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On the theory of fixed point theorems for convex contraction mappings

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ABSTRACT. Based on the concepts and problems introduced in [Rus, I. A., *The theory of a metrical fixed point theorem: theoretical and applicative relevances,* Fixed Point Theory, **9** (2008), No. 2, 541–559], in the present paper we consider the theory of some fixed point theorems for convex contraction mappings. We give some results on the following aspects: data dependence of fixed points; sequences of operators and fixed points; well-posedness of a fixed point problem; limit shadowing property and Ulam-Hyers stability for fixed point equations.

1. INTRODUCTION

The class of convex contraction mappings and some applications have been introduced in [9] and studied in many papers [6], [10], [11], [16] - [19], [22], [23], [26] - [29]. On the other hand, I. A. Rus [25], has formulated many questions like: "what does it mean the theory of a theorem ?" or "what does it mean the theory of a fixed point theorem ?"

For some classes of mappings, there have been given various results about the theory of a fixed point theorem, see [8], [14], [15], [25], [27] and the papers cited therein. More specifically, in the paper [20], M. Păcurar obtained several results about the fixed point theory for some cyclic Berinde operators, while in [21] M. Păcurar and I. A. Rus have studied the fixed point theory for some cyclic φ-contractions.

Starting from the results in [24] and [25], the aim of this paper is to state and study some problems about asymptotic fixed point theorems like: data dependence, sequences of operators and fixed points, well-posedness of fixed point problem, limit shadowing property and Ulam-Hyers stability of fixed point equation. So, we give partial answers to the above question.

2. NEEDED NOTIONS AND RESULTS

Let (X, d) be a complete metric space and let $f : X \to X$ be an operator.

Definition 2.1. ([9]) Let (X, d) be a metric space. A self map $f : X \to X$ is called a **convex contraction** if

(2.1) $$d(f^2(x), f^2(y)) \leq a \cdot d(f(x), f(y)) + b \cdot d(x, y), \forall x, y \in X,$$

where a, b are constants satisfying $0 < a, b < 1$ and $a + b < 1$.

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Example 2.1. If $b = 0$, then by the convex contraction condition (2.1) we obtain the Banach contraction condition:

$$d(f(x), f(y)) \leq a \cdot d(x, y), \forall x, y \in X,$$

subject to a change of notation.

If $a = 0$, then by the convex contraction condition (2.1), we obtain the well known "asymptotic" contraction condition:

$$d(f^2(x), f^2(y)) \leq b \cdot d(x, y),$$

that ensures the existence of a fixed point (even in the case when 2 is replaced by a given integer n).

Example 2.2. ([9])

Let $X = [0, 1]$ with the usual metric and let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \frac{x^2 + 1}{2}, x \in [0, 1].$$

Then f is not a Banach contraction, although $F_f = \{1\}$.

But f is a convex contraction, as we have

$$\left|f^2(x) - f^2(y)\right| \leq \frac{1}{2} \left|f(x) - f(y)\right| + \frac{1}{4} \left|x - y\right|, x, y \in [0, 1],$$

with $a = \dfrac{1}{2}$ and $b = \dfrac{1}{4}$.

The first main result in [9] is the following fixed point theorem.

Theorem 2.1. ([9]) *Let (X, d) be a complete metric space and $f : X \to X$ a continuous (a, b)-convex contraction, i.e., a mapping satisfying*

$$d(f^2(x), f^2(y)) \leq a \cdot d(f(x), f(y)) + b \cdot d(x, y), \forall x, y \in X,$$

where $0 < a, b < 1$ and $a + b < 1$. Then

1) $F_f = \{x \in X : f(x) = x\} = \{x^*\}$;

2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = f(x_n)$, $n = 0, 1, 2, ...$, converges to x^*.

For other classes of contractive type mappings presented in the following, we refer to [22]-[27], and [9]-[11].

