CARPATHIAN J. MATH. **Online version available at** `http://carpathian.ubm.ro`
31 (2015), No. 3, 389 - 394 **Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401**

Some fixed point theorems via partial order relations without the monotone property

WARUT SAKSIRIKUN and NARIN PETROT

ABSTRACT. The main aim of this paper is to consider some fixed point theorems via a partial order relation in complete metric spaces, when the considered mapping may not satisfy the monotonic properties. Furthermore, we also obtain some couple fixed point theorems, which can be viewed as an extension of a result that was presented in [V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 7347–7355].

1. INTRODUCTION

Banach contraction principle is one of the essential results of analysis. It is widely considered as the source of metric fixed point theory. In 2003, Ran and Reurings [8] considered some fixed point theorems in partially ordered metric spaces, which can be viewed as an improvement of the Banach contraction principle, by assuming the continuity and monotonicity of considered mappings. Afterward, Bhaskar and Lakshmikantham [6] obtained some coupled fixed point theorems on partially ordered metric spaces and showed that it can be applied for considering a periodic boundary value problem. Recently, Berinde [2] improved the contractive condition, which was considered by Bhaskar and Lakshmikantham [6], by showing a following coupled fixed point theorem.

Theorem 1.1. ([2]) *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mixed monotone mapping for which there exists a constant $k \in [0, 1)$ such that for each $x \succeq u$ and $y \preceq v$,*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \le k[(d(x, u) + d(y, v))].$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \ \ \textit{and} \ \ y_0 \succeq F(y_0, x_0),$$

or

$$x_0 \succeq F(x_0, y_0) \ \ \textit{and} \ \ y_0 \preceq F(y_0, x_0),$$

then, there exist $\bar{x}, \bar{y} \in X$ such that

$$\bar{x} = F(\bar{x}, \bar{y}) \ \ \textit{and} \ \ \bar{y} = F(\bar{y}, \bar{x}).$$

For more works on fixed point theory on partially ordered metric spaces, we refer the readers to [1, 3, 5, 7, 10, 11], for example.

Motivated by the above literature, in this paper we present some fixed point theorems in partially ordered metric space but without assuming the monotonic properties on

Received: 01.09.2014; In revised form: 02.03.2015; Accepted: 05.03.2015
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *Fixed point, coupled fixed point, R-function, monotone property, mixed monotone property.*
Corresponding author: N. Petrot; narinp@nu.ac.th

the considered mapping. Further, we provide some coupled fixed point theorems, which is an extended version of Theorem 1.1 to a more widely class of mapping.

Indeed, we are planning to consider a new class of mappings by using the following concept, namely, R-function (see [9]). A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an R-function if

$$\limsup_{s \to t^+} \varphi(s) < 1 \ \text{ for all } \ t \in [0, \infty).$$

Here are examples that related to the class of R-functions.

Example 1.1. Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) := \begin{cases} 2t, & \text{if } t \in [0, \frac{1}{2}), \\ 0, & \text{if } t \in [\frac{1}{2}, \infty). \end{cases}$$

Then, φ is an R-function.

Example 1.2. Let $\varphi : [0, \infty) \to [0, 1)$ be defined by

$$\varphi(t) := \begin{cases} \frac{\sin t}{t}, & \text{if } t \in (0, \frac{1}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

Since $\limsup_{s \to 0^+} \varphi(s) = 1$, φ is not an R-function.

Remark 1.1. If $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is an R-function. So the set of R-functions is a rich class, and subsequently, the results those related to R-functions are of interesting.

The following lemma is very useful for our works.

Lemma 1.1. [4] *A function $\varphi : [0, \infty) \to [0, 1)$ is an R-function if and only if for each $t \in [0, \infty)$, there exist $r_t \in [0, 1)$ and $\epsilon_t > 0$ such that $\varphi(s) \leq r_t$ for all $s \in [t, t + \epsilon_t)$.*

For more information and a characterization of R-function, one may consult a paper by Du [4].

2. Fixed point theorems

Let X be a nonempty set and $T : X \to X$ be a mapping. For $x_0 \in X$, we define an iterative sequence $\mathcal{O}_T(x_0)$ by

$$\mathcal{O}_T(x_0) := \{x_0, Tx_0, T^2x_0, T^3x_0, ...\}.$$

We start with a key lemma.

