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Fixed point theorems for correspondences with properties weaker than lower semicontinuity

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ABSTRACT. In this paper, we study existence of fixed points for correspondences having weak continuity properties. The obtained results extend or improve the corresponding results present in literature. We use continuous selection technique and also well-known KKM principle in order to establish our fixed point theorems.

1. INTRODUCTION

In order to find the particularities of some recent results concerning the existence of the fixed points, the reader is referred to [1]-[3], [5]-[22].

Systematic studies apply analytical methods and approaches that attempt to give advantages in expressing simple and general statements. A reader can be overwhelmed by the many techniques out here. This paper tries to be innovative in our approach, which involves two essential steps. The first one is a sustainable development of known theorems, by providing conditions which imply the existence of the fixed points for correspondences for which it is already proved that they have almost fixed points. In this way, we show, for instance, that under our assumptions, the almost lower semicontinuous correspondences enjoy the beautiful property of having fixed points. The second step of our research is distinguished by the way how we embed the KKM principle to obtain relevant and new applications in the domain. Our three last theorems have been built upon this well-known principle. This fact allows us to remain open to the challenge of exploiting further this method to get other possible results.

The rest of the paper is organized as follows. In the following section, some notational and terminological conventions are given. The fixed point theorems for correspondences with weak continuity properties are stated in Section 3. Section 4 contains fixed point results obtained by using the KKM principle. Section 5 presents the conclusions of our research.

2. PRELIMINARIES

Throughout this paper, we shall use the following notation:

2^D denotes the set of all non-empty subsets of the set D. If $D \subset X$, where X is a topological space, cl$D$ denotes the closure of D. We also denote $C(X)$ the family of all non-empty and closed subsets of X. A *paracompact* space is a Hausdorff topological space in which every open cover admits an open locally finite refinement. Metrizable and compact topological spaces are paracompact.

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Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be *upper semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. T is said to be *lower semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. $T : X \to 2^Y$ *has open lower sections* if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

Let X be a subset of a topological vector space and D a nonempty subset of X such that co$D \subset X$.

$T : D \to 2^X$ is called a *KKM correspondence* if co$N \subset T(N)$ for each $N \in \langle D \rangle$, where $\langle D \rangle$ denotes the class of all nonempty finite subsets of D.

Many modern essential results in different areas of mathematical sciences can be derived from the KKM principle. We recall it here. We note that its open version is due to Kim [9] and Shih and Tan [22].

KKM principle Let D be a set of vertices of a simplex S and $T : D \to 2^S$ a correspondence with closed (respectively open) values such that

co$N \subset T(N)$ for each $N \subset D$.

Then, $\bigcap_{z \in D} T(z) \neq \emptyset$.

The following lemma is a consequence of the KKM principle. It will be used to obtain new fixed point theorems in Section 4.

Lemma 2.1. *Let X be a subset of a topological vector space, D a nonempty subset of X such that co$D \subset X$ and $T : D \to 2^X$ a KKM correspondence with closed (respectively open) values. Then $\{T(z)\}_{z \in D}$ has the finite intersection property.*

Remark 2.1. In [14], the author proved that the KKM principle is equivalent to the Fan-Browder fixed point property.

The Fan-Browder fixed point property [14]. Let $R : X \to 2^D$ and $T : X \to 2^X$ be correspondences satisfying:

1) for each $x \in X$, co$R(x) \subset T(x)$;
2) $R^{-1}(z)$ is open (resp. closed) for each $z \in D$;
3) $X = \cup_{z \in M} R^{-1}(z)$ for some $M \in \langle D \rangle$.

Then, T has a fixed point $x_0 \in X$.

Let (X, d) be a metric space, C be a non-empty subset of X and $T : C \to 2^X$ be a correspondence. We denote by $B(x, r) = \{y \in C : d(y, x) < r\}$. If B_0 is a subset of X, then, we will denote $B(B_0, r) = \{y \in C : d(y, B_0) < r\}$, where $d(y, B_0) = \inf_{x \in B_0} d(y, x)$.

3. New fixed point theorems

In this section we focus on establishing new fixed point theorems concerning mainly the almost lower semicontinuous correspondences. Other types of correspondences having weak continuity properties are also condidered. We provide new conditions which assure the existence of the fixed points for a correspondence for which it is easy to prove that it has almost fixed points.

We are starting by presenting the almost lower semicontinuous correspondences.

