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Some fixed points results on Branciari metric spaces via implicit functions

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ABSTRACT. In this paper, we introduce the notion of α-implicit contractive mapping of integral type in the context of Branciari metric spaces. The results of this paper, generalize and improve several results on the topic in literature. We give an example to illustrate our results.

1. INTRODUCTION AND PRELIMINARIES

Branciari proposed a new distance function, as a generalization of the notion of metric, by substituting the triangle inequality with the quadrilateral inequality. In what follows that we recall the notion of a Branciari metric space.

Definition 1.1. [6] Let X be a nonempty set and let $d : X \times X \longrightarrow [0, \infty)$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y.

$$\begin{array}{ll} (b1) & d(x,y) = 0 \text{ if and only if } x = y \\ (b2) & d(x,y) = d(y,x) \\ (b3) & d(x,y) \leq d(x,u) + d(u,v) + d(v,y). \end{array} \tag{1.1}$$

Then, the map d is called a Branciari metric (or generalized metric) and abbreviated as "BMS" Here, the pair (X, d) is called a Branciari metric space.

Notice that, in the literature, generalized metric space represents several different notions (see e.g. [3, 4, 5, 6, 8, 9, 10, 13, 14, 15, 16, 17, 19, 20, 21, 22, 29, 32]). In this note, we prefer to rename and use it as Branciari metric space to avoid from the confusion.

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of non-negative integers. Analogously, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively.

The concepts of convergence, Cauchy sequence, completeness and continuity on a BMS are defined below.

Definition 1.2.

(1) A sequence $\{x_n\}$ in a BMS (X, d) is BMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in a BMS (X, d) is BMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.
(3) A BMS (X, d) is called complete if every BMS Cauchy sequence in X is BMS convergent.

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(4) A mapping $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

For BMS, Branciari successfully defined an open ball, closed ball and hence a topology. Despite the analogy, the topology of BMS is different than metric space. In particular, (p1) Branciari metric is not necessarily continuous, (p2) BMS is not necessarily Haussdorf (limit is not necessarily unique), (p3)open ball need not to open set, (p4) a convergent sequence in BMS needs not to be Cauchy. Furthermore, the mentioned topologies are incompatible (for more details see e.g. [32]).

Inspired from the examples in [31, 14], we state the following illustrative example.

Example 1.1. Let $X = Y \cup Z$ where $Y = \{0, 2, 3\}$ and $Z = \{\frac{1}{2n+1} : n \in \mathbb{N}\}$. Consider the function $d : X \times X \to [0, \infty)$ in the following way:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } [\{x,y\} \subset Y \text{ or } \{x,y\} \subset Z], \\ y, & \text{if } x \in Y, y \in Z. \end{cases}$$

Notice that $d(y, z) = d(z, y) = z$ whenever $y \in Y$ and $z \in Z$. Furthermore, (X, d) is a complete BMS. Clearly, we have $(P1)$-$(P4)$. Indeed, the sequence $\{\frac{1}{2n+1} : n \in \mathbb{N}\}$ converges to $0, 2$ and 3. There is no $r > 0$ such that $B_r(0) \cap B_r(3) = \emptyset$ and hence it is not Hausdorff. It is clear that the ball $B_{\frac{2}{3}}(\frac{1}{5}) = \{0, \frac{1}{5}, 2, 3\}$ since there is no $r > 0$ such that $B_r(0) \subset B_{\frac{2}{3}}(\frac{1}{3})$, that is, open balls may not be an open set. The function d is not continuous since $\lim_{n\to\infty} d(\frac{1}{2n+1}, \frac{1}{5}) \neq d(0, \frac{1}{5})$ although $\lim_{n\to\infty} \frac{1}{2n+1} = 0$.

Example 1.2. (See [32]) Let $X = \{(0,0)\} \cup ((0,1] \times [0,1])$. Define a function $d : X \times X \to \mathbb{R}_0^+$ by

$$\begin{aligned} d(x,x) &= 0 \\ d((0,0),(s,0)) &= d((s,0),(0,0)) = s \text{ if } s \in (0,1], \\ d((s,0),(p,q)) &= d((p,q),(s,0))) = |s-p| + q \text{ if } s,p,q \in (0,1]. \end{aligned}$$

Then the followings hold:

(i) (X, d) is not a metric space.
(ii) (X, d) is a generalized metric space.
(iii) X does not have a topology which is compatible with d.