Lemma 2.2. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Assume that there exist an R-function $\varphi : [0, \infty) \to [0, 1)$ and a nonempty T-invariant set C (i.e. $T(C) \subset C$) in X such that*

$$d(Tx, T^2x) \leq \varphi(d(x, Tx))d(x, Tx), \qquad \textit{for all} \quad x \in C. \tag{2.1}$$

If $x_0 \in C$, then $\mathcal{O}_T(x_0)$ is a Cauchy sequence.

Proof. Let us start by an element $x_0 \in C$. Thus, by the T-invariantness of C, it follows that $\mathcal{O}_T(x_0) \subset C$. Subsequently, by (2.1), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(T^{n+1}x_0, T^{n+2}x_0) \\ &\leq \varphi(d(T^n x_0, T^{n+1}x_0))d(T^n x_0, T^{n+1}x_0) \\ &= \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}), \quad \text{for each} \quad n \in \mathbb{N}, \end{aligned}$$

Using this one, together with a fact that $0 < \varphi(t) < 1$ for all $t \in [0, \infty)$, we can conclude that the sequence $\{d(x_n, x_{n+1})\}$ is a strictly decreasing of nonnegative real numbers. Let $\delta \geq 0$ be such that $\delta = \lim_{n\to\infty} d(x_n, x_{n+1})$. In view of Lemma 1.1, by using this δ, we can find $\gamma \in [0, 1)$ and $\epsilon > 0$ such that

$$\varphi(s) \leq \gamma \quad \text{for each} \quad s \in [\delta, \delta + \epsilon). \tag{2.2}$$

Now, we choose $\ell \in \mathbb{N}$ be a natural number such that

$$\delta \leq d(x_n, x_{n+1}) < \delta + \epsilon \quad \text{for all} \quad n \geq \ell. \tag{2.3}$$

By using (2.2) and (2.3), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &< \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \\ &\leq \gamma d(x_n, x_{n+1}), \quad \text{for all} \quad n \geq \ell. \end{aligned} \tag{2.4}$$

This gives,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_n, x_{n+1}) &= \sum_{n=1}^{\ell} d(x_n, x_{n+1}) + \sum_{n=\ell}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{\ell} d(x_n, x_{n+1}) + \sum_{n=\ell}^{\infty} \gamma^n d(x_0, x_1) < \infty, \end{aligned}$$

since $\gamma \in [0, 1)$. This implies that $\mathcal{O}_T(x_0)$ is a Cauchy sequence, and the proof is completed. □

Now, we are in position to present our main results. In fact, the next theorem is followed immediately from Lemma 2.2.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous mapping. If there are a nonempty T-invariant subset C of X and an R-function $\varphi : [0, \infty) \to [0, 1)$ such that the condition (2.1) is fulfilled, then T has a fixed point.*

Next, we will consider some fixed point theorems in the context of partially ordered set.

Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a mapping. A mapping $T : X \to X$ is said to be a comparable mapping if T maps comparable elements into comparable elements (i.e., for any $x, y \in X$, $x \preceq y$ or $x \succeq y \Rightarrow Tx \preceq Ty$ or $Tx \succeq Ty$).

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \to X$ is called nonincreasing if for all $x, y \in X$ such that $x \preceq y$, one has $Tx \preceq Ty$. Likewise, a function is called nondecreasing if whenever $x \preceq y$, then $Ty \preceq Tx$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a mapping. A function $T : X \to X$ is called monotonic if it is either entirely nonincreasing or nondecreasing.

Remark 2.2. The class of comparable mappings is obviously larger than those of monotone mappings

From now on, for the sake of simplicity, we will use the following notation

$$C^T_{\preceq} := \{x \in X : x \preceq Tx \ \text{ or } \ x \succeq Tx\}.$$

Note that the set of all fixed point of T is contained in $C^T_{\preceq}$. Thus, in order to consider the existence theorem, we see that $C^T_{\preceq} \neq \emptyset$ is a necessary condition.