Let X be a topological space and Y be a normed linear space. The correspondence $T : X \to 2^Y$ is said to be *almost lower semicontinuous (a.l.s.c.) at $x \in X$* (see [4]), if, for any

$\varepsilon > 0$, there exists a neighborhood $U(x)$ of x such that $\bigcap_{z \in U(x)} B(T(z), \varepsilon) \neq \emptyset$. T is *almost lower semicontinuous* if it is a.l.s.c. at each $x \in X$.

Deutsch and Kenderov [4] established the following characterization of a.l.s.c. correspondences.

Lemma 3.2. (see [4]) Let *X be a paracompact topological space, Y be a normed vector space and $T : X \to 2^Y$ be a correspondence having convex values. Then, T is a.l.s.c. if and only if, for each $\varepsilon > 0$, T admits a continuous $\varepsilon-$approximate selection f; that is, $f : X \to Y$ is a continuous single-valued function such that $f(x) \in B(T(x), \varepsilon)$ for each $x \in X$.*

The main result of this section is the following theorem. It concerns the existence of the fixed points for the almost lower semicontinuous correspondences defined on Banach spaces.

Theorem 3.1. *Let C be a compact convex subset of a Banach space X and let $T : C \to 2^C$ be a correspondence such that there exists $n_0 \in \mathbb{N}^*$ with the property that $B(T(C), \frac{1}{n_0}) \subseteq C$. Suppose that T is almost lower semicontinuous with non-empty convex closed values and $T^{-1} : C \to 2^C$ is closed valued.*

Then, T has a fixed point.

Proof. Firstly, let us define $T_n : C \to 2^C$ by $T_n(x) = B(T(x); 1/(n+n_0-1))$ for each $x \in C$ and $n \in \mathbb{N}^*$. Since T is almost lower semicontinuous, according to Lemma 3.2, for each $n \in N$, there exists a continuous function $f_n : C \to C$ such that $f_n(x) \in T_n(x)$ for each $x \in C$. Brouwer-Schauder fixed point theorem assures that, for each $n \in N$, there exists $x_n \in C$ such that $x_n = f_n(x_n)$ and then, $x_n \in T_n(x_n)$.

Then, $d(x_n, T(x_n)) \to 0$ when $n \to \infty$ and since C is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_0 = \lim_{n_k \to \infty} x_{n_k}$. It follows that $d(x_0, T(x_{n_k})) \to 0$ when $n_k \to \infty$.

Let us assume that $x_0 \notin T(x_0)$. Since $\{x_0\} \cap T^{-1}(x_0) = \emptyset$ and X is a regular space, there exists $r_1 > 0$ such that $B(x_0, r_1) \cap T^{-1}(x_0) = \emptyset$. Consequently, for each $z \in B(x_0, r_1)$, we have that $z \notin T^{-1}(x_0)$, which is equivalent with $x_0 \notin T(z)$ or $\{x_0\} \cap T(z) = \emptyset$. The closedness of each $T(z)$ and the regularity of X imply the existence of a real number $r_2 > 0$ such that $B(x_0, r_2) \cap T(z) = \emptyset$ for each $z \in B(x_0, r_1)$, which implies $x_0 \notin B(T(z); r_2)$ for each $z \in B(x_0, r_1)$. Let $r = \min\{r_1, r_2\}$. Hence, $x_0 \notin B(T(z); r)$ for each $z \in B(x_0, r)$, and then, there exists $N^* \in \mathbb{N}$ such that for each $n_k > N^*$, $x_0 \notin B(T(x_{n_k}); r)$ which contradicts $d(x_0, T(x_{n_k})) \to 0$ as $n \to \infty$. It follows that our assumption is false.

Hence, $x_0 \in T(x_0)$. □

Corollary 3.1. *Let C be a compact convex subset of a Banach space X and let $T : C \to 2^C$ be a correspondence. Suppose that T is lower semicontinuous with non-empty convex closed values and $T^{-1} : C \to 2^C$ is closed valued.*

Then, T has a fixed point.

We will obtain some results related to Theorem 3.1 in cases when the correspondences are sub-lower semicontinuous or transfer open-valued. The first property differs very slightly from the one we presented in the beginning of this section. The definitions are given below.

The sub-lower semicontinuous correspondences were defined by Zheng in [25].

Let X be a topological space and Y be a topological vector space. A correspondence $T : X \to 2^Y$ is called *sub-lower semicontinuous* [25] if, for each $x \in X$ and for each neighborhood V of 0 in Y, there exist $z \in T(x)$ and a neighborhood $U(x)$ of x in X such that, for each $y \in U(x)$, $z \in T(y) + V$.