Definition 2.2. The operator f is called a **graphic contraction** if there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)),\ (\forall)\ x \in X.$$

Definition 2.3. The operator f is called **contractive** if

$$d(f(x), f(y)) < d(x, y),\ (\forall)\ x, y \in X,\ \ x \neq y.$$

Definition 2.4. The operator f is called **convex contractive of order 2** if there exist $a_1, a_2 \in [0, 1)$ with $a_1 + a_2 = 1$, such that

$$d(f^2(x), f^2(y)) < a_1 d(x, y) + a_2 d(f(x), f(y)),\ (\forall)\ x, y \in X,\ \ x \neq y.$$

A well known result of V. Nemytskii [18] states that, if f is a contractive operator, defined on a compact space X, then $F_f \neq \emptyset$.

In [9] V.I. Istrăţescu proved some extensions of Nemytskii (see [18]) and of Edelstein (see [7]) results as follows (see [9], Theorem 1.7, Theorem 1.8, Theorem 2.3 and Theorem 2.4):

Theorem 2.2. *Let $f : X \to X$ be a continuous convex contractive operator of order 2. If X is a compact space then f has a unique fixed point x_f^*, i.e. $F_f = \{x_f^*\}$.*

Theorem 2.3. *Let $f : X \to X$ be a continuous convex contractive operator of order 2. We suppose that any orbit $(f^n(x))_0^\infty$, $x \in X$, has a limit point ξ. Then ξ is the unique fixed point of f, i.e. $x_f^* = \xi$.*

Definition 2.5. The operator f is said to be a **two-sided convex contraction** if there exist $a_1, a_2, b_1, b_2 \in [0, 1)$, with $a_1 + a_2 + b_1 + b_2 < 1$, such that

$$d(f^2(x), f^2(y)) \le a_1 d(x, f(x)) + a_2 d(f(x), f^2(x)) +$$
$$+ b_1 d(y, f(y)) + b_2 d(f(y), f^2(y)),\ (\forall)\, x, y \in X,\ \ x \neq y.$$

Definition 2.6. The operator f is said to be a **convex contraction of type 2,** if there exist $c_0, c_1, a_1, a_2, b_1, b_2 \in [0, 1)$, with $c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$, such that

$$d(f^2(x), f^2(y)) \le c_0 d(x, y) + c_1 d(f(x), f(y)) + a_1 d(x, f(x)) +$$
$$+ a_2 d(f(x), f^2(x)) + b_1 d(y, f(y)) + b_2 d(f(y), f^2(y)),\ (\forall)\, x, y \in X.$$

Theorem 2.4. *Any continuous two-sided convex contraction operator has a unique fixed point.*

Theorem 2.5. *Any continuous convex contraction operator of type 2 has a unique fixed point.*

Following I. A. Rus [25] we present some needed definitions and results.

Definition 2.7. The operator f is called a **weakly Picard operator (WPO)** if the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit, denoted by $f^\infty(x) = x_f^*$, is a fixed point of f.

Remark 2.1. If f *is a weakly Picard operator and, there exists $c > 0$ a real number such that*

$$d(x, f^\infty(x)) \le c\, d(x, f(x)),\ (\forall)\, x \in X$$

where $f^\infty(x) = x_f^*$, then the operator f is a c**-weakly Picard operator**.

Definition 2.8. The operator f is called a **Picard operator (PO)** if $F_f = \{x_f^*\}$ and $f^n(x) \to x_f^*$ as $n \to \infty$, for all $x \in X$.

Remark 2.2. If f is a WPO and $F_f = \{x_f^*\}$, then f is a PO.

Definition 2.9. The operator f is called a **Bessaga operator (BO)** if $F_f = F_{f^n} = \{x_f^*\}$, for all $n \in \mathbb{N}^*$.

Definition 2.10. The operator f is called a **Janos operator (JO)** if $\cap_{n\in\mathbb{N}^*} f^n(X) = \{x_f^*\}$.

Definition 2.11. The fixed point problem for the operator f is **well possed** if the following conditions are satisfied:

(i) $F_f = \{x_f^*\}$;

(ii) if $x_n \in X$, $n \in \mathbb{N}$ are such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $x_n \to x_f^*$ as $n \to \infty$.

