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A general convergence theorem for multiple-set split feasibility problem in Hilbert spaces

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ABSTRACT. We establish strong convergence result of split feasibility problem for a family of quasi-nonexpansive multi-valued mappings and a total asymptotically strict pseudo-contractive mapping in infinite dimensional Hilbert spaces.

1. INTRODUCTION

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP has attracted the attention of many authors due to its application in signal processing (see, for example, [14, 16]). Note that the split feasibility problem (1.1) can be formulated as a fixed-point equation:

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \tag{1.2}$$

that is, x^* solves the SFP (1.1) if and only if x^* solves the fixed point equation (1.2), where A^* is the adjoint of A, P_C and P_Q are metric projections onto C and Q, respectively. This implies that we can use fixed-point algorithms ([4]) to solve SFP. A popular algorithm that solves the SFP (1.1) is CQ algorithm due to Byrne [2]. Byrne [3] applied Krasnosel'skii-Mann (KM) iteration to the CQ algorithm, and Zhao and Yang [17] applied KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

Let us recall some definitions:

Definition 1.1. Let $T : H \to H$ be a mapping and $F(T)$ be its set of fixed points. T is said to be
(a) *nonexpansive* if $||Tx - Ty|| \leq ||x - y||, \forall x, y \in H$. (b) *quasi-nonexpansive* if $||Tx - Tp|| \leq ||x - p||, \forall x \in H, p \in F(T)$. (c) *firmly nonexpansive* mapping if $||Tx - Ty||^2 \leq ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \forall x, y \in H$. (d) *quasi-firmly nonexpansive* mapping if $||Tx - Tp||^2 \leq$

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$||x-p||^2-||x-Tx||^2, \forall x \in H, p \in F(T)$. (e) *strictly pseudocontractive* mapping if there exists a constant $k \in [0,1)$ such that $||Tx-Ty||^2 \leq ||x-y||^2+k||(x-y)-(Tx-Ty)||^2, \forall x,y \in H$. (f) *total asymptotically strict pseudocontractive mapping*, if there exist a constant $k \in [0,1)$ and sequences $\{\mu_n\} \subset [0,\infty), \{\xi_n\} \subset [0,\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $n \geq 1, x, y \in H$

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(x - y) - (Tx - Ty)||^2 + \mu_n\phi(||x - y||) + \xi_n, \forall x, y \in H.$$

The class of firmly-nonexpansive mappings is well known to include resolvents and projection operators, while the class of quasi-firmly nonexpansive mappings contains subgradient projection operators (see, e.g., [11] and the reference therein).

In the sequel, we denote by $CB(H)$ the collection of all nonempty, closed and bounded subsets of H. The *Hausdorff metric* $\tilde{H}$ on $CB(H)$ is defined by
$\tilde{H}(A,B) := \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \forall A,B \in CB(H)$, where $D(x,K) := \inf_{y\in K} d(x,y)$.

Definition 1.2. Let $S : H \to CB(H)$ be a multi-valued mapping. An element $p \in H$ is said to be *a fixed point of* S, if $p \in Sp$. The mapping S is said to be: (i) *nonexpansive*, if $\tilde{H}(Sx,Sy) \leq ||x-y||, \forall x,y \in H$; (ii) *quasi-nonexpansive*, if $F(S) \neq \emptyset$ and $\tilde{H}(Sx,Sp) \leq ||x-p||, \forall x \in H, p \in F(S)$.