Zheng proved in [25] a continuous selection result for the sub-lower semicontinuous correspondences, which can be used in order to obtain Theorem 3.2 Here is his result.

Lemma 3.3. [25] *Let X be a paracompact topological space, Y be a locally convex topological vector space and let $T : X \to 2^Y$ be a correspondence with convex values. Then, T is sub-lower semicontinuous if and only if, for each neighborhood V of 0 in Y, there exists a continuous function $f : X \to Y$ such that, for each $x \in X$, $f(x) \in T(x) + V$.*

The existence of the fixed points for the sub-lower semicontinuous correspondences is stated now.

Theorem 3.2. *Let C be a compact convex subset of a locally convex regular topological vector space X, and let $T : C \to 2^C$ be a correspondence. Suppose that T is sub-lower semicontinuous with non-empty convex closed values and $T^{-1} : C \to 2^C$ is closed valued.*

Then, T has a fixed point.

The proof of Theorem 3.2 is similar to the one of Theorem 3.1, but it relies on Lemma 3.3

Corollary 3.2. *Let C be a compact convex subset of a locally convex regular topological vector space X, and let $T : C \to 2^C$ be a correspondence. Suppose that T is lower semicontinuous with non-empty convex closed values and $T^{-1} : C \to 2^C$ is closed valued.*

Then, T has a fixed point.

Theorem 3.3 concerns the transfer open-valued correspondences. Lemma 3.4 is crucial for the proof.

Let X and Y be two topological spaces. The correspondence $T : X \to 2^Y$ is said to be *transfer open-valued* (see [1]) if, for any $x \in X$ and $y \in T(x)$, there exists an $z \in X$ such that $y \in \mathrm{int}_Y T(z)$.

The proof of the next lemma is included in the proof of Theorem 3.1 in [1], in the particular case when $I = \{1\}$, $S = T$ and K is compact.

Lemma 3.4. *Let K be a non-empty compact convex subset of a Hausdorff topological vector space E and let $T : K \to 2^K$ be a correspondence with non-empty convex values. If $K = \bigcup\{int_K T^{-1}(y) : y \in K\}$ (or T^{-1} is transfer open-valued), then, T has a continuous selection.*

Theorem 3.3. *Let C be a compact convex subset of a regular topological vector space X and let $T : C \to 2^C$ be a correspondence with non-empty closed convex values, such that $T^{-1} : C \to 2^C$ is closed valued. Suppose that, for each open neighborhood V of the origin, the correspondence $(S^V)^{-1} : C \to 2^C$ is transfer open-valued, where $S^V(x) = (T(x) + V) \cap C$ for each $x \in C$.*

Then, T has a fixed point.

Proof. By using Lemma 3.4, we prove that for each open neighborhood V of the origin, there exists a continuous function $f_V : C \to C$ such that $f_V(x) \in S^V(x) = (T(x) + V) \cap C$ for each $x \in C$. The proof in similar to the proof of Theorem 3.1. $\square$

In [13], Park established the following result. Further, we will use it to derive Theorem 3.5.

Theorem 3.4. (see [13]) *Let X be a convex subset of a topological vector space E. Let $T : X \to 2^X$ be an upper semicontinuous (respectively a lower semicontinuous) correspondence with nonempty convex values such that the following holds:.*

(Z) for each neighborhood U of 0 in E, there exists a neighborhood V of the origin in E such that
co$(V \cap (T(X) - T(X)) \subset U$.
If $T(X)$ is totally bounded, then, for each neighborhood U of 0 in E, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + V) \neq \emptyset$.

We state Theorem 3.5, by using Park's theorem enunciated above. We note that the regularity of the space X is essential in the proof.

Theorem 3.5. *Let X be a compact convex subset of a regular topological vector space E. Let $T : X \to 2^X$ be a lower semicontinuous correspondence with nonempty closed convex values such that the following holds:.*

i) (Z) for each neighborhood U of 0 in E, there exists a neighborhood V of the origin in E such that
co$(V \cap (T(X) - T(X)) \subset U$ and
ii) $T^{-1} : X \to 2^X$ is closed valued.
Then, T has fixed points.

Proof. For each symmetric open neighborhood W of 0 in E, there exists a symmetric open neighborhood U of 0 in E such that $U + U \subset W$. According to Theorem 3.4, for each such a neighborhood U, there exists points x_U, $y_U \in X$ such that $x_U \in T(x_U) + U$ and $x_U \in y_U + U$.