Definition 2.12. The operator f has the **limit shadowing property** if the following implication holds

[$x_n \in X, n \in \mathbb{N}$ such that $d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$]
implies that
[there exists $x \in X$ such that $d(x_n, f^n(x)) \to 0$ as $n \to \infty$].

Definition 2.13. The equation $x = f(x)$ is **Ulam-Hyers stable** if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$d(y, f(y)) \le \varepsilon$, there exists a solution x^* of the equation $x = f(x)$, such that $d(y^*, x^*) \le c_f \varepsilon$.

3. Main results

By using the above definitions we state and prove the following results.

Regarding the data dependence of the fixed point in the case of two-sided convex contraction operator, we have

Theorem 3.6. *Let $f : X \to X$ be a continuous two-sided convex contraction operator and let $g : X \to X$ be, such that:*

(a) g has at least a fixed point, say $x_g^* \in F_g$,

(b) there exists $\eta_1 > 0$ such that $d(f(x), g(x)) \leq \eta_1$, for any $x \in X$,

(c) there exists $\eta_2 > 0$ such that $d(f^2(x), g^2(x)) \leq \eta_2$, for any $x \in X$.

If $x_f^* \neq x_g^*$ then $d(x_f^*, x_g^*) \leq (b_1 + b_2)\eta_1 + (1 + b_2)\eta_2$.

Proof. Because f is a continuous two-sided convex contraction operator it results that $F_f = \{x_f^*\}$. Then, if we suppose that $x_f^* \neq x_g^*$, we have

$$d(x_f^*, x_g^*) = d(f^2(x_f^*), g^2(x_g^*)) \leq d(f^2(x_f^*), f^2(x_g^*)) + d(f^2(x_g^*), g^2(x_g^*)) \leq$$

$$\leq a_1 d(x_f^*, f(x_f^*)) + a_2 d(f(x_f^*), f^2(x_f^*)) + b_1 d(x_g^*, f(x_g^*)) + b_2 d(f(x_g^*), f^2(x_g^*)) + \eta_2 \leq$$

$$\leq b_1 d(x_g^*, f(x_g^*)) + b_2 d(f(x_g^*), f^2(x_g^*)) + \eta_2 \leq$$

$$\leq b_1\eta_1 + b_2[d(f(x_g^*), g(x_g^*)) + d(g(x_g^*), f^2(x_g^*)) + \eta_2 \leq$$

$$\leq b_1\eta_1 + b_2\eta_1 + b_2\eta_2 + \eta_2.$$

So, the theorem is proved. □

Remark 3.3. Moreover, if in the previous theorem, the operator f is a graphic contraction (see [14]), then

$$d(x_f^*, x_g^*) \leq \frac{b_1\eta_1 + \eta_2 + b_2\alpha d(x_f^*, f(x_g^*))}{1 - b_2\alpha}.$$

Theorem 3.7. *Let $f : X \to X$ be a continuous two-sided convex contraction operator and let $f_n : X \to X$ be, $n \in \mathbb{N}$, such that:*

(a) for each $n \in \mathbb{N}$ there exists $x_n^* \in F_{f_n}$,

(b) $f_n \rightrightarrows f$, as $n \to \infty$.

Then $x_n^* \to x_f^*$, as $n \to \infty$.

Proof. Because f is a continuous two-sided convex contraction we have $F_f = \{x_f^*\}$.

As $\{f_n\}_{n\geq 0}$ converges uniformly to f, there exist $\eta_{1n} \in \mathbb{R}_+, n \in \mathbb{N}$, such that $\eta_{1n} \to 0, n \to \infty$ and $d(f_n(x), f(x)) \leq \eta_{1n}$ for any $x \in X$.

As $\{f_n^2\}_{n\geq 0}$ converges uniformly to f^2, there exist $\eta_{2n} \in \mathbb{R}_+, n \in \mathbb{N}$, such that $\eta_{2n} \to 0, n \to \infty$ and $d(f_n^2(x), f^2(x)) \leq \eta_{2n}$ for any $x \in X$.