To remove the weakness of BMS, we need the following proposition and lemma.

Proposition 1.1. [20] *In a semimetric space, the assumption* (W) *is equivalent to the assertion that limits are unique.*

Proposition 1.2. [20] *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a BMS* (X, d) *with* $\lim_{n\to\infty} d(x_n, u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

Lemma 1.1. (See e.g.[13, 14]) *Let* (X, d) *be a BMS and let* $\{x_n\}$ *be a Cauchy sequence in* X *such that* $x_m \neq x_n$ *whenever* $m \neq n$. *Then the sequence* $\{x_n\}$ *can converge to at most one point.*

Let Ψ denote the set of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying:
$(\psi_1)\psi$ is nondecreasing,
$(\psi_2) \sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ.

Remark 1.1. It is easy to see that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

Now, we introduce the following implicit relation (see also e.g.[23, 24, 25, 26, 27, 33]).

Definition 1.3. Let Γ be the set of all continuous functions $F(t_1, ..., t_4) : \mathbb{R}^4_+ \to \mathbb{R}$ such that
$(F1)$: F is nondecreasing in variable t_1 and nonincreasing in variables t_3 and t_4,
$(F2)$: There exists $\psi \in \Psi$ such that for all $p, q \geq 0$, (a) $F(p, q, q, p) \leq 0$ implies $p \leq \psi(q)$, and (b) $F(p, q, p, q) \leq 0$ implies $p \leq \psi(q)$, and (c) $F(p, q, t, t) \leq 0$ implies $p \leq \psi(q)$ or $p \leq \psi(t)$ for all $t > 0$.
$(F3)$: For all $t > 0$, we have $0 < F(t, t, 0, 0)$.

We give the following examples.

Example 1.3. $F(t_1, ..., t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a, b, c \geq 0$ such that $a + b + c < 1$.

Example 1.4. $F(t_1, ..., t_4) = t_1 - k \max\{t_2, t_3, t_4\}$, where $k \in [0, \frac{1}{2})$.

Example 1.5. $F(t_1, ..., t_4) = t_1 - k \max\{t_2, \frac{t_3+t_4}{2}\}$, where $k \in [0, 1)$.

Example 1.6. $F(t_1, ..., t_4) = t_1 - at_2 - b \max\{t_3, t_4\}$, where $a + 2b < 1$ and $a, b \geq 0$.

The notion of α-admissible maps was introduced by Samet *et al.* [30], (see also [3, 1, 11, 12, 18, 15, 16, 17, 30]).

Definition 1.4. [30] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+_0$ be mappings. We say that the self-mapping T on X is α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

In what follows we recall the following class of functions: $\Phi = \{\varphi : \varphi : \mathbb{R}^+_0 \to \mathbb{R}\}$ such that φ is nonnegative, Lebesgue integrable, and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad for \quad each \quad \epsilon > 0. \tag{1.3}$$

Definition 1.5. (See [2]) We say that $\phi \in \Phi$ is an integral sub-additive if for each $a, b > 0$, we have

$$\int_0^{a+b} \phi(t)\, dt \leq \int_0^{a} \phi(t)\, dt + \int_0^{b} \phi(t)\, dt.$$

We denote by Φ_s, the class of all integral sub-additive functions $\phi \in \Phi$.

Example 1.7. (See [2])Let $\phi_1(t) = \frac{1}{2}(t + 1)^{-\frac{1}{2}}$ for all $t \geq 0$, $\phi_2(t) = \frac{2}{3}(t + 1)^{-\frac{1}{3}}$ for all $t \geq 0$, and $\phi_3(t) = e^{-t}$ for all $t \geq 0$. Then $\phi_i \in \Phi_s$, where $i = 1, 2, 3$.

In what follows that we introduce the concept of α-implicit contractive mapping of integral type in the setting of BMS.

Definition 1.6. Let (X, d) be a BMS and $T : X \to X$ be a given mapping. We say that T is a α-implicit contractive mapping of integral type if there exist three functions $\alpha : X \times X \to \mathbb{R}^+_0$, $\phi \in \Phi_s$ and $F \in \Gamma$ such that

$$F\left(\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)dt, \int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt \right) \leq 0, \tag{1.4}$$

for all $x, y \in X$.

In this paper, we investigate fixed point of the α-implicit contractive mapping of integral type (see e.g. [7, 28]).

2. Fixed point theorems

In this section, we shall state and prove our main results.