We assume that H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ bounded linear operator and $A^* : H_2 \to H_1$ the adjoint of A. We investigate the following multiple-set split feasibility problem (MSSFP) for a family of multi-valued quasi-nonexpansive mappings and a total asymptotically pseudo-contractive mapping in infinite dimensional Hilbert spaces, i.e.,

$$\text{find } x \in C \text{ such that } Ax \in Q, \tag{1.3}$$

where, $\{S_i\}_{i=1}^\infty : H_1 \to CB(H_1)$ is a family of multi- valued quasi-nonexpansive mappings and $T : H_2 \to H_2$ is a total asymptotically strict pseudo-contractive mapping with nonempty fixed-point sets $C := \cap_{i=1}^\infty F(S_i)$; and $Q := F(T)$, and denote the solution set of (MSSFP) by

$$\Gamma := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q). \tag{1.4}$$

Recall that $\cap_{i=1}^\infty F(S_i)$ and $F(T)$ are nonempty, closed and convex subsets of H_1 and H_2, respectively ([18]). If $\Gamma \neq \emptyset$, we have that Γ is closed and convex subset of H_1.

Theorem 1.1. [9] *Let $\{S_i\}_{i=1}^\infty : H_1 \to CB(H_1)$ be a family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, S_i is demi-closed at 0. Let $T : H_2 \to H_2$ be a uniformly L-Lipschitzan continuous and total asymptotically strict pseudocontractive mapping satisfying $\sum_{n=1}^\infty \mu_n < \infty$ and $\sum_{n=1}^\infty \xi_n < \infty$. Suppose there exist constants $M_0 > 0, M_1 > 0$ such that $\phi(\lambda) \leq M_0\lambda^2, \forall \lambda > M_1$. Let $C := \cap_{i=1}^\infty F(S_i) \neq \emptyset$; and $Q := F(T)$. Assume that for each $p \in C$; $S_ip = \{p\}$, for each $i \geq 1$. Let $\{x_n\}$ be the sequence generated by:* $x_1 \in H_1$ chosen arbitrarily,

$$\begin{cases} x_{n+1} = \alpha_{0,n}y_n + \sum_{i=1}^\infty \alpha_{i,n}w_{i,n}, \ w_{i,n} \in S_iy_n \\ y_n = x_n + \gamma A^*(T^n - I)Ax_n, n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_{i,n}\} \subset (0,1)$ *and* $\gamma > 0$ *satisfy the following conditions:*
(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, *for each* $n \geq 1$; (b) *for each* $i \geq 1$, $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$ (c) $\gamma \in \left(0, \frac{1-k}{||A||^2}\right)$.
If Ω *(the set of solutions of multiple-set split feasibility problem 1.3) is nonempty, then both* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *converge weakly to some point* $x \in \Omega$.

If in addition, there exists some positive integer m such that S_m is semi-compact, then both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $x \in \Omega$.
Question: Can we modify the iterative scheme (1.5) so that the strong convergence is guaranteed without compactness of the operator?
In this paper, we modify the iterative scheme (1.5) and prove strong convergence result for multiple-set split feasibility problem (1.3) for a family of multi-valued quasi-nonexpansive mappings and a total asymptotically pseudo-contractive mapping in infinitely dimensional Hilbert spaces without compactness condition on the operator. Our results improve the corresponding results of Chang *et al.* [9], Censor *et al.* [6, 7], Yang [16], Moudafi [13], Xu [14], Censor and Segal [8], Masad and Reich [12] and others.

2. Preliminaries

Definition 2.3. A mapping $T : H \to H$ is called (1) *demiclosed at the origin* if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ (weak convergence) and $||x_n - Tx_n|| \to 0$, we have $Tx = x$ (2) *uniformly L-Lipschitzian continuous* if there exists a constant $L > 0$ such that $||T^n x - T^n y|| \leq L||x - y||, \forall x, y \in H, n \geq 1$ (3) A multi-valued mapping $S : H \to CB(H)$ is said to be *demiclosed at origin* if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $d(x_n, Sx_n) \to 0$, we have $x \in Sx$.

Next, we state some well-known lemmas which will be used in the sequel.

Lemma 2.1. *Let* H *be a real Hilbert space. Then* $\forall x, y \in H$, *the following results hold:* (i) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$; (ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$.

