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A new type of contractions that characterize metric completeness

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ABSTRACT. We prove that a new type of contractions characterizes the metric completeness of the underlying space. We also discuss the Meir-Keeler fixed point theorem.

1. INTRODUCTION

Let (X, d) be a complete metric space and T a selfmap of X. Then T is called a *contraction* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$.

T is called *Kannan* if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$$

for all $x, y \in X$.
The following famous theorem is referred to as the Banach contraction principle.

Theorem 1.1. (Banach [1]) *Let (X, d) be a complete metric space and let T be a contraction on X. Then T has a unique fixed point.*

This theorem is a very forceful and simple, and it became a classical tool in nonlinear analysis. It has many generalizations, see [4], [5], [6], [12], [17], [18], [21], [23], [31], [33]. Connel [9] gave an example of a noncomplete metric space X on which every contraction on X has a fixed point. Thus, Theorem 1 cannot characterize the metric completeness of X which means the notion of contraction is too strong from this point of view. Kannan [13] proved that if X is complete, then every Kannan mapping has a fixed point. Kannan's theorem is independent of the Banach contraction principle. Subrahmanyiam [32] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point. Also several mathematicians have studied the metric completeness. For other results in this setting, see [11], [24], [27], [34] and others.

In 2008 Suzuki [33] introduced a new type of mapping and presented a generalization of the Banach contraction principle in which the completeness can also be characterized by the existence of fixed point of these mappings.

Theorem 1.2. ([33]) *Let (X, d) be a complete metric space and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

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$$\theta(r) = \begin{cases} 1 & \text{if} \quad 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)/r^2 & \text{if} \quad (\sqrt{5}-1)/2 \le r \le 1/\sqrt{2}, \\ 1/(1+r) & \text{if} \quad 1/\sqrt{2} \le r < 1. \end{cases} \tag{1.1}$$

Assume that there exists $r \in [0,1)$ such that $\theta(r)d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le rd(x,y)$ for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Its further outcomes by Kikkawa and Suzuki [14], [15], Moţ and Petruşel [22], Dhompongsa and Yingtaweesittikul [10], Popescu [25], Singh and Mishra [29], [30] are important contributions to metric fixed point theory.

Suzuki [33] also generalized the Meir-Keeler fixed point theorem [21].

Theorem 1.3. ([33]) *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that for each $\epsilon > 0$, there exists $\delta > 0$ such that*
- *$(1/2)d(x,Tx) < d(x,y)$ and $d(x,y) < \epsilon + \delta$ imply $d(Tx,Ty) \le \epsilon$ and*
- *$(1/2)d(x,Tx) < d(x,y)$ implies $d(Tx,Ty) < d(x,y)$*

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

The Meir-Keeler fixed point theorem [21] is a generalization of the Banach contraction principle(Theorem 1.1), but Theorem 1.3 is not a generalization of Theorem 1.2. However, we note that $\lim_{r\to 1-0} \theta(r) = 1/2$ and $1/2$ is the best constant.

In this paper, we prove that a new type of contractions characterizes the metric completeness. The direction of our extension is new, very simple and inspired by Theorem 1.2. We also generalize the Meir-Keeler fixed point theorem [21] and Theorem 1.3.

2. (S, R)-CONTRACTIONS

Popescu [26] introduced a new type of contractive operator and proved the following theorem.

Theorem 2.4. ([26]) *Let (X, d) be a complete metric space and $T : X \to X$ be a (s, r)-contractive singlevalued operator:*
there exist $r \in [0,1)$ and $s > r$ such that

$$x, y \in X \text{ with } d(y,Tx) \le sd(y,x) \text{ implies } d(Tx,Ty) \le rM_T(x,y)$$

where

$$M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then T has a fixed point. Moreover, if $s \ge 1$ then T has a unique fixed point.

The quantity $M_T(x,y)$ appearing in Theorem 2.4 is due to Ćirić [8] and is involved in Rhoades classification [28] as condition number (21). For more bibliografic details regarding this condition, see Berinde [2] for self mape case and Berinde and Păcurar [3]. As a direct consequences of Theorem 2.4, we obtain the following result.