Theorem 2.1. *Let (X,d) be a complete BMS and $T : X \to X$ be an α-implicit contractive mapping of integral type. Suppose that*

(i) T is $\alpha-$admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;
(iii) T is continuous.

Then there exists a $u \in X$ such that $Tu = u$.

Proof. The assumption (ii) guarantees that there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. By using this initial point $x_0 \in X$, we set up an iterative sequence $\{x_n\}$ in X in the

$$x_{n+1} = Tx_n = T^{n+1}x_0 \text{ for all } n \in \mathbb{N}_0.$$

Throughout the proof, we assume that consecutive terms of the sequence $\{x_n\}$ are distinct, that is,

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}_0. \tag{2.5}$$

Notice that if $x_{n_0} = x_{n_0+1}$ for some n_0, then the proof is completed in this case. Indeed, we have $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$.

Since T is $\alpha-$admissible, the assumption (ii) yields that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we derive that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.6}$$

Due to same arguments, used above, we observe also that

$$\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1.$$

Consequently, we have that

$$\alpha(x_n, x_{n+2}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.7}$$

We divide the rest of the proof into 4 steps.

Step 1: We shall proved that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.8}$$

By letting $x = x_{n-1}$ and $y = x_n$ in (3.34), we get that

(2.9)
$$F\left(\alpha(x_{n-1}, x_n)\int_0^{d(x_n,x_{n+1})} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_n,x_{n+1})}\varphi(t)dt\right) \leq 0,$$

Since $\phi \in \Phi_s$, the above expression yields that

(2.10)
$$F\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \right) \leq 0.$$

Due to $(F2)$, there exists $\psi \in \Psi$ which implies

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \quad \leq \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt\right) \leq \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)dt\right), \tag{2.11}$$

for all $n \in \mathbb{N}$. On the other hand, by recalling the property (iii) of the auxiliary function ψ, we have

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt < \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \quad \text{for all } n \in \mathbb{N}. \tag{2.12}$$

By property of ψ again in (2.11), we deduce that $\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0,$ and hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Step 2: We show that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.13}$$

If we take $x = x_{n-1}$ and $y = x_{n+1}$ in (3.34), we get that

$$F\Big(\alpha(x_{n-1}, x_{n+1})\int_0^{d(x_n,x_{n+2})}\varphi(t)dt, \int_0^{d(x_{n-1},x_{n+1})}\varphi(t)dt,$$

$$\int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_{n+1},x_{n+2})}\varphi(t)dt\Big) \leq 0, \tag{2.14}$$

By $(F1)$ and (2.12) we have

(2.15)
$$F\Big(\int_0^{d(x_n,x_{n+2})} \varphi(t)dt, \int_0^{d(x_{n-1},x_{n+1})} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \Big) \leq 0.$$

Due to $(F2_c)$ the inequality (2.15) implies two cases. For the first case, $\int_0^{d(x_n,x_{n+2})} \varphi(t)dt \leq \psi(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$, we have the desired result due to Step1. Let us examine the second case.

(2.16)
$$\int_0^{d(x_n,x_{n+2})} \varphi(t)dt \leq \psi(\int_0^{d(x_{n-1},x_{n+1})} \varphi(t)dt) < \int_0^{d(x_{n-1},x_{n+1})} \varphi(t)dt \quad \text{for all } n \in \mathbb{N}.$$

Recursively, we observe that

$$\int_0^{d(x_n,x_{n+2})} \varphi(t)dt \leq \psi^n(\int_0^{d(x_0,x_2)} \varphi(t)dt), \text{ for all } n \in \mathbb{N}. \tag{2.17}$$

Due to the property of ψ again, we conclude that

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+2})} \varphi(t)dt = 0, \text{ and hence } \lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Step 3: We shall prove that

$$x_n \neq x_m \quad \text{for all } n \neq m. \tag{2.18}$$

We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, so without loss of generality, assume that $m > n + 1$. Therefore, from (3.34) we obtain that

$$F\Big(\alpha(x_{m-1}, x_m) \int_0^{d(Tx_{m-1},Tx_m)} \varphi(t)dt, \int_0^{d(x_{m-1},x_m)} \varphi(t)dt,$$

$$\int_0^{d(x_{m-1},Tx_{m-1})} \varphi(t)dt, \int_0^{d(x_m,Tx_m)} \varphi(t)dt\Big) \leq 0 \ . \tag{2.19}$$