Theorem 2.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a continuous mapping. Assume that there is an R-function such that the condition (2.1) is fulfilled for all $x \in C^T_\preceq$. If T is a comparable mapping, then T has a fixed point.*

Proof. In view of Theorem 2.2, we only have to show that $C^T_\preceq$ is an T-invariant set. Indeed, the T-invariantness of $C^T_\preceq$ is followed immediately from the assumption that T is a comparable mapping. □

Remark 2.3. In Theorem 2.3, we do not assume the monotonic property to the considered mapping T.

In view of Remark 2.2 and Theorem 2.3, we have the following result.

Theorem 2.4. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a continuous mapping. Assume that there is an R-function such that the condition (2.1) is fulfilled for all $x \in C^T_\preceq$. If T is a monotone mapping, then T has a fixed point.*

The next result can be deduced from Theorem 2.4.

Theorem 2.5. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a continuous monotone mapping. Assume that there exist an R-function $\varphi : [0, \infty) \to [0, 1)$ such that for any comparable pair $x, y \in X$ there holds*

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y). \tag{2.5}$$

If $C^T_\preceq \neq \emptyset$, then T has a fixed point.

Remark 2.4. If the partial order relation, that was considered in Theorem 2.5, is satisfied the following property:

(H) For every $x, y \in X$, there exist $z \in X$ which is comparable to x and y,

then we can assert that T has the unique fixed point. Indeed, supposing that there exist $z, y \in X$ which are fixed points of T, from (H) there exists $x \in X$ which is comparable to z and y. Subsequently, by the monotonicity of T, we see that $T^n(x)$ is comparable to $T^n(y)$ and $T^n(z)$, for $n = 1, 2, 3, ...$
Moreover,

$$\begin{aligned} d(z, T^n(x)) = d(T^n(z), T^n(x)) &\leq \varphi(d(T^{n-1}(z), T^{n-1}(x)))d(T^{n-1}(z), T^{n-1}(x)) \\ &\leq d(T^{n-1}(z), T^{n-1}(x)) \\ &= d(z, T^{n-1}(x)). \end{aligned} \tag{2.6}$$

This means that, the sequence $\{d(z, T^n(x))\}$ is a nonincreasing. Let $\gamma \geq 0$ be such that

$$\lim_{n\to\infty} d(z, T^n(x)) = \gamma.$$

By (2.6), we have

$$\gamma \leq \limsup_{s\to\gamma^+} \varphi(s)\gamma.$$

Thus, since φ is an R-function, we must have $\gamma = 0$. That is $\lim_{n\to\infty} d(z, T^n(x)) = 0$.
Similarly, we can show that $\lim_{n\to\infty} d(y, T^n(x)) = 0$.
Subsequently, since

$$d(z, y) \leq d(z, T^n(x)) + d(T^n(x), y), \quad \text{for each} \quad n \in \mathbb{N},$$

we can conclude that $d(z, y) = 0$.

3. COUPLED FIXED POINT THEOREMS

In this section, we will apply a result that has presented in Section 2 to obtain some coupled fixed point theorems. If $(X, \preceq)$ is a partially ordered set, we will endow the product set $X \times X$ with the partial order:

$$(x, y) \preceq_2 (u, v) \Leftrightarrow x \preceq u, y \succeq v, \qquad \text{for all } (x, y), (u, v) \in X \times X.$$

Let us start by recalling some basic concepts.

Definition 3.3. ([6]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \qquad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$
$$\text{and, } \; y_1, y_2 \in X, \qquad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 3.4. ([6]) We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping F if $F(x, y) = x, \;\; F(y, x) = y$.

Remark 3.5. Let (X, d) be a complete metric space. Let $D : X^2 \times X^2 \to [0, \infty)$ be a function which is defined by

$$D((x, y), (u, v)) = d(x, u) + d(y, v), \qquad \text{for all } \;(x, y), (u, v) \in X^2.$$

We know that (X^2, D) is a complete metric space. Further, let us define the mapping $\tilde{T} : X^2 \to X^2$ by

$$\tilde{T}(x, y) = (F(x, y), F(y, x)), \qquad \text{for all } \;(x, y) \in X^2.$$

Then we have the following facts:

(a) F has the mixed monotone property if and only if $\tilde{T}$ is monotone nondecreasing with respect to $\preceq_2$.

(b) $(x, y) \in X \times X$ is a coupled fixed point of F if and only if (x, y) is a fixed point of $\tilde{T}$.