Lemma 2.2. (Yang *et al.*, [18]) *Let* $T : H \to H$ *be a uniformly L-Lipschitzian continuous and total asymptotically strict pseudocontractive mapping, then* T *is demi-closed at origin.*

Lemma 2.3. (Chang *et al.*, [10]) *Let* E *be a uniformly convex Banach space,* $B_r(0) := \{x \in E : ||x|| \leq r\}$ *be a closed ball with center 0 and radius* $r > 0$. *For any given sequence* $\{x_1, x_2, \ldots, x_n, \ldots\} \subset B_r(0)$ *and any given number sequence* $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ *with* $\lambda_i \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1$, *there exists a continuous strictly increasing and convex function* $g : [0, 2r) \to [0, \infty)$ *with* $g(0) = 0$ *such that for any* $i, j \in \mathbb{N}, i < j$ *the following holds:*

$$\Big|\Big|\sum_{n=1}^{\infty} \lambda_n x_n\Big|\Big|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

Lemma 2.4. ([1]) *Let* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *be nonnegative,* $\{\alpha_n\}$ *be positive real numbers such that* $\lambda_{n+1} \leq \lambda_n - \alpha_n\lambda_n + \gamma_n, n \geq 1$. *Let for all* $n > 1$, $\frac{\gamma_n}{\alpha_n} \leq c_1$ and $\alpha_n \leq \alpha$. *Then* $\lambda_n \leq \max\{\lambda_1, K_*\}$, *where* $K_* = (1 + \alpha)c_1$.

Lemma 2.5. ([15]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the relation:* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n$, $n \geq 0$, *where (i)* $\{a_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; *(ii)* $\limsup \sigma_n \leq 0$; *(iii)* $\gamma_n \geq 0$; $(n \geq 1)$, $\Sigma\gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

3. Main results

We assume that the following conditions are satisfied:

(1) $\{S_i\}_{i=1}^{\infty} : H_1 \to H_1$ is a family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, S_i is demiclosed at origin; (2) $T : H_2 \to H_2$ is a uniformly L-Lipschitzan continuous and total asymptotically strict pseudocontractive mapping satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$; (ii) $\{\alpha_n\}$ is a real sequence in (0,1) such that $\mu_n = o(\alpha_n); \xi_n = o(\alpha_n)$; $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iii) there exist constants $M_0 > 0, M_1 > 0$ such that $\phi(\lambda) \leq M_0\lambda^2, \forall \lambda > M_1$; (3) $C := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$; and $Q := F(T) \neq \emptyset$;
(4) for each $p \in C; S_i p = \{p\}$, for each $i \geq 1$.
A modification of the algorithm (1.5) is given in the following result to have the desired strong convergence.

Theorem 3.2. *Let $\{S_i\}_{i=1}^{\infty}, T, C, Q, k, \{\mu_n\}, \{\xi_n\}, \phi$ and L satisfy the above conditions (1)-(5). Let $\{x_n\}$ be the sequence generated by $x_1 \in H_1$,*

$$\begin{cases} u_n = (1-\alpha_n)x_n, \; y_n = u_n + \gamma A^*(T^n - I)Au_n, \\ x_{n+1} = \beta_{0,n} y_n + \sum_{i=1}^{\infty} \beta_{i,n} w_{i,n}, \; w_{i,n} \in S_i y_n, n \geq 1. \end{cases} \tag{3.1}$$

where $\{\beta_{i,n}\} \subset (0,1)$ and $\gamma > 0$ satisfy the following conditions:
(a) $\sum_{i=0}^{\infty} \beta_{i,n} = 1$, for each $n \geq 1$; (b) for each $i \geq 1$, $\liminf_{n\to\infty} \beta_{0,n}\beta_{i,n} > 0$ (c) $\gamma \in \left(0, \frac{1-k}{||A||^2}\right)$. If Ω (the set of solutions of multiple-set split feasibility problem in (3.1)) is nonempty, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element of Ω.