Regarding $(F1)$ we get that

(2.20)
$$F\Big(\int_0^{d(x_m,x_{m+1})} \varphi(t)dt, \int_0^{d(x_{m-1},x_m)} \varphi(t)dt, \int_0^{d(x_{m-1},x_m)} \varphi(t)dt, \int_0^{d(x_m,x_{m+1})} \varphi(t)dt, \Big) \leq 0.$$

Owing to the property $(F2_a)$, we derive that

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})}\varphi(t)dt = & \int_0^{d(x_n,Tx_n)}\varphi(t)dt = \int_0^{d(x_m,Tx_m)}\varphi(t)dt \\
= & \int_0^{d(x_m,x_{m+1})}\varphi(t)dt \leq \alpha(x_m,x_{m+1})\int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)dt \\
\leq & \psi(\int_0^{d(x_{m-1},x_m))}\varphi(t)dt \leq \psi^{m-n}(\int_0^{d(x_n,x_{n+1})}\varphi(t)dt).
\end{aligned} \tag{2.21}$$

Due to a property of ψ, the inequality (2.21) yields that

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leq \psi^{m-n}(\int_0^{d(x_n,x_{n+1})}\varphi(t)dt) < \int_0^{d(x_n,x_{n+1})}\varphi(t)dt, \tag{2.22}$$

which is a contradiction. Thus, we have (2.18).

Step 4: We shall prove that $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n\to\infty}\int_0^{d(x_n,x_{n+k})}\varphi(t)dt = 0 \quad \text{for all } k\in\mathbb{N}. \tag{2.23}$$

The cases $k=1$ and $k=2$ are proved respectively by (2.8) and (2.13). Now, take $k\geq 3$ arbitrary. It is sufficient to examine two cases.

Case (I): Suppose that $k=2m+1$ where $m\geq 1$. Then, by using Step 1, Step 3 and the quadrilateral inequality together with the fact that $\varphi\in\Phi_s$, we find that

(2.24)

$$\begin{aligned}
\int_0^{d(x_n,x_{n+k})}\varphi(t)dt &= \int_0^{d(x_n,x_{n+2m+1})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{n+2m},x_{n+2m+1})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+1})}\varphi(t)+\int_0^{d(x_{n+1},x_{n+2})}\varphi(t)+\ldots+\int_0^{d(x_{n+2m},x_{n+2m+1})}\varphi(t)dt \\
&\leq \sum_{p=n}^{n+2m}\psi^p(\int_0^{d(x_0,x_1)}\varphi(t)dt) \leq \sum_{p=n}^{+\infty}\psi^p(\int_0^{d(x_0,x_1)}\varphi(t)dt) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Case (II): Suppose that $k=2m$ where $m\geq 2$. Again, by applying the quadrilateral inequality and Step 1 - Step 3 combining with the fact that $\varphi\in\Phi_s$,, we find

(2.25)

$$\begin{aligned}
\int_0^{d(x_n,x_{n+k})}\varphi(t)dt &= \int_0^{d(x_n,x_{n+2m})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+\ldots+d(x_{n+2m-1},x_{n+2m})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+2})}\varphi(t)dt+\int_0^{d(x_{n+2},x_{n+3})}\varphi(t)dt+\ldots+\int_0^{d(x_{n+2m-1},x_{n+2m})}\varphi(t)dt \\
&\leq \int_0^{d(x_n,x_{n+2})}\varphi(t)dt+\sum_{p=n+2}^{n+2m-1}\psi^p(\int_0^{d(x_0,x_1)}\varphi(t)dt) \\
&\leq \int_0^{d(x_n,x_{n+2})}\varphi(t)dt+\sum_{p=n}^{+\infty}\psi^p(\int_0^{d(x_0,x_1)}\varphi(t)dt) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

By combining the expressions (2.24) and (2.25), we have

$$\lim_{n\to\infty}\int_0^{d(x_n,x_{n+k})}\varphi(t)dt = 0 \quad \text{for all } k\geq 3.$$

Hence, we have

$$\lim_{n\to\infty}d(x_n,x_{n+k}) = 0 \quad \text{for all } k\geq 3.$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, there exists $u\in X$ such that

$$\lim_{n\to\infty} d(x_n,u)=0. \tag{2.26}$$

Since T is continuous, we obtain from (2.26) that

$$\lim_{n\to\infty} d(x_{n+1},Tu)=\lim_{n\to\infty} d(Tx_n,Tu)=0, \tag{2.27}$$

that is, $\lim_{n\to\infty} x_{n+1}=Tu$. Taking Proposition 1.2 into account, we conclude that $Tu=u$, that is u is a fixed point of T. □