Now, we present a coupled fixed point theorem, which is an extension of a result in [2].

Theorem 3.6. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a continuous mapping which satisfying the mixed monotone property. Assume that there exist an R-function $\varphi : [0, \infty) \to [0, 1)$ such that for all $(x, y), (u, v) \in X^2$ with $x \succeq u$ and $y \preceq v$,*

(3.7)
$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \varphi(d(x, u) + d(y, v))[(d(x, u) + d(y, v))].$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

Proof. By condition (3.7), it follows that, the mapping $\tilde{T}$ satisfies

$$D(\tilde{T}(x, y), \tilde{T}(u, v)) \leq \varphi(D((x, y), (u, v)))D((x, y), (u, v))$$

for all $(x, y), (u, v) \in X^2$ with $(u, v) \preceq_2 (x, y)$, where $\tilde{T}$ and D are defined as in Remark 3.5. Further, since F has mixed monotone property, we know that $\tilde{T}$ is a nondecreasing mapping with respect to $\preceq_2$. While condition (iii), implies that $(x_0, y_0) \preceq_2 \tilde{T}(x_0, y_0)$. Meanwhile, the continuity of F implies that $\tilde{T}$ is a continuous mapping. By using these facts, in view of Theorem 2.5, we can assert $\tilde{T}$ has a fixed point. Finally, by Remark 3.5 (b), we complete the proof. □

Remark 3.6. (a) In addition, if the partial order relation $\preceq_2$, which has considered in Theorem 3.6, satisfies the following property: For all $(x, y), (u, v) \in X \times X$, there exists $(z_1, z_2) \in X \times X$ which is comparable to (x, y) and (u, v), then we can show that the considered mapping F has the unique coupled fixed point.

(b) Taking $\varphi(t) = k, 0 < k < 1,$ in Theorem 3.6 we obtain Theorem 1.1.

Remark 3.7. In view of the technique that we have presented for proving the Theorem 3.6, one may see that some couple fixed point theorems can be obtained from the original (single)fixed point theorems.

Acknowledgement. The authors would like to thank the referees for the complete reading of the first version of this work and for the pertinent suggestions allowing us to improve the presentation of the paper. This work is partially supported by the National Research Council of Thailand (Project No. R2558B005). W. Saksirikun is supported by the Thailand Research Fund through the Royal Golden Jubilee PhD Program (Grant No. PHD/0248/2553).

REFERENCES

[1] Amini-Harandi, A., *Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem*, Mathematical and Computer Modelling, vol. **57**, No. 9–10, pp. 2343-2348, 2012
[2] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 7347–7355
[3] Chang, S. S., Cho, Y. J. and Huang, N. J., *Coupled fixed point theorem with applications*, J. Korean Math. Soc., **33** (1996), No. 3, 575-585
[4] Du, W.-S., *Some new results and generalizations in metric fixed point theory*, Nonlinear Analysis: Theory, Method and Applications, (2010)
[5] Du, W.-S., *Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashis condition in quasiordered metric spaces*, Fixed Point Theory Appl 2010, **9** (2010), Article ID 876372
[6] Gnana Bhaskar, T. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006) 1379–1393
[7] Nieto, J. and Rodriguez-Lopez, R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order., **22**, 223-239 (2005), doi: 10.1007/s11083-005-0769-0
[8] Ran, A. C. M and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equation*, Proc. Am. Math. Soc., **132** (2004), 1435–1443
[9] Reich, S., *Some problems and results in fixed point theory*, in Topological Methods in Nonlinear Functional Analysis(Toronto, Ont., 1982). In Contemp Math, vol. **21** (1983) 179-187. American Mathematical Society, Providence, RI, USA
[10] Samet, B. Karapinar, E., Aydi, H. and Rajic, V., *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl., **50** (2013)
[11] Sintunavarat, W., Kumam, P. and Cho, Y. J., *Coupled fixed point theorems for nonlinear contractions without mixed monotone property*, Fixed Point Theory and Applications 2012, 2012:170

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
NARESUAN UNIVERSITY
PHITSANULOK, 65000, THAILAND
E-mail address: w_saksirikun@hotmail.co.th
E-mail address: narinp@nu.ac.th