Proof. Note that ϕ is continuous, it follows that ϕ attains maximum (say M) in $[0, M_1]$ and by our assumption, $\phi(\lambda) \leq M_0\lambda^2, \forall \lambda > M_1$. In either case, we have that $\phi(\lambda) \leq M + M_0\lambda^2, \forall \lambda \in [0, \infty)$. Let $x^* \in \Omega$. Then from the convexity of $||.||^2$, we obtain

$$\begin{aligned} ||u_n - x^*||^2 &= ||(1-\alpha_n)x_n - x^*||^2 = ||(1-\alpha_n)(x_n - x^*) + \alpha_n(-x^*)||^2 \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2. \end{aligned} \tag{3.2}$$

From (3.2) and Lemma 2.1 (i), we obtain that

$$\begin{aligned} &||y_n - x^*||^2 = ||u_n - x^* + \gamma A^*(T^n - I)Au_n||^2 \\ &= ||u_n - x^*||^2 + 2\gamma\langle u_n - x^*, A^*(T^n - I)Au_n\rangle + \gamma^2||A^*(T^n - I)Au_n||^2. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} \gamma^2||A^*(T^n - I)Au_n||^2 &= \gamma^2\langle A^*(T^n - I)Au_n, A^*(T^n - I)Au_n\rangle \\ &= \gamma^2\langle AA^*(T^n - I)Au_n, (T^n - I)Au_n\rangle \\ &\leq \gamma^2||A||^2||(T^n - I)Au_n||^2, \end{aligned} \tag{3.4}$$

As $Ax^* \in Q = F(T)$ and T is a total asymptotically strict pseudocontractive mapping, so we obtain

$$\begin{aligned}\langle u_n - x^*, A^*(T^n - I)Au_n\rangle &= \langle A(u_n - x^*), (T^n - I)Au_n\rangle \\ &= \langle A(u_n - x^*) + (T^n - I)Au_n - (T^n - I)Au_n, (T^n - I)Au_n\rangle \\ &= \langle T^n Au_n - Ax^*, (T^n - I)Au_n\rangle - ||(T^n - I)Au_n||^2 \\ &= \frac{1}{2}\Big[||T^n Au_n - Ax^*||^2 + ||(T^n - I)Au_n||^2 - ||Au_n - Ax^*||^2\Big] \\ &\quad - ||(T^n - I)Au_n||^2 \\ &\leq \frac{1}{2}\Big[||Au_n - Ax^*||^2 + k||(T-I)Ax_n||^2 + \mu_n\phi(||Au_n - Ax^*||) + \xi_n\Big] \\ &\quad + \frac{1}{2}\Big[||(T^n - I)Au_n||^2 - ||Au_n - Ax^*||^2\Big] \\ &\quad - ||(T^n - I)Au_n||^2 \\ &\leq \frac{k-1}{2}||(T^n - I)Au_n||^2 + \frac{\mu_n}{2}\Big(M + M_0||Au_n - Ax^*||^2) + \frac{\xi_n}{2}. \qquad (3.5)\end{aligned}$$

Substituting (3.5) and (3.4) into (3.3), we have

$$\begin{aligned}||y_n - x^*||^2 \leq\ & ||u_n - x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ & +\mu_n\gamma(M + M_0||Au_n - Ax^*||^2 + \gamma\xi_n. \qquad (3.6)\end{aligned}$$