Theorem 3.1 remains true if we replace the continuity hypothesis by a proper condition on the iterative sequence as follows:

Theorem 2.2. *Let (X,d) be a complete BMS and $T:X\to X$ be an α-implicit contractive mapping of integral type. Suppose that*

- (i) *T is $\alpha-$admissible;*
- (ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ and $\alpha(x_0,T^2x_0)\geq 1$;*
- (iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\geq 1$ for all k.*

Then there exists a $u\in X$ such that $Tu=u$.

Proof. Following the lines in the proof of Theorem 3.1, we deduce that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\geq 0$ is Cauchy and converges to some $u\in X$. In view of Proposition 1.2,

$$\lim_{k\to\infty} d(x_{n(k)+1},Tu)=d(u,Tu). \tag{2.28}$$

By using the method of *reductio ad absurdum*, we shall show that $Tu=u$. Suppose, on the contrary, that $Tu\neq u$, i.e, $d(Tu,u)>0$. From (2.6) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},u)\geq 1$ for all k. Hence, by the implicit relation, we have

$$\begin{aligned} F\Big(\ &\alpha(x_{n(k)},u)\int_0^{d(Tx_{n(k)},Tu)}\varphi(t)dt, \int_0^{d(x_{n(k)},u)}\varphi(t)dt,\\ &\int_0^{d(x_{n(k)},Tx_{n(k)})}\varphi(t)dt, \int_0^{d(u,Tu)}\varphi(t)dt\Big)\leq 0\ . \end{aligned} \tag{2.29}$$

Due to $F1$, we find that

$$F\Big(\ \int_0^{d(x_{n(k)}+1,Tu)}\varphi(t)dt, \int_0^{d(x_{n(k)},u)}\varphi(t)dt, \int_0^{d(x_{n(k)},x_{n(k)+1})}\varphi(t)dt, \int_0^{d(u,Tu)}\varphi(t)dt,\ \Big)\leq 0, \tag{2.30}$$

Since F is continuous, letting $n\to\infty$ by regarding (2.26) and (2.28)

$$F\Big(\ \int_0^{d(u,Tu)}\varphi(t)dt,0,0,\int_0^{d(u,Tu)}\varphi(t)dt\ \Big)\leq 0. \tag{2.31}$$

Due to $(F2_a)$ for $q=0$, we have $\int_0^{d(u,Tu)}\varphi(t)dt\leq 0$. Thus, u is a fixed point of T, that is, $Tu=u$. □

For the uniqueness, an additional condition was considered.

(U)**:** For all $x,y\in Fix(T)$, we have $\alpha(x,y)\geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.3. *Adding condition (U) to the hypotheses of Theorem 3.1 (resp. Theorem 2.2), we obtain that u is the unique fixed point of T.*

Proof. By using the method of *reductio ad absurdum*, we shall show that u is the unique fixed point of T. Let v be another fixed point of T with $v \neq u$. By hypothesis (U),

$$1 \leq \alpha(u,v) = \alpha(Tu,Tv).$$

Now, owing to (3.34), we have

$$F\left(\alpha(u,v)\int_0^{d(Tu,Tv)}\varphi(t)dt, \int_0^{d(u,v)}\varphi(t)dt, \int_0^{d(u,Tu)}\varphi(t)dt, \int_0^{d(v,Tv)}\varphi(t)dt \right) \leq 0, \tag{2.32}$$

By elementary calculation and taking $(F1)$ into account, we observe that

$$F\left(\int_0^{d(u,v)}\varphi(t)dt, \int_0^{d(u,v)}\varphi(t)dt, 0, 0 \right) \leq 0, \tag{2.33}$$

which contradicts $(F3)$. Hence u is a unique fixed point of T. □

3. Consequences

The following is the main result of [3].

Corollary 3.1. *Let (X,d) be a complete BMS and $T : X \to X$ be continuous mapping. If there exist $\phi \in \Phi_s$ and $F \in \Gamma$ such that*

$$F\left(\int_0^{d(Tx,Ty)}\varphi(t)dt, \int_0^{d(x,y)}\varphi(t)dt, \int_0^{d(x,Tx)}\varphi(t)dt, \int_0^{d(y,Ty)}\varphi(t)dt \right) \leq 0, \tag{3.34}$$

then, there exists a $u \in X$ such that $Tu = u$.