Corollary 2.1. *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exist $r \in [0,1)$ and $s > r$ such that*

$$d(y,Tx) \le sd(y,x) \text{ implies } d(Tx,Ty) \le rd(x,y), \tag{2.2}$$

for all $x, y \in X$. Then there exists a fixed point z of T. Further, if $s \geq 1$, then there exists a unique fixed point of T.

It is obvious that the set of our contractions in Corollary 2.1 includes that of the usual contractios. However, our contractions and Kannan mappings are independent. We next show it.

Example 2.1. Define a complete metric space X by $X = \{(0,3),(0,2)\} \cup \{(a,0) : a \in [0,6]\}$ and let d be the euclidian metric. Define a mapping T on X by

$$T(x_1, x_2) = \begin{cases} (0,0) & \text{if} \quad (x_1,x_2) = (6,0), \\ (0,3) & \text{if} \quad (x_1,x_2) = (a,0) : a \in (0,6), \\ (0,2) & \text{if} \quad x_1 = 0. \end{cases} \tag{2.3}$$

Then T is a Kannan mapping, but T does not satisfy the assumption in Corollary 2.1.

Proof. Let $A = d(Tx, Ty)$ and $B = (1/3)[d(x, Tx) + d(y, Ty)]$. We have the following cases:
1) $x = (6,0)$. If $y = (a,0), a \in (0,6)$, then $A = 3$ and $B \geq (1/3)(6+3) = 3$. If $y = (0,3)$, then $A = 2$ and $B = 7/3$. If $y = (0,2)$, then $A = 2$ and $B = 2$. If $y = (0,0)$, then $A = 2$ and $B = 8/3$.
2) $x = (a,0), a \in (0,6)$. If $y = (b,0), b \in (0,6)$, then $A = 0$. If $y = (0,0)$, then $A = 1, B \geq 5/3$. If $y = (0,3)$, then $A = 1, B \geq 4/3$. If $y = (0,2)$, then $A = 1, B \geq 1$. If $y = (6,0)$, then $A = 3, B \geq 3$.
3) $x = (0,0)$. If $y = (0,3)$ or $y = (0,2)$, then $A = 0$. If $y = (6,0)$, then $A = 2, B = 8/3$. If $y = (a,0), a \in (0,6)$, then $A = 1, B \geq 5/3$.
4) $x = (0,3)$. If $y = (6,0)$, then $A = 2$ and $B = 7/3$. If $y = (a,0), a \in (0,6)$, then $A = 1, B \geq 4/3$. If $y = (0,0)$ or $y = (0,2)$, then $A = 0$.
5) $x = (0,2)$. If $y = (6,0)$, then $A = 2$ and $B = 2$. If $y = (a,0), a \in (0,6)$, then $A = 1, B \geq 1$. If $y = (0,0)$ or $y = (0,3)$, then $A = 0$.
6) $x = y$. Then $A = 0$.
In all cases we have $A \leq B$. Therefore T is a Kannan mapping with $\alpha = 1/3$.

Now we show that for every $r \in [0,1)$ there exist $x_r, y_r \in X$ such that

$$d(y_r, Tx_r) \leq rd(x_r, y_r) \text{ and } d(Tx_r, Ty_r) > rd(x_r, y_r).$$

We fix $r \in (0,1)$. Let $x_r = (6,0), y_r = (a,0), a = \frac{6r}{1+r} \in (0,6)$. Then we have $d(y_r, Tx_r) = a, d(x_r, y_r) = 6 - a = \frac{6}{1+r}, d(Tx_r, Ty_r) = 3$, so $d(y_r, Tx_r) = rd(x_r, y_r)$. Since $r < 1$ we get $\frac{6r}{1+r} < 3$, hence $d(Tx_r, Ty_r) > rd(x_r, y_r)$. If $r = 0$ we can take $x_r = (6,0), y_r = (0,3)$. Therefore T does not satisfy the assumption in Corollary 2.1. □

Example 2.2. Define a complete metric space X by $X = \{-1, 0, 1, 2\}$ and a mapping T on X by $Tx = 0$ if $x \in \{0,1,2\}$ and $Tx = 1$ if $x = -1$. Then T satisfy the assumption in Corollary 2.1, but T is not a Kannan mapping.