Applying the previous theorem for each pair f and f_n, $n \in \mathbb{N}$, it follows that we have $d(x_n^*, x_f^*) = d(f_n^2(x_n^*), f^2(x_f^*)) \leq (b_1 + b_2)\eta_{1n} + (1 + b_2)\eta_{2n}$.

Since $\eta_{1n} \to 0$ and $\eta_{2n} \to 0$ as $n \to \infty$ the conclusion of theorem follows immediately. □

Theorem 3.8. *Let $f : X \to X$ be a continuous two-sided convex contraction operator. If there exists $\alpha > 0$ such that*

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)),\ (\forall)\ x \in X,\ (1)$$

then the fixed point problem for f is well possed, that is, assuming there exist $z_n \in X, n \in \mathbb{N}$ such that $[d(z_n, f(z_n)) \to 0$ as $n \to \infty]$ implies $[z_n \to x_f^*$ as $n \to \infty]$.

Proof. Because f is a continuous two-sided convex contraction operator, $F_f = \{x_f^*\}$.

Let $z_n \in X$, $n \in \mathbb{N}$ such that $d(z_n, f(z_n)) \to 0$ as $n \to \infty$.

Therefore, we obtain

$$d(z_n, x_f^*) \le d(z_n, f(z_n)) + d(f(z_n), f^2(z_n)) + d(f^2(z_n), f^2(x_f^*)) \le$$
$$\le d(z_n, f(z_n)) + \alpha d(z_n, f(z_n)) + a_1 d(z_n, f(z_n)) +$$
$$+ a_2 d(f(z_n), f^2(z_n)) + b_1 d(x_f^*, f(x_f^*)) + b_2 d(f(x_f^*), f^2(x_f^*)) \le$$
$$\le (1 + \alpha + a_1 + a_2\alpha) d(z_n, f(z_n)).$$

From these relationships we get that

$$d(z_n, x_f^*) \le (1 + \alpha + a_1 + a_2\alpha) d(z_n, f(z_n)),$$

which obviously implies that

$$z_n \to x_f^* \text{ as } n \to \infty.$$

So the fixed point problem for f is well possed. □

Remark 3.4. Relative to the Theorem 3.6 and to the Remark 3.3 we have: If f is a continuous two-sided convex contraction operator which satisfies the condition (1) then f is a c-PO with $c = 1 + \alpha + a_1 + \alpha a_2$ and for c-POs we have (see [24])

$$d(x_g^*, x_f^*) \le d(x_g^*, f(x_f^*)) = c d(g(x_g^*), f(x_g^*)) \le c\eta_1,$$

therefore we have data dependence without the condition (c).

Remark 3.5. An operator f that is a continuous two-sided convex contraction, in generally, isn't a Bessaga operator. But, because we have $\{x_f^*\} = F_f \subset F_{f^2}$, if there exists another fixed point of f^2, say $y_f^* \neq x_f^*$, then the following estimation holds:

$$d(x_f^*, y_f^*) \le (b_1 + b_2) d(y_f^*, f(y_f^*)).$$

Example 3.3. Let $f : [0,1] \to [0,1]$ be, given by

$$f(x) = \begin{cases} \frac{1}{7}x, & x \in [0, \frac{1}{2}] \\ \frac{1}{14}, & x \in (\frac{1}{2}, 1] \end{cases}.$$

It results that $f^2(x) = \frac{1}{7} f(x)$.

The operator f is a two-sided convex contraction with $a_1 = \frac{1}{5}$, $a_2 = \frac{1}{6}$, $b_1 = \frac{2}{5}$, $b_2 = \frac{1}{6}$ and we have $F_f = F_{f^2} = \{0\}$. Moreover, how $f^n(x) = (\frac{1}{7})^n f(x)$, this f is a Picard operator, a Bessaga operator and a Janos operator with $x_f^* = 0$.