Using (3.6) and (3.2) in (3.1), we obtain

$$\begin{aligned}||x_{n+1} - x^*||^2 &\leq \beta_{0,n}||y_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_{i,n}||w_{i,n} - x^*||^2 \\ &= \beta_{0,n}||y_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_{i,n}(d(w_{i,n}, S_i(x^*))^2 \\ &\leq \beta_{0,n}||y_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_{i,n}(H(S_i y_n, S_i(x^*))^2 \\ &= ||y_n - x^*||^2 \leq ||u_n - x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad +\mu_n\gamma(M + M_0||Au_n - Ax^*||^2 + \gamma\xi_n \\ &\leq (1 - \alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad +\mu_n\gamma\Big(M + M_0||A||^2((1 - \alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2)\Big) + \gamma\xi_n \\ &= ||x_n - x^*||^2 - (\alpha_n - \mu_n\gamma M_0||A||^2(1 - \alpha_n))||x_n - x^*||^2 + \alpha_n||x^*||^2 \\ &\quad +\mu_n\gamma M + \gamma\alpha_n M_0||A||^2||x^*||^2 + \gamma\xi_n \\ &\quad -\gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\leq ||x_n - x^*||^2 - (\alpha_n - \mu_n\gamma M_0||A||^2(1 - \alpha_n))||x_n - x^*||^2 + \alpha_n||x^*||^2 \\ &\quad +\mu_n\gamma M + \gamma\alpha_n M_0||A||^2||x^*||^2 + \gamma\xi_n \\ &= ||x_n - x^*||^2 - (\alpha_n - \mu_n\gamma M_0||A||^2(1 - \alpha_n))||x_n - x^*||^2 + \sigma_n, \qquad (3.7)\end{aligned}$$

where $\sigma_n = \alpha_n||x^*||^2\mu_n\gamma M + \gamma\alpha_n M_0||A||^2||x^*||^2 + \gamma\xi_n$. Since $\mu_n = o(\alpha_n)$ and $\xi_n = o(\alpha_n)$, we may assume without loss of generality that there exist constants $k_0 \in (0, 1)$ and $M_2 > 0$ such that for all $n \geq 1$, we have $\frac{\mu_n}{\alpha_n} \leq \frac{1-k_0}{M_0\gamma||A||^2(1-\alpha_n)}$ and $\frac{\sigma_n}{\alpha_n} \leq M_2$. Thus, we obtain $||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \alpha_n k_0||x_n - x^*||^2 + \sigma_n$. By Lemma 2.4, we have that

$||x_n - x^*||^2 \leq \max\{||x_n - x^*||^2, (1+k_0)M_2\}$. Therefore, $\{x_n\}$ is bounded. Furthermore, the sequences $\{y_n\}$ and $\{u_n\}$ are bounded.
The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - x^*||\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{||x_n - x^*||\}_{n=1}^{\infty}$ converges and $||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 \to 0,\ n \to \infty$. From (3.6) and (3.2), we have that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||y_n - x^*||^2 \leq ||u_n - x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2 + \gamma\xi_n \\ &\leq ||x_n - x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2) + \gamma\xi_n + \alpha_n||x^*||^2. \end{aligned}$$

So we get $\gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2) + \gamma\xi_n + \alpha_n||x^*||^2$. This implies that $\gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \to 0,\ n \to \infty$. Hence, we obtain

$$||(T^n - I)Au_n|| \to 0,\ n \to \infty. \tag{3.8}$$

Also, we observe that $||y_n - u_n|| = \gamma A^*||(T^n - I)Au_n|| \to 0,\ n \to \infty$ and $||u_n - x_n|| = \alpha_n||x_n|| \to 0,\ n \to \infty$. Furthermore, $||y_n - x_n|| \leq ||y_n - u_n|| + ||u_n - x_n|| \to 0,\ n \to \infty$. Using (3.6) and Lemma 2.3 in (3.1), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq \beta_{0,n}||y_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_{i,n}||w_{i,n} - x^*||^2 - \beta_{0,n}\beta_{i,n}g(||y_n - w_{i,n}||) \\ &\leq ||u_n - x^*||^2 - \gamma(1 - k - \gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2) + \gamma\xi_n - \beta_{0,n}\beta_{i,n}g(||y_n - w_{i,n}||). \end{aligned}$$

This implies by (3.2) and condition (c) that

$$\begin{aligned} \beta_{0,n}\beta_{i,n}g(||y_n - w_{i,n}||) &\leq ||u_n - x^*||^2 - ||x_{n+1} - x^*||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2) + \gamma\xi_n \\ &\leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \alpha_n||x^*||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Ax^*||^2) + \gamma\xi_n. \end{aligned}$$

This implies by condition (b) that $\lim_{n\to\infty} g(||y_n - w_{i,n}||) = 0$. As g is continuous and strictly increasing with $g(0) = 0$, so we have $\lim_{n\to\infty}||y_n - w_{i,n}|| = 0\ \forall w_{i,n} \in S_i y_n$.