Proof. We show that T satisfy the assumption in Corollary 2.1 with $r = 1/2$ and $s = 2/3$. We have the following cases:
1) $x = y$. Then $d(Tx, Ty) = 0 \leq rd(x, y)$.
2) $x = -1$. If $y = 0$, then $d(y, Tx) = 1 > (2/3)d(x, y)$. If $y \in \{1, 2\}$, then $d(Tx, Ty) = 1 \leq (1/2)d(x, y)$.
3) $x = 0$. If $y \in \{-1, 1\}$, then $d(y, Tx) = 1 > (2/3)d(x, y) = 2/3$. If $y = 2$, then $d(y, Tx) = 2 > (2/3)d(x, y) = 4/3$.
4) $x = 1$. If $y = -1$, then $d(Tx, Ty) = 1 \leq (1/2)d(x, y) = 1$. If $y \in \{0, 2\}$, then $d(Tx, Ty) = 0 \leq rd(x, y)$.

5) $x = 2$. If $y = -1$, then $d(Tx, Ty) = 1 \leq (1/2)d(x, y) = 3/2$. If $y \in \{0, 1, 2\}$, then $d(Tx, Ty) = 0 \leq (1/2)d(x, y)$.
Since $d(T-1, T0) = 1 \geq (1/2)[d(-1, T-1) + d(0, T0)] = 1$, T is not a Kannan mapping. We note that $\theta(r)d(0, T0) = 0 \leq d(0, -1)$ and $d(T0, T-1) = 1 > r = rd(-1, 0)$, so T does not satisfy the assumption in Theorem 1.2. □

3. Metric completeness

In this section, we discuss the metric completnes.

Theorem 3.5. *For a metric space (X, d), the following are equivalent:*
(i) X is complete.
(ii) There exist $r \in (0, 1)$, $s > r$ such that every mapping T on X satisfying the following has a fixed point:

$$d(y, Tx) \leq sd(y, x) \text{ implies } d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$.

Proof. By Corollary 2.1, (i) implies (ii). Let us prove (ii) implies (i). We assume (ii). Arguing by contradiction, we also assume that X is not complete. That is, there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function from X to $[0, \infty)$ by $f(x) = \lim_n d(x, u_n)$ for every $x \in X$. We also note that:
(a) $f(x) - f(y) \leq d(x, y) \leq f(x) + f(y)$ for $x, y \in X$,
(b) $f(x) > 0$ for all $x \in X$,
(c) $\lim_n f(u_n) = 0$.
Define a mapping T on X as follows: for each $x \in X$ since (b) and (c) hold, there exists an integer $\nu \geq 1$ such that $f(u_\nu) \leq \frac{r}{3+3s} f(x)$. We put $Tx = u_\nu$. Then it is obvious that $f(Tx) \leq \frac{r}{3+3s} f(x)$ and $Tx \in \{u_n : n \geq 1\}$ for all $x \in X$. Since $\frac{r}{3+3s} < 1$, by (b) we have $f(Tx) < f(x)$, so $Tx \neq x$ for all $x \in X$. Hence T does not have a fixed point. Fix $x, y \in X$ with $d(y, Tx) \leq sd(y, x)$. Then $d(x, y) \geq (1/s)d(y, Tx)$. In the case where $f(x) > 2f(y)$, we have

$$\begin{aligned} d(Tx, Ty) &\leq f(Tx) + f(Ty) < (r/3)[f(x) + f(y)] \\ &< (r/3)[f(x) + f(y)] + (2r/3)[f(x) - 2f(y)] \\ &= r[f(x) - f(y)] \leq rd(x, y). \end{aligned}$$

In the other case, where $f(x) \leq 2f(y)$, we have

$$\begin{aligned} d(x, y) &\geq (1/s)d(y, Tx) \geq (1/s)[f(y) - f(Tx)] \geq (1/s)[f(y) - \frac{r}{3+3s} f(x)] \\ &\geq (1/s)[f(y) - \frac{2r}{3+3s} f(y)] = (1/s)\frac{3+3s-2r}{3+3s} f(y) > \frac{3}{3+3s} f(y). \end{aligned}$$

and hence

$$\begin{aligned} d(Tx, Ty) &\leq f(Tx) + f(Ty) < \frac{r}{3+3s}[f(x) + f(y)] \\ &\leq \frac{3r}{3+3s} f(y) \leq rd(x, y). \end{aligned}$$