Proof. It is sufficient to take $\alpha(x,y) = 1$ for all $x, y \in X$ in Theorem 2.3. □

Corollary 3.2. (See [3]) *Let (X,d) be a complete BMS and $T : X \to X$ be a mapping. Suppose that there exist two functions $\alpha : X \times X \to \mathbb{R}_0^+$ and $\psi \in \Psi$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(\max\{d(x,y), d(x,Tx), d(y,Ty)\}), \text{ for all } x, y \in X. \tag{3.35}$$

Suppose also that

- (i) *T is $\alpha-$admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
- (iii) *T is continuous or*
  if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists a $u \in X$ such that $Tu = u$.

Proof. It is sufficient to take $F(t_1,t_2,t_3,t_4) = t_1 - \psi(\max\{t_2,t_3,t_4\})$ with $\varphi(t) = 1$ in Theorem 3.1 and Theorem 2.2. □

Remark 3.2. The above corollaries are the main results of Aydi *et al.* [3] and Karapınar[16] in which several existing fixed point theorems in the context of BMS have been listed. We underline that all consequences of [3, 16] can be added here. On the other hand, to avoid the repetition, we do not listed here.

Example 3.8. Let $X = [0,1]$ and $A = \{x_n : n \in \mathbb{N}\} \subset (0,1)$. We define the distance function $d : X \times X \to \mathbb{R}_0^+$ as follows:

$$\left\{ \begin{array}{rlrl} d(x,y) & = 0 \text{ if } x = y, & d(x,y) & = d(y,x) \text{ for all } x,y, \\ d(x_1,x_2) & = d(x_3,x_4) = \frac{1}{5} & d(x_1,x_4) & = d(x_2,x_3) = \frac{2}{5} \\ d(x_1,x_3) & = d(x_2,x_4) = 1 & d(x,y) & = |x-y| \text{ otherwise.} \end{array} \right. \tag{3.36}$$

It is clear that (X,d) is a Branciari metric space. Notice also that d is not a metric since

$$1 = d(x_1,x_3) > d(x_1,x_2) + d(x_2,x_3) = \frac{3}{5}.$$

We define $T : X \to X$ as $Tx = 1 - x$: Furthermore, let $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be defined as $\psi(t) = \frac{t}{4}$. Now, we define $\alpha : X \times X \to [0,\infty)$ as follows:

$$\alpha(x,y) = \left\{ \begin{array}{cl} 1 & \text{if } x = y, \\ 5x & \text{if } x,y \in \{x_1,x_2,x_3,x_4\} \text{ with } x \neq y, \\ 0 & \text{otherwise.} \end{array} \right. \tag{3.37}$$

Hence, all conditions of Corollary 3.2, (and hence Theorem 2.3 with $\varphi(t) = 1$) is satisfied and $x = \frac{1}{2}$ is a unique fixed point of T.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this article.

References

[1] Ali, M. U. and Kamran, T., *On (α^*,ψ)-contractive multi-valued mappings*, Fixed Point Theory Appl., 2013 2013:137

[2] Ali, M. U., Kamran, T., and Karapınar, E., *An Approach to Existence of Fixed Points of Generalized Contractive Multivalued Mappings of Integral Type via Admissible Mapping*, Applied and Abstract Analysis, (2014), Article Id:141489

[3] Aydi, H., Karapınar, E.,and Samet, B., *Fixed points for generalized (α,ψ)-contractions on generalized metric spaces*, Journal of Inequalities and Applications, (2014)

[4] Aydi, H., Karapınar, E. and Lakzian, H., *Fixed point results on the class of generalized metric spaces*, Math. Sciences, 2012, 6:46

[5] Azam, A. and Arshad, M., *Kannan fixed point theorems on generalized metric spaces*, J. Nonlinear Sci. Appl., **1** (2008), 45–48

[6] Branciari, A. , *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57** (2000), 31–37

[7] Branciari, A. , *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (2002), No. 9, 531–536

[8] Bilgili, N. and Karapınar, E., *A note on "common fixed points for (ψ,α,β)-weakly contractive mappings in generalized metric spaces"*, Fixed Point Theory Appl., 2013, 2013:287

