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Common best proximity points for proximity commuting mapping with Geraghty's functions

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ABSTRACT. In this paper, we prove new common best proximity point theorems for proximity commuting mapping by using concept of Geraghty's theorem in complete metric spaces. Our results improve and extend recent result of Sadiq Basha [Basha, S. S., *Common best proximity points: global minimization of multi-objective functions*, J. Glob Optim, **54** (2012), No. 2, 367-373] and some results in the literature.

1. INTRODUCTION

The significance of fixed point theory stems from the fact that it furnishes an unified treatment and is a vital tool for solving equations of form $Tx = x$ where T is a self-mapping defined on a subset of a metric space, a normed linear space, topological vector spaces or some suitable spaces. Some applications of fixed point theory can be found in [5, 8, 14]. One such generalizations is due to Geraghty [7] as follows:

Theorem 1.1. ([7]) *Let (X, d) be a complete metric space and f be a self mapping on X such that for each $x, y \in X$ satisfying $d(fx, fy) \leq \beta(d(x, y))d(x, y)$, where $\beta \in \mathcal{S}$, that $\mathcal{S}$ is the families of functions from $[0, \infty)$ into $[0, 1)$ which satisfies the condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$. Then the sequence $\{f_n\}$ converges to the unique fixed point of f in X.*

In 1969, Fan [6] introduced the classical best approximation theorem, that is, if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [10], Reich [11], Sehgal and Singh [12, 13], have derived extensions of Fan's Theorem in many directions. Other works of of the existence of a best proximity point for contractions can be seen in [1, 2, 9]. Recently, Sadiq Basha [4] gave common best proximity point theorems for proximity commuting mapping of multi-objective function. The aim of this paper is to study the common best proximity point theorem for a classes of Geraghty's condition with a commute proximally mapping and we also give an illustrative example for support our main result. The result of this paper extends and generalizes the corresponding results given by Sadiq Basha [4] and some authors in the literature.

2. PRELIMINARIES

Given nonempty subsets A and B of a metric space (X, d), we recall the following notations and notions that will be used in what follows.

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$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$, $A_0 := \{x \in A : d(x, y) = d(A, B)\}$ for some $\{y \in B\}$ and $B_0 := \{y \in B : d(x, y) = d(A, B)$for some $x \in A\}$. If $A \cap B \neq \emptyset$, then A_0 and B_0 are non-empty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B respectively provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [3]).

Definition 2.1. A point $x \in A$ is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the following condition $d(x, Sx) = d(A, B)$.

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.2. Let $S : A \to B$ and $T : A \to B$. An element $x^* \in A$ is said to be a *common best proximity* point if it satisfies the following condition $d(x^*, Sx^*) = d(x^*, Tx^*) = d(A, B)$.

Observed that a common best proximity point is an element at which the multi-objective functions $x \mapsto d(x, Sx)$ and $x \mapsto d(x, Tx)$ attain common global minimum, since $d(x, Sx) \geq d(A, B)$ and $d(x, Tx) \geq d(A, B)$ for all x.

Definition 2.3. [4]A mapping $S : A \to B$ and $T : A \to B$ is said to be a *commute proximally* if they satisfy the following condition $[d(u, Sx) = d(v, Tx) = d(A, B)] \Longrightarrow Sv = Tu$ for all $u, v, x, \in A$.

It is easy to see that proximal commutativity of self-mappings become commutativity of the mappings.

Definition 2.4. [4] A mapping $S : A \to B$ and $T : A \to B$ is said to be a *swapped proximally* if they satisfy the following condition $[d(y, u) = d(y, v) = d(A, B)$ and $Su = Tv] \Longrightarrow Sv = Tu$ for all $u, v, \in A$ and $y \in B$.

Definition 2.5. A is said to be *approximatively compact with respect to* B if every sequence $\{x_n\}$ in A satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself, and that every compact set is approximatively compact. Moreover, A_0 and B_0 are non-empty set if A is compact and B is approximatively compact with respect to A.

3. Main results

In this section, we prove the existence of a common best proximity point for proximally commuting non-self mappings by using Geraghty's condition and we also give some example for support our main results.