By (ii), we get that T has a fixed point. This is a contradiction. Hence we obtain that X is complete. □

4. The Meir-Keeler theorem

In this section, we prove a generalization of the Meir-Keeler fixed point theorem [21] and Theorem 1.3. See also [7], [19], [20].

Theorem 4.6. *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that for each $\epsilon > 0$, there exists $\delta > 0$ such that*
• *$d(y, x) \geq d(y, Tx)$ and $d(y, x) \neq (1/2)d(x, Tx)$ imply $d(Tx, Ty) < d(x, y)$*
• *$d(y, x) \geq d(y, Tx), d(y, x) \neq (1/2)d(x, Tx)$ and $d(x, y) < \epsilon + \delta$ imply $d(Tx, Ty) \leq \epsilon$,*
for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Proof. If $Tx \neq x$, then it is obvious that $d(Tx, x) \geq d(Tx, x) > 0$ and $d(Tx, x) \neq (1/2)d(Tx, x)$. So, by hypothesis,
$$d(Tx, T^2x) < d(x, Tx),$$
for all $x \in X$ with $Tx \neq x$. Hence for all $x \in X$ we have
$$d(Tx, T^2x) \leq d(x, Tx).$$
Fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$ for $n \in N$. Since $\{d(u_n, u_{n+1})\}$ is a nonincreasing sequence, then $\{d(u_n, u_{n+1})\}$ converges to some $\alpha \geq 0$. Arguing by contradiction, we assume $\alpha > 0$. Then $\{d(u_n, u_{n+1})\}$ is strictly decreasing. Hence $d(u_n, u_{n+1}) > \alpha$ for every $n \in N$. By hypothesis, there exists $\delta > 0$ such that
• $d(y, x) \geq d(y, Tx), d(y, x) \neq (1/2)d(x, Tx)$ and $d(x, y) < \alpha + \delta$ imply $d(Tx, Ty) \leq \alpha$.
From the definition of α, there exists $j \in N$ such that $d(u_j, u_{j+1}) < \alpha + \delta$. So we have $d(u_{j+1}, u_{j+2}) \leq \alpha$. This is a contradiction. Therefore $\alpha = 0$. That is, $\lim d(u_n, u_{n+1}) = 0$ holds.

Now let $\epsilon > 0$. Then there exists $\delta \in (0, \epsilon)$ such that
• $d(y, x) \geq d(y, Tx), d(y, x) \neq (1/2)d(x, Tx)$ and $d(x, y) < \epsilon + \delta$ imply $d(Tx, Ty) \leq \epsilon$.
Let $\nu \in N$ such that $d(u_n, u_{n+1}) < \delta/4$ for all $n \in N$ with $n \geq \nu$. We shall show by induction that

$$d(u_n, u_{n+m}) \leq \epsilon + \delta/2 \tag{4.4}$$

for all $n \in N$ with $n \geq \nu, m \geq 1$. It is obvious that (4) holds when $m = 1$. We assume (4) holds for all $n \in N$ with $n \geq \nu$, and some $m \in N, m \geq 1$. In the case where $d(u_{n+m+1}, u_n) < d(u_{n+m+1}, u_{n+1})$, we have
$$d(u_{n+m+1}, u_{n+1}) \leq \epsilon + \delta/2,$$
so
$$d(u_{n+m+1}, u_n) < \epsilon + \delta/2.$$
In the other case, where $d(u_{n+m+1}, u_n) \geq d(u_{n+m+1}, u_{n+1})$, we have two subcases: $d(u_{n+m+1}, u_n) = (1/2)d(u_n, u_{n+1})$ or $d(u_{n+m+1}, u_n) \neq (1/2)d(u_n, u_{n+1})$. In the first subcase
$$d(u_{n+m+1}, u_n) \leq d(u_{n+m+1}, u_{n+1}) + d(u_n, u_{n+1}) < \delta/8 + \delta/4 < \epsilon + \delta/2.$$
In the second subcase, taking $y = u_{n+m+1}, x = u_n$ in the hyphotesis, we get
$$d(u_{n+m+2}, u_{n+1}) < d(u_{n+m+1}, u_n).$$
But
$$d(u_{n+m+1}, u_n) \leq d(u_{n+m}, u_n) + d(u_{n+m+1}, u_{n+m}) < \epsilon + \delta/2 + \delta/4 < \epsilon + \delta.$$
By hypothesis, we obtain
$$d(u_{n+m+2}, u_{n+1}) \leq \epsilon.$$