Remark 3.6. The operator f in the previous example is a graphic contraction, that is, with $\alpha = \frac{6}{35}$, we have

$$d(f(x), f^2(x)) \le \alpha d(x, f(x)), \ (\forall)\ x \in [0,1].$$

Remark 3.7. There exist operators f that are two-sided convex contractions, but they aren't continuous operators. As an example we consider the operator $f : [0,1] \to [0,1]$, given by

$$f(x) = \begin{cases} \frac{x}{7}, & x \in [0, \frac{1}{2}] \\ \frac{1}{7}, & x \in (\frac{1}{2}, 1] \end{cases}.$$

This operator isn't continuous on $[0,1]$ but it is a two-sided convex contraction with $a_1 = \frac{1}{9}$, $a_2 = \frac{1}{6}$, $b_1 = \frac{2}{9}$, $b_2 = \frac{1}{6}$. Moreover, how $f^n(x) = (\frac{1}{7})^n f(x)$, f is a Picard operator, a Bessaga operator and a Janos operator with $x_f^* = 0$. This operator is a graphic contraction, too, with $\alpha = \frac{3}{7}$.

Theorem 3.9. *Let $f : X \to X$ be a continuous two-sided convex contraction operator. We suppose that:*

(i) the operator f is a graphic contraction;

(ii) the sequence $(z_n)_{n\geq 0}$ is convergent in X and $d(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$. Then the limit of the sequence $(z_n)_{n\geq 0}$ is the unique fixed point of f .

Proof. Let $(z_n)_{n\geq 0}$ be a sequence such that $d(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$ and let x_z be its limit, i.e. $x_z = \lim_{n\to\infty} z_n$.

We can prove that $x_z = x_f^*$, where x_f^* is the unique fixed point of f. □

Remark 3.8. If, for an operator f, the conclusion of the previous theorem remains true without asking "the sequence $(z_n)_{n\geq 0}$ is convergent in X" then this operator f has the limit shadowing property, where x is any element of X.

We have

Theorem 3.10. *Let $f : X \to X$ be a continuous two-sided convex contraction opera-tor. We suppose that X is a compact space. Then the operator f has the limit shadowing property, i.e. for each sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in X$, $n \in \mathbb{N}$, such that $d(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$ there exists $x \in X$ such that $d(z_n, f^n(x)) \to 0$, as $n \to \infty$.*

Proof. We have that $F_f = \{x_f^*\}$, and for each element $x \in X$, $\lim_{n\to\infty} f^n(x) = x_f^*$.

Because the space X is compact and $z_n \in X$, $n \in \mathbb{N}$, it results that there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ which is convergent. Let x_z be its limit, i.e. $\lim_{k\to\infty} z_{n_k} = x_z$.

How f is a continuous operator we get that $\lim_{k\to\infty} f(z_{n_k}) = f(x_z)$. But, from $\lim_{n\to\infty} d(z_{n+1}, f(z_n)) = 0$ it results that $\lim_{k\to\infty} d(z_{n_k+1}, f(z_{n_k})) =$
$= d(x_z, f(x_z)) = 0$, that is $f(x_z) = x_z$, therefore x_z is a fixed point of f. How $F_f = \{x_f^*\}$, we have that $x_z = x_f^*$.

Let $x \in X$ be. We have

$$d(z_n, f^n(x)) \leq d(z_n, x_f^*) + d(x_f^*, f^n(x)) \to 0 \text{ as } n \to \infty.$$

The theorem is proved. □

Theorem 3.11. *If f is a continuous two-sided convex contraction operator which satisfies the condition (1), then the fixed point equation $x = f(x)$ is Ulam-Hyers stable.*

Proof. Indeed, let $\varepsilon > 0$ and y^* a solution of the inequation $d(y, f(y)) \leq \varepsilon$. Since f is a c -weakly Picard operator with $c = 1 + \alpha + a_1 + \alpha a_2$, we have that

$$d(x, f^\infty(x)) \leq c\, d(x, f(x)), \text{ for all } x \in X.$$

If we take $x := y^*$ and $f^\infty(x) := x^*$, then we have that $d(y^*, x^*) \leq c\,\varepsilon$. □

Remark 3.9. Some similar results can be obtained for the convex contractive operators of order 2 and for the convex contraction operators of type 2.

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