Since X is compact, $\{x_U\}$ has a convergent subsequence $\{x_{U'}\}$. Let x_0 be the limit of $\{x_{U'}\}$. It follows that $x_0 \in T(x_{U'}) + W'$ for each symmetric open neighborhood U' of 0 with the property that $U' + U' \subset W'$.

Let us assume that $x_0 \notin T(x_0)$. Since $\{x_0\} \cap T^{-1}(x_0) = \emptyset$ and X is a regular space, there exists V_1 an open neighborhood of 0 such that $(x_0 + V_1) \cap T^{-1}(x_0) = \emptyset$. Consequently, for each $z \in (x_0 + V_1)$, we have that $z \notin T^{-1}(x_0)$, which is equivalent with $x_0 \notin T(z)$ or $\{x_0\} \cap T(z) = \emptyset$. The closedness of each $T(z)$ and the regularity of X imply the existence of V_2, an open neighborhood of 0, such that $(x_0 + V_2) \cap T(z) = \emptyset$ for each $z \in x_0 + V_1$, which implies $x_0 \notin T(z) + V_2$ for each $z \in x_0 + V_1$. Let $V = V_1 \cap V_2$. Hence, $x_0 \notin T(z) + V$ for each $z \in x_0 + V$, and then, there exists U^*, an open neighborhood of 0 such that for each symmetric open neighborhood of 0, U', with the property that $U' \subset U^*$, it is true that $x_0 \notin T(x_{U'}) + V$ and therefore, $x_0 \notin T(x_{U' \cap V}) + W' \cap V$. The last assertion contradicts $x_0 \in T(x_{U'}) + W'$ for each symmetric open neighborhood U' of 0 with the property that $U' + U' \subset W'$. It follows that our assumption is false.

Hence, $x_0 \in T(x_0)$. □

Remark 3.2. The case of the upper semicontinuous correspondences is treated by Park in [13].

Corollary 3.3 is obtained easily from Theorem 3.4, if we take into consideration that any convex subset of a locally convex topological vector space is of the Zima type (Z). For a discussion of this fact, the reader is referred to [5] and [6].

Corollary 3.3. *Let X be a compact convex subset of a locally convex regular topological vector space E. Let $T : X \to 2^X$ be a lower semicontinuous correspondence with nonempty closed convex values such that $T^{-1} : X \to 2^X$ is closed valued.*

Then, T has fixed points.

4. APPLICATIONS OF THE KKM PRINCIPLE

This section is mainly dedicated to establishing some new fixed point theorems for correspondences, by using the KKM principle and Lemma 2.1 derived from it. For other results obtained in this way, the reader is referred, for instance, to [7], [8], [12], [13].

Our first result in this section is stated now. It concerns the existence of the fixed points and its particularity is given by the transfer open-valuedness of the involved correspondences.

Theorem 4.6. *Let X be a subset of a topological vector space E, D a nonempty subset of X such that co$D \subset X$ and $S : D \to 2^X$, $T : X \to 2^X$ be correspondences.*

Assume that S and T satisfy the following conditions:
i) S is transfer-open valued;
ii) there exists $M \in \langle D\rangle$ and for each $x \in M$ and $y \in S(x)$, there exists $z_{x,y} \in X$ *such that* $y \in int_X S(z_{x,y}) \cap S(x)$ and $\bigcup_{x\in M}(\bigcup_{y\in S(x)} int_X S(z_{x,y})) = X$;
iii) $\bigcup_{y\in S(x)} int_X S(z_{x,y}) \subset T(x)$ *for each* $x \in D$;
iv) T^{-1} is convex valued.
Then, T has fixed points.

Proof. Let us define $F : D \to 2^X$ by $F(x) := X\setminus \bigcup_{y\in S(x)} \text{int}_X S(z_{x,y})$ for each $x \in D$.
Then, F is closed valued and
$\bigcap_{x\in M} F(x) = X\setminus \bigcup_{x\in M}(\bigcup_{y\in S(x)} \text{int}_X S(z_{x,y})) = \emptyset$.
According to Lemma 1, we can conclude that F is not a KKM correspondence. Thus, there exists $N \in \langle D\rangle$ such that co$N \subsetneq F(N) = \bigcup_{x\in N}(X\setminus \bigcup_{y\in S(x)} \text{int}_X S(z_{x,y}))$.
Hence, there exists $x_0 \in$co$N$ with the property that $x_0 \in \bigcup_{y\in S(x)} \text{int}_X S(z_{x,y})$ for each $x \in N$. Therefore, there exists $x_0 \in$co$N$ such that $x_0 \in T(x)$ for each $x \in N$, which implies $N \subset T^{-1}(x_0)$. Further, it is true that co$N \subset$co$T^{-1}(x_0) = T^{-1}(x_0)$. Consequently, $x_0 \in$co$N \subset$co$T^{-1}(x_0) = T^{-1}(x_0)$, which means that $x_0 \in T(x_0)$, that is, x_0 is a fixed point for T. □

The second theorem established in this section is also based on the application of the KKM principle. It asserts the existence of the fixed points in case when the correspondences have the local intersection property.