Hence

$d(u_{n+m+1}, u_n) \leq d(u_{n+m+1}, u_{n+m+2}) + d(u_{n+m+2}, u_{n+1}) + d(u_{n+1}, u_n) < \epsilon + \delta/4 + \delta/4 = \epsilon + \delta/2.$

So, by induction, (9) holds for every $n \in N$ with $n \geq \nu, m \geq 1$. Therefore we have shown

$$\lim_{n\to\infty} \sup_{m>n} d(u_n, u_{n+1}) = 0.$$

This implies that $\{u_n\}$ is Cauchy. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

We shall show that z is a fixed point of T, dividing the following two cases:

- There exists $\nu \in N$ such that $u_\nu = u_{\nu+1}$.
- $u_n \neq u_{n+1}$ for all $n \in N$.

In the first case, $u_n = u_\nu$ for all $n \in N$ with $n \geq \nu$. Since $\{u_n\}$ converges to z, we have $u_n = z$ for all $n \in N$. This implies $Tz = z$. In the second case, we note $u_n \neq Tu_n$, for $n \in N$, so $\{d(u_n, u_{n+1})\}$ is strictly decreasing. If we assume that there exists a subsequence $\{u_{n(k)}\}$ such that
$d(z, u_{n(k)+1}) \leq d(z, u_{n(k)})$ and $d(z, u_{n(k)}) \neq (1/2)d(u_{n(k)}, u_{n(k)+1})$,
then by hypothesis

$$d(Tz, u_{n(k)+1}) < d(z, u_{n(k)})$$

for all $k \in N$. Letting k tend to ∞ we get $d(Tz, z) \leq 0$, that is, $Tz = z$. In other case, there exists $\nu \in N$ such that $d(z, u_{n+1}) > d(z, u_n)$ or $d(z, u_n) = (1/2)d(u_n, u_{n+1})$ for every $n \geq \nu$. But $d(z, u_n) = (1/2)d(u_n, u_{n+1})$ implies

$$d(z, u_{n+1}) \geq d(u_n, u_{n+1}) - d(z, u_n) = (1/2)d(u_n, u_{n+1}) = d(z, u_n).$$

So, for every $n \geq \nu$ we have

$$d(z, u_{n+1}) \geq d(z, u_n).$$

Hence $\{d(z, u_n)\}$ is an increasing sequence. This is a contradiction, because $\lim_n d(z, u_n) = 0$, and $d(z, u_n) > 0$ for every $n \in N$. Therefore, we have shown that z is a fixed point of T.

Finally, arguing by contradiction, suppose that y is another fixed point of T. We have $d(y, Tz) = d(y, z)$ and $d(y, z) \neq (1/2)d(z, Tz) = 0$,
so, by hypothesis

$$d(y, z) = d(Ty, Tz) < d(y, z),$$

which is a contradiction. Therefore, the fixed point of T is unique. This completes the proof. □

We note that $d(y, Tx) \leq d(y, x)$, implies

$$d(y, x) \geq d(x, Tx) - d(y, Tx) \geq d(x, Tx) - d(y, x),$$

so

$$d(y, x) \geq (1/2)d(x, Tx).$$

If additionally, $d(y, x) \neq (1/2)d(x, Tx)$, we get

$$d(y, x) > (1/2)d(x, Tx).$$

Therefore our conditions from Theorem 4.6 are weaker than the conditions from Theorem 1.3.

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