[9] Das, P. and Lahiri, B. K. , *Fixed point of a Ljubomir Ćirić's quasi-contraction mapping in a generalized metric space*, Publ. Math. Debrecen, **61** (2002), 589–594

[10] Erhan, I. M., Karapınar, E. and Sekulić, T., *Fixed points of (ψ,ϕ) contractions on rectangular metric spaces*, Fixed Point Theory Appl., 2012, 2012:138

[11] Jleli, M., Karapınar, E. and Samet, B., *Best proximity points for generalized $\alpha-\psi$-proximal contractive type mappings*, Journal of Applied Mathematics, Article ID 534127, (2013)

[12] Jleli, M., Karapınar, E. and Samet, B., *Fixed point results for $\alpha-\psi_\lambda$ contractions on gauge spaces and applications*, Abstract and Applied Analysis, (2013) Article Id, 730825

[13] Jleli, M. and Samet, B., *The Kannan's fixed point theorem in a cone rectangular metric space*, J. Nonlinear Sci. Appl., **2** (2009), No. 3, 161–167

[14] Kadeburg, Z. and Radenovič, S., *On generalized metric spaces: a survey*, TWMS J. Pure Appl. Math., **5** (2014), No. 1, 3–13

[15] Karapınar, E., *Discussion on* (α, ψ) *contractions on generalized metric spaces*, Abstract and Applied Analysis, (2014) Article ID 962784

[16] Karapınar, E., *Fixed points results for alpha-admissible mapping of integral type on generalized metric spaces*, Abstract and Applied Analysis, (2014), Article Id: 141409

[17] Karapınar, E., On (α, ψ) contractions of integral type on generalized metric spaces, in *Proceedings of the 9th ISAAC Congress*, V. Mityushevand, M. Ruzhansky, Eds., Springer, Krakow, Poland, 2013

[18] Karapınar, E. and Samet, B., *Generalized* α-ψ*-contractive type mappings and related fixed point theorems with applications*, Abstract Appl. Analysis, 2012 (2012), Article ID 793486

[19] Kikina, L. and Kikina, K., *A fixed point theorem in generalized metric space*, Demonstrateio Mathematica, **XLVI** (2013), No. 1, 181–190

[20] Kirk, W. A. and Shahzad, N., *Generalized metrics and Caristis theorem*, Fixed Point Theory Appl., 2013, 2013:129

[21] Kirk, W. A. and Shahzad, N., *Correction: Generalized metrics and Caristis theorem*, Fixed Point Theory Appl.,2014, 2014:177

[22] Mihet, D., *On Kannan fixed point principle in generalized metric spaces*, J Nonlinear Sci Appl., **2** (2009), No. 2, 92-96

[23] Popa, V., *Implicit Contractive Maps in Ordered Metric Spaces*, Topics in Mathematical Analysis and Applications Springer Optimization and Its Applications, Volume **94**, 2014, pp 715–746

[24] Popa, V., *Fixed point theorems for implicit contractive mappings*, St. Cerc. t. Univ. Bacu (Ser. Math.), **7** (1997) 129–133

[25] Popa, V., *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Dem. Math., **32** (1999) 157–163

[26] Popa, V., *On some fixed point theorems for mappings satisfying a new type of implicit relation*, Math. Moravica, **7** (2003), 61–66

[27] Popa, V. and Mocanu, M., *Altering distance and common fixed points under implicit relations*, Hacettepe J. Math. Stat., **38** (2009) 329–337

[28] Samet, B., *A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type*, Int. J. Math. Anal., **26** (2009), No. 3, 1265–1271

[29] Samet, B., *Discussion on: a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari*, Publ. Math. Debrecen, **76** (2010), No. 4, 493-494

[30] Samet, B., Vetro, C. and Vetro, P., *Fixed point theorems for* α-ψ*-contractive type mappings*, Nonlinear Anal., **75** (2012) 2154–2165

[31] Sarma, I. R., Rao, J. M. and Rao, S. S., *Contractions over generalized metric spaces*, J. Nonlinear Sci. Appl., **2** (2009), No.3, 180-182

[32] Suzuki, T., *Generalized metric space do not have the compatible topology*, Abstract Appl. Anal., vol. **2014**, Article ID 458098, 5 pages, 2014

[33] Turinici, M., *Fixed points of implicit contraction mappings*, An. Şt. Univ. "A. I. Cuza" Iaşi (S I-a, Mat), **22** (1976), 177–180

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