The correspondences which satisfy the local intersection property are defined below.
Let X, Y be topological spaces. The correspondence $T : X \to 2^Y$ has the *local intersection property (see* [23]) if $x \in X$ with $T(x) \neq \emptyset$ implies the existence of an open neighborhood $U(x)$ of x such that $\cap_{z\in U(x)} T(z) \neq \emptyset$.

Theorem 4.7. *Let X be a subset of a topological vector space E, D a nonempty subset of X such that co$D \subset X$ and $S : D \to 2^X$, $T : X \to 2^X$ be correspondences.*

Assume that S and T satisfy the following conditions:
i) S is closed valued;

ii) there exists W, *an open neighborhood of* 0 in X, *such that* $R(x) = \bigcap_{z\in x+W} S(z)$ *is nonempty for each* $x \in X$ *with the property that* $S(x) \neq \emptyset$;
iii) there exists $M \in \langle D\rangle$ *such that* $\bigcup_{x\in M} R(x) = X$;
iv) $\mathrm{co}R^{-1}(y) \subset T^{-1}(y)$ *for each* $y \in X$.
Then, T *has fixed points.*

Proof. Let us define $F : D \to 2^X$ by $F(x) := X\setminus \bigcap_{z\in x+W} S(z)$ for each $x \in D$.
Then, F is open valued and assumption iii) implies
$\bigcap_{x\in M} F(x) = \bigcap_{x\in M}(X\setminus \bigcap_{z\in x+W} S(z))) = X\setminus \bigcup_{x\in M}(\bigcap_{z\in x+W} S(z)) =$
$= X\setminus \bigcup_{x\in M} R(x) = \emptyset$.
According to Lemma 2.1, we can conclude that F is not a KKM correspondence. Thus, there exists $N \in \langle D\rangle$ such that $\mathrm{co}N \subsetneq F(N) = \bigcup_{x\in N}(X\setminus \bigcap_{z\in x+W} S(z))$.

Hence, there exists $x_0 \in \mathrm{co}N$ with the property that $x_0 \in \bigcap_{z\in x+W} S(z)$ for each $x \in N$. Therefore, $R(x)$ is nonempty for each $x \in N$ and $N \subset R^{-1}(x_0) \subset \mathrm{co}R^{-1}(x_0)$. Consequently, it is true that $x_0 \in \mathrm{co}N \subset \mathrm{co}R^{-1}(x_0) \subset T^{-1}(x_0)$, which means that $x_0 \in T(x_0)$, that is, x_0 is a fixed point for T. □

In a similar way, we establish Theorem 4.8, which concerns the existence of the fixed points for the sub-lower semicontinuous correspondences.

Theorem 4.8. *Let* X *be a subset of a topological vector space* E, D *a nonempty subset of* X *such that coD* $\subset X$ *and* $S : D \to 2^X$, $T : X \to 2^X$ *be correspondences.*

Assume that S *and* T *satisfy the following conditions:*
i) S *is closed valued;*
ii) there exist W and V, *open neighborhoods of* O in X, *such that* $R(x) = \bigcap_{z\in x+W}(S(z)+V)$ *is nonempty for each* $x \in X$ *with the property that* $S(x) \neq \emptyset$;
iii) there exists $M \in \langle D\rangle$ *such that* $\bigcup_{x\in M} R(x) = X$;
iv) $\mathrm{co}R^{-1}(y) \subset T^{-1}(y)$ *for each* $y \in X$.
Then, T *has fixed points.*

Remark 4.3. Since the proofs of Theorems 4.7 and Theorem 4.8 are based on the KKM principle, in view of Remark 1, we can conclude that the specified results are consequences of the Fan-Browder fixed point property.

5. Concluding remarks

In this paper, we have firstly proved the existence of fixed points for almost lower semicontinuous correspondences and correspondences with weak continuity properties. Our research extends on some results which exist in literature. Secondly, we obtained new fixed point theorems by applying the KKM principle. It is an interesting problem to find new applications of this well-known principle in the fixed point theory. This study will be continued by considering abstract convex spaces and generalized KKM theorems.

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