Theorem 3.2. *Let A and B be non-empty closed subsets of a complete metric space X such that A is approximatively compact with respect to B, and A_0 is non-empty set. Let the non-self mapping $S : A \to B, T : A \to B$ satisfy the following conditions:*

(a) There is a function $\beta \in \mathcal{S}$ such that

$$d(Sx, Sy) \leq \beta(d(Tx, Ty))d(Tx, Ty), \text{ for all } x, y \in A;$$

(b) T is continuous;
(c) S and T commute proximally;
(d) S and T can be swapped proximally;

(e) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.

Then, there exists an element $x \in A$ such that $d(x, Tx) = d(A, B)$ and $d(x, Sx) = d(A, B)$ Moreover, if x^* is another common best proximity point of the mappings S and T, then it is necessary that $d(x, x^*) \leq 2d(A, B)$.

Proof. Let x_0 a fixed element in A_0. In view of the fact that $S(A_0) \subseteq T(A_0)$ it is ascertained that there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By similar fashion we can find x_n in A_0 such that

$$Sx_{n-1} = Tx_n, \tag{3.1}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \beta(d(Tx_n, Tx_{n+1}))d(Tx_n, Tx_{n+1}) \\ &= \beta(d(Sx_{n-1}, Sx_n))d(Sx_{n-1}, Sx_n) \\ &\leq d(Sx_{n-1}, Sx_n) \end{aligned}$$

this mean that the sequence $\{d(Sx_n, Sx_{n+1})\}$ is non-increasing and converges to some nonnegative r, that is, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = r$. If $r > 0$, from the fact that

$$\frac{d(Sx_n, Sx_{n+1})}{d(Sx_{n-1}, Sx_n)} \leq \beta(d(Sx_{n-1}, Sx_n)). \tag{3.2}$$

Taking $n \to \infty$, in inequality (3.2), we get $\lim_{n\to\infty} \beta(d(Sx_{n-1}, Sx_n)) \to 1$ and since $\beta \in \mathcal{S}$ implies that $r = 0$. Therefore

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = 0. \tag{3.3}$$

Next, we will prove that $\{d(Sx_n, Sx_{n+1})\}$ is Cauchy sequence. We distinguish two cases.
Case I Suppose there exits $n \in \mathbb{N}$ such that $Sx_n = Sx_{n+1}$, we get

$$\begin{aligned} d(Sx_{n+1}, Sx_{n+2}) &\leq \beta(d(Tx_{n+1}, Tx_{n+2}))d(Tx_{n+1}, Tx_{n+2}) \\ &\leq \beta(d(Sx_n, Sx_{n+1}))d(Sx_n, Sx_{n+1}) \\ &= 0, \end{aligned}$$

which implies that $Sx_{n+1} = Sx_{n+2}$. So, for every $m > n$, we conclude that $Sx_m = Sx_n$ and hence $\{Sx_n\}$ is a Cauchy sequence in B.
Case II The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon. \tag{3.4}$$

By using (3.4) and triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(Sx_{m_k}, Sx_{n_k}) \\ &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + d(Sx_{n_k-1}, Sx_{n_k}). \end{aligned} \tag{3.5}$$

Using (3.3) and (3.5), we have

$$d(Sx_{m_k}, Sx_{n_k}) \to \varepsilon \quad \text{as} \quad k \to \infty. \tag{3.6}$$

Again, by the triangular inequality, we get

$$d(Sx_{m_k}, Sx_{n_k}) \leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) \tag{3.7}$$

and

$$d(Sx_{m_k+1}, Sx_{n_k+1}) \leq d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k}, Sx_{n_k+1}). \tag{3.8}$$

From, (3.3), (3.6), (3.7) and (3.8), we get $d(Sx_{m_k+1}, Sx_{n_k+1}) \to \varepsilon$ as $k \to \infty$.
In view of the fact that