Hence, we have

$$\lim_{n\to\infty} d(y_n, S_i y_n) \leq \lim_{n\to\infty}||y_n - w_{i,n}|| = 0. \tag{3.9}$$

Now obtain from (3.1) that

$$\begin{aligned} ||x_{n+1} - x_n||^2 &\leq \beta_{0,n}||y_n - x_n||^2 + \sum_{i=1}^{\infty}\beta_{i,n}||w_{i,n} - x_n||^2 \\ &\leq \beta_{0,n}||y_n - x_n||^2 + \sum_{i=1}^{\infty}\beta_{i,n}\Big(||w_{i,n} - y_n|| + ||y_n - x_n||\Big)^2. \end{aligned}$$

Since $\lim_{n\to\infty}||y_n - x_n|| = 0$ and $\lim_{n\to\infty}||y_n - w_{i,n}|| = 0$, we have $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$.
Consequently,

$$\begin{aligned} ||u_{n+1} - u_n|| &= ||(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n|| \\ &\le |\alpha_{n+1} - \alpha_n|||x_{n+1}|| + (1-\alpha_n)||x_{n+1} - x_n|| \to 0,\ n \to \infty. \end{aligned} \tag{3.10}$$

Using the fact that T is uniformly L-Lipschitzian, we have

$$\begin{aligned} ||TAu_n - Au_n|| &\le ||TAu_n - T^{n+1}Au_n|| + ||T^{n+1}Au_n - T^{n+1}Au_{n+1}|| \\ &\quad + ||T^{n+1}Au_{n+1} - Au_{n+1}|| + ||Au_{n+1} - Au_n|| \\ &\le L||Au_n - T^nAu_n|| + (L+1)||Au_{n+1} - Au_n|| + ||T^{n+1}Au_{n+1} - Au_{n+1}|| \\ &\le L||Au_n - T^nAu_n|| + (L+1)||A||||u_{n+1} - u_n|| + ||T^{n+1}Au_{n+1} - Au_{n+1}||. \end{aligned}$$

By (3.8) and (3.10), we obtain

$$||(T-I)Au_n|| \to 0,\ n \to \infty. \tag{3.11}$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in H_1$. Using the fact that $x_{n_j} \rightharpoonup z \in H_1$ and $||y_n - x_n|| \to 0,\ n \to \infty$, we have that $y_{n_j} \rightharpoonup z \in H_1$. Similarly, $u_{n_j} \rightharpoonup z \in H_1$ since $||u_n - x_n|| \to 0,\ n \to \infty$. By the demiclosedness of S_i at origin and (3.9), we have that $z \in F(S_i)$. Hence, $z \in \cap_{i=1}^{\infty} F(S_i) = C$. On the other hand, since A is a linear bounded operator, it follows from $u_{n_j} \rightharpoonup z \in H_1$ that $Au_{n_j} \rightharpoonup Az \in H_2$. Hence, from (3.11), we have that $||TAu_{n_j} - Au_{n_j}|| = ||TAu_{n_j} - Au_{n_j}|| \to 0,\ j \to \infty$. Since T is demiclosed at zero, therefore we have that $Az \in F(T) = Q$. Hence, $z \in \Omega$.

Next, we prove that $\{x_n\}$ converges strongly to z. From (3.6) and Lemma 2.1 (ii), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\le ||y_n - z||^2 \le ||u_n - z||^2 - \gamma(1-k-\gamma||A||^2)||(T^n - I)Au_n||^2 \\ &\quad + \mu_n\gamma(M + M_0||Au_n - Az||^2) + \gamma\xi_n \\ &\le ||u_n - z||^2 + \mu_n\gamma(M + M_0||Au_n - Az||^2) + \gamma\xi_n \\ &\le ||u_n - z||^2 + M^*\mu_n + \gamma\xi_n \\ &= ||(1-\alpha_n)(x_n - z) - \alpha_n z||^2 + M^*\mu_n + \gamma\xi_n \\ &\le (1-\alpha_n)^2||x_n - z||^2 - 2\alpha_n\langle u_n - z, z\rangle + M^*\mu_n + \gamma\xi_n \\ &\le (1-\alpha_n)||x_n - z||^2 - 2\alpha_n\langle u_n - z, z\rangle + M^*\mu_n + \gamma\xi_n, \end{aligned} \tag{3.12}$$

