satisfy H1 of Theorem 3.3.
On the other hand, for $t \in [0,1]$ and $x,y,z,u,v,w \in \mathbb{R}$, we have

$$\begin{aligned} |f(t,x,y,z) - f(t,u,v,w)| \le \\ &\le \frac{1}{3}[|\arctan x - \arctan u| + |\arctan y - \arctan v| + |\arctan z - \arctan w|] \\ &\le \frac{1}{3}[\arctan(|x-u|) + \arctan(|y-v|) + \arctan(|z-w|)] \,, \end{aligned}$$

where we have used Lemma 3.2.
Since the inverse tangent function is increasing, from the last inequality if follows that

$$\begin{aligned} |f(t,x,y,z) - f(t,u,v,w)| &\le \frac{1}{3}[3 \cdot \arctan[\max(|x-u|,|y-v|,|z-w|)]] \\ &= \arctan[\max(|x-u|,|y-v|,|z-w|)] \,. \end{aligned}$$

Notice that $\varphi(x) = \arctan x$ for $x \ge 0$ is an increasing function satisfying $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\arctan x < x$ for $x > 0$.
Since φ is continuous, by Lemma 3.3, $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$, and, consequently, φ is a comparison function.
This proves that assumption H2 of Theorem 3.3 is satisfied.
Finally, for $t \in [0,1]$ and $x,y,u,v \in \mathbb{R}$, we have

$$|g(t,x,y) - g(t,u,v)| \le \alpha[|x-u| + |y-v|] \le 2\alpha \max(|x-u|,|y-v|) \,.$$

If $0 < \alpha \le \frac{1}{2}$, we obtain

$$|g(t,x,y) - g(t,u,v)| \le \max(|x-u|,|y-v|) \,,$$

and assumption H3 of Theorem 3.3 is satisfied.
Therefore, by Theorem 3.3, if $0 < \alpha \le \frac{1}{2}$ then our coupled system 3.3 has a unique solution $(x_0,y_0) \in \mathcal{C}[0,1] \times \mathcal{C}[0,1]$.

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