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq \beta(d(Tx_{m_k+1}, Tx_{n_k+1}))d(Tx_{m_k+1}, Tx_{n_k+1}) \\ &\leq \beta(d(Sx_{m_k}, Sx_{n_k}))d(Sx_{m_k}, Sx_{n_k}), \end{aligned}$$

it follow that

$$\frac{d(Sx_{m_k+1}, Sx_{n_k+1})}{d(Sx_{m_k}, Sx_{n_k})} \leq \beta(d(Sx_{m_k}, Sx_{n_k})),$$

we get $\lim_{k\to\infty} \beta(d(Sx_{m_k}, Sx_{n_k})) \to 1$ and since $\beta \in \mathcal{S}$ implies that $\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = 0$ which is a contradiction. Then, we deduce that $\{Sx_n\}$ is a Cauchy sequence in B. Since B is closed subset a complete metric space X, then there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, we have that the sequence $\{Tx_n\}$ also converges to y. From $S(A_0) \subseteq B_0$, for each $n \in \mathbb{N}$ there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B). \tag{3.9}$$

So, it follows from (3.1) and (3.9) that

$$d(Tx_n, u_{n-1}) = d(Sx_{n-1}, u_{n-1}) = d(A, B), \tag{3.10}$$

for all $n \in \mathbb{N}$. By (3.9), (3.10) and the fact that the mappings S and T are commuting proximally, we obtain $Tu_n = Su_{n-1}$ for all $n \in \mathbb{N}$. Moreover, we have

$$d(y, A) \quad \leq d(y, u_n) \quad = d(y, Sx_n) + d(A, B) \leq d(y, Sx_n) + d(y, A).$$

Therefore $d(y, u_n) \to d(y, A)$ as $n \to \infty$. Since A is approximatively compact with respect to B, then there exists subsequence $\{u_{n_k}\}$ of sequence $\{u_n\}$ such that converging to some element $u \in A$. Further, since $d(y, u_{n_k-1}) \to d(y, A)$ and again, A is approximatively compact with respect to B, then there exists subsequence $\{u_{n_{k_j}-1}\}$ of sequence $\{u_{n_k-1}\}$ such that converging to some element $v \in A$. Since T is continuous, consequently S is continuous and thus $Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{j\to\infty} Su_{n_{k_j}-1} = Sv$, $d(y, u) = \lim_{k\to\infty} d(Sx_{n_k}, u_{n_k}) = d(A, B)$ and $d(y, v) = \lim_{j\to\infty} d(Tx_{n_{k_j}}, u_{n_{k_j}-1}) = d(A, B)$. Because S and T can be swapped proximally, we get $Tv = Su$. Next, to prove $Su = Sv$, suppose the contrary, it follow that

$$d(Su, Sv) \quad \leq \beta(d(Tu, Tv))d(Tu, Tv) \quad \leq \beta(d(Sv, Su))d(Sv, Su) < d(Sv, Su)$$

which is a contradiction. Thus $Su = Sv$ and hence $Tu = Su$. Since $S(A_0)$ is contained in B_0, there exists an element x in A such that $d(x, Tu) = d(A, B)$ and $d(x, Su) = d(A, B)$. By the commuting proximally of S and T, we get $Sx = Tx$. Consequently, we have

$$d(Su, Sx) \quad \leq \beta(d(Tu, Tx))d(Tu, Tx) \quad \leq \beta(d(Su, Sx))d(Su, Sx). \tag{3.11}$$

In inequality (3.11), if $Su \neq Sx$ then

$$1 = \frac{d(Su, Sx)}{d(Su, Sx)} \leq \beta(d(Su, Sx)) < 1, \tag{3.12}$$

it is impossible. So, we have $Su = Sx$ and hence $Tu = Tx$. It follow that
$d(x, Tx) = d(x, Tu) = d(A, B)$ and $d(x, Sx) = d(x, Su) = d(A, B)$.

Therefore, x is a common best proximity point of S and T. Suppose that x^* is another common best proximity point of the mappings S and T, so that $d(x^*, Tx^*) = d(A, B)$ and $d(x^*, Sx^*) = d(A, B)$. By the commuting proximally of S and T, we get $Sx = Tx$ and $Sx^* = Tx^*$. Thus,

$$d(Sx^*, Sx) \leq \beta(d(Tx^*, Tx))d(Tx^*, Tx) \leq \beta(d(Sx^*, Sx))d(Sx^*, Sx). \tag{3.13}$$