where $M^* > \gamma \sup\limits_{n\ge1}(M + M_0||Au_n - Az||^2) > 0$. It is clear that $-2\langle u_n - z, z\rangle \to 0,\ n \to \infty$ and $\sum\limits_{n=1}^{\infty} M^*\mu_n < \infty$; $\sum\limits_{n=1}^{\infty}\gamma\xi_n < \infty$. Now, applying Lemma 2.5 to (3.12), we have $||x_n - z|| \to 0$. That is, $x_n \to z, n \to \infty$.

Case 2: Assume that $\{||x_n - x^*||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||x_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by $\tau(n) := \max\{k \in \mathbb{N} : k \ge n, \Gamma_n \le \Gamma_{n+1}\}$. Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} - \Gamma_{\tau(n)} \forall n \ge n_0$. By (3.9), it is easy to see that $\lim\limits_{n\to\infty} d(y_{\tau(n)}, S_i y_{\tau(n)}) \le \lim\limits_{n\to\infty} ||y_{\tau(n)} - w_{i,\tau(n)}|| = 0$. Furthermore, we can show that $||(T-I)Au_{\tau(n)}|| \to 0,\ n \to \infty$. By an argument similar to the one in case 1, we conclude immediately that $x_{\tau(n)}, y_{\tau(n)}$ and $u_{\tau(n)}$ weakly converge to z as $\tau(n) \to \infty$. At the same time, from (3.12), we note that, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - z||^2 - ||x_{\tau(n)} - z||^2 \\ &\leq \alpha_{\tau(n)}[-2\langle u_{\tau(n)} - z, z\rangle - ||x_{\tau(n)} - z||^2] + M^*\mu_{\tau(n)} + \gamma\xi_{\tau(n)}, \end{aligned}$$

which implies $||x_{\tau(n)} - z||^2 \leq -2\langle u_{\tau(n)} - z, z\rangle + M^*\mu_{\tau(n)} + \gamma\xi_{\tau(n)}$. Hence, we deduce that $\lim_{n\to\infty} ||x_{\tau(n)} - z|| = 0$. Therefore, $\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$),because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$, $0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$. Hence $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to z. □

Corollary 3.3. *Let $\{S_i\} : H_1 \to H_1$ be a family of single-valued quasi-nonexpansive mappings, and for each $i \geq 1$, $I - S_i$ is demi-closed at 0. Let $A, A^*, T, C, Q, k, \{\mu_n\}, \{\xi_n\}, L, \phi, \{\beta_{i,n}\}$ and $\gamma > 0$ be the same as in Theorem 3.2. Let $\{y_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be sequences generated by $x_1 \in H_1$,*

$$\begin{cases} u_n = (1 - \alpha_n)x_n, \; y_n = u_n + \gamma A^*(T^n - I)Au_n, \\ x_{n+1} = \beta_{0,n}y_n + \sum\limits_{i=1}^{\infty} \beta_{i,n}S_iy_n, n \geq 1. \end{cases}$$

Then the conclusions in Theorem 3.2 still hold.

Remark 3.1. The asymptotic behaviour of the sequence (3.1) is like that of (1.5). Note that $\{\alpha_n\}$ in (3.1) satisfies the conditions: $\lim\limits_{n\to\infty} \alpha_n = 0; \sum\limits_{n=1}^{\infty} \alpha_n = \infty,$ which show that $\{\alpha_n\}$ is a slowly decreasing sequence and that explains the good behaviour of our iterative scheme (3.1).

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