In inequality (3.13) , if $Sx^* \neq Sx$, by similar argument of (3.12), it is impossible. Therefore, $Sx = Sx^*$. Moreover, it can be concluded that

$$d(x, x^*) \leq d(x, Sx) + d(Sx, Sx^*) + d(Sx^*, x^*) = 2d(A, B)$$

and the proof is completes. □

If take $\beta(t) = k$, where $0 \leq k < 1$ in Theorem 3.2, we obtain following corollary:

Corollary 3.1. [4] *Let A and B be non-empty closed subsets of a complete metric space X such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are non-empty. Let the non-self mapping $S : A \to B$, $T : A \to B$ satisfy the following conditions.*

(a) There is a non-negative real number $\alpha < 1$ such that

$$d(Sx_1, Sx_2) \leq kd(Tx_1, Tx_2) \text{ for all } x_1, x_2 \in A.$$

(b) T is continuous.
(c) S and T commute proximally.
(d) S and T can be swapped proximally.
(e) $S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$. Then, there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B) \text{ and } d(x, Sx) = d(A, B).$$

Further, if x^* is another common best proximity point of the mappings S and T, then it is necessary that $d(x, x^*) \leq 2d(A, B)$.

Now, below we give an example to illustrate Theorem 3.2.

Example 3.1. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let $A = \{(0, y) : 0 \leq y < 4\}$ and $B = \{(2, y) : 0 \leq y < 4\}$. Define two mappings $S : A \to B$, $T : A \to B$ as follows: $S((0, y)) = \big(2, \ln(1 + y)\big)$, $T((0, y)) = \big(2, y\big)$. Then it is easy to see that $d(A, B) = 2$, $A_0 = A$, $B_0 = B$ and satisfying condition (b) and (e). First, we will show that S and T satisfying condition (a) of Theorem 3.2 with $\beta \in \mathcal{S}$ defined by

$$\beta(t) = \begin{cases} 1 & ; \quad t = 0 \\ \dfrac{\ln(1+t)}{t}, & ; \quad t > 0. \end{cases}$$

Let $(0, y_1), (0, y_2) \in A$. If $y_1 = y_2 = 0$, we are done. Assume that $y_1, y_2 > 0$, we have

$$\begin{aligned} d(S(0, y_1), S(0, y_2)) &= \big|\ln(1 + y_1) - \ln(1 + y_2)\big| \\ &\leq \frac{\big|\ln(1 + |y_1 - y_2|)\big|}{|y_1 - y_2|}|y_1 - y_2| \\ &= \beta(d(T(0, y_1), T(0, y_2)))d(T(0, y_1), T(0, y_2)) \end{aligned}$$

Hence, S and T satisfying (a). Next, we will show that S and T commute proximally. Let $(0, u), (0, v), (0, y) \in A$ are satisfying $d((0, u), S(0, y)) = d(A, B) = 2$ and $d((0, v), T(0, y)) = d(A, B) = 2$, it follow that $u = \ln(1 + y)$ and $v = y$, and hence $S(0, v) = T(0, u)$. Finally, we will show that S and T swapped proximally. Suppose that $(0, u), (0, v), (0, y) \in A$ are satisfying $d((0,y),(0,u)) = d((0,y),(0,v)) = d(A,B) = 2$ and $S(0,u) = T(0,v)$, then we get $u = v$ and $v = \ln(1+u)$ which is implies that $S(0,v) = T(0,u)$. Therefore, all hypothesis of Theorem 3.2 are satisfied. Furthermore, $(0,0) \in A$ is a common best proximity point of S and T, because $d((0,0),S(0,0)) = d((0,0),(2,0)) = d((0,0),T((0,0)) = d(A,B)$.

On the other hand, suppose that there exists non-negative real number $k < 1$ such that for each $(0,x^*)$ and $(0,y^*)$ are an element in A satisfying $d(S(0,x^*),S(0,y^*)) \leq kd(T(0,x^*), T(0,y^*))$. So that $\big|\ln(1+x^*) - \ln(1+y^*)\big| \leq k|x^* - y^*|$. Consider $y^* = 0$ and $x^* > 0$, we get $1 = \lim_{x^* \to 0^+} \frac{\ln(1+x^*)}{x^*} \leq k < 1$ which is a contradiction. Therefore, the results of [4] can not be applied to this example and Theorem 3.2.

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