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A new approach to α-ψ-contractive mappings and generalized Ulam-Hyers stability, well-posedness and limit shadowing results

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ABSTRACT. In this paper, we introduce the new concept of weakly α-admissible mapping and give example to show that our concept is different from the concept corresponding existing in the literature. We also establish fixed point theorems by using such concept along with α-ψ-contractive condition and give some example which support our main result while previous results in literature are not applicable. Moreover, we study the generalized Ulam-Hyers stability, the well-posedness and the limit shadowing for fixed point problems satisfy our conditions.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers and real numbers, respectively. Also, we denote Ψ by the class of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ. For each $\psi \in \Psi$, we can easily to see that the following assertions holds:

- $\psi(t) < t$ for each $t > 0$;
- $\psi(0) = 0$
- ψ is continuous at $t = 0$.

In 2012, Samet et al. [16] introduced the concepts of α-ψ-contractive mapping and α-admissible mapping as follows:

Definition 1.1 ([16]). Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-*contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

Definition 1.2 ([16]). Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that T is α-*admissible* if the following condition holds:

$$\text{for } x, y \in X \text{ with } \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

They studied the fixed point results for such mappings in metric spaces and also showed that these results can be utilized to derive fixed point theorems in partially ordered metric spaces. Since concept of α-admissible mapping has many usefulness for fixed point analysis in various problems such as fixed point problems in metric spaces endowed with

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arbitrary relation, fixed point problems in metric spaces endowed with graph, fixed point problems for cyclic mappings etc., so several fixed point results via the concepts of α-admissible mappings occupies a prominent place in many aspects.

In this work, inspired by the concept of α-admissible mapping, we introduce the concept of weakly α-admissible mapping and show that such mapping is a real generalization of α-admissible mapping by given some example. The fixed point results for weakly α-admissible mapping along with α-ψ-contractive condition are established. We furnish some interesting examples which support our main theorems while results of Samet et al. [16] are not applicable. Further, we study the generalized Ulam-Hyers stability, the well-posedness and the limit shadowing of the fixed point problem satisfy our conditions.

2. Fixed point results

In this section, we introduce concept of weakly α-admissible mapping and prove fixed point results for such mapping along with α-ψ-contractive condition.

Definition 2.3. Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that T is *weakly α-admissible* if the following condition holds:

$$\text{for } x \in X \text{ with } \alpha(x, Tx) \geq 1 \Longrightarrow \alpha(Tx, TTx) \geq 1.$$

Remark 2.1. If T is an α-admissible mapping, then T is also a weakly α-admissible mapping. In general, the converse of the previous statement is not true.

Next, we give some example to show the real generalization of concept of weakly α-admissible mapping.

Example 2.1. Let $X = \{1, 2, 3, ...\}$. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} 3 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 1 & \text{if } x = 3, \\ x - 1 & \text{if } x = 4, 5, 6, ..., \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} x + y & \text{if } x, y \in \{1, 2\}, \\ \dfrac{|x - y|}{x + y} & \text{otherwise .} \end{cases}$$

It is easy to see that T is not an α-admissible mapping. Indeed, for $x = 1, y = 2$, we see that

$$\alpha(x, y) = \alpha(1, 2) = 3$$

but

$$\alpha(Tx, Ty) = \alpha(T1, T2) = \alpha(3, 2) = \frac{1}{5} < 1.$$

Next, we show that T is a weakly α-admissible. Suppose that $x \in X$ such that $\alpha(x, Tx) \geq 1$ and so $x = 2$. Now we obtain that

$$\alpha(Tx, TTx) = \alpha(T2, TT2) = \alpha(2, 2) = 4.$$

Now we give the first our result in this paper.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is weakly α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then the fixed point problem of T has a solution, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Starting from x_0 in (ii), we have $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n^\star} = x_{n^\star-1}$ for some $n^\star \in \mathbb{N}$, then $x_{n^\star}$ is a fixed point for T. We have nothing to prove. So we may assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Since T is weakly α-admissible, we have $\alpha(Tx_0, TTx_0) = \alpha(x_1, x_2) \geq 1$. By induction, we get

$$\alpha(x_{n-1}, x_n) \geq 1 \tag{2.2}$$

for all $n \in \mathbb{N}$. By α-ψ-contractive condition of T, we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$. By induction, we have

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$$

for all $n \in \mathbb{N}$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Fix $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$. By using the triangular inequality, we obtain that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, x_1)) \\ &< \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d). By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of T , it follows that

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\big(\lim_{n\to\infty} x_n\big) = Tx^*,$$

that is, x^* is a fixed point of T. This completes the proof. □

In next theorem, we will replace the continuity hypothesis of T by condition of α-regularity of metric space X.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is weakly α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *X is α-regular, i.e., if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then the fixed point problem of T has a solution, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Following the proof of Theorem 2.1, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From (2.2) and (iii), we get

$$\alpha(x_n, x^*) \geq 1 \tag{2.3}$$

for all $n \in \mathbb{N}$. By (2.3), α-ψ-contractive condition of T and the triangle inequality, we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \alpha(x_n, x^*) d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \psi(d(x_n, x^*)) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in above inequality, since ψ is continuous at $t = 0$, we obtain that $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$. Therefore, x^* is a fixed point of T. This completes the proof. □

Remark 2.2. From Remark 2.1, Theorem 2.1 and Theorem 2.2 are generalize and complementary of Theorem 2.1 and Theorem 2.2 of Samet et al. [16].

Next, we give some interesting examples which support our main theorems while results of Samet et al. [16] are not applicable.

Example 2.2. Let $X = [0, \infty)$ with the usual metric d. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} 1.9 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ x^2 & \text{if } x \notin \{1, 2\}, \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} x + y & \text{if } x, y \in \{1, 2\}, \\ 0 & \text{otherwise .} \end{cases}$$

It is easy to see that T is not an α-admissible mapping. Therefore, *main results of Samet et al.* [16] *is not applicable here*.

Next, we show that Theorem 2.2 can be guarantee the existence of fixed point of T. First, we can easily to see that T is weakly α-admissible. Clearly T is an α-ψ-contractive mapping with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we get $x_n \in \{1, 2\}$ for all $n \in \mathbb{N}$ and $x \in \{1, 2\}$. Then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Therefore, all the required hypotheses of Theorem 2.2 are satisfied, and so T has a fixed point. Here, 0 and 2 are fixed points of T.

We obtain that Theorem 2.1 and 2.2 don't claim the uniqueness of fixed point. To assure the uniqueness of the fixed point, we will add the following properties:

$$(H_0) : \alpha(a, b) \geq 1 \text{ for all } a, b \in X, \text{ where } a, b \text{ are fixed points of } T.$$

Theorem 2.3. *Adding condition (H_0) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2) we obtain the uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed point of T. From (1.1), we can conclude that $d(x^*, y^*) = 0$. So we get uniqueness of the fixed point of T. □

3. Generalized Ulam-Hyers stability, well-posedness and limit shadowing results

In 1940, the stability problem for functional equations first initial from a question of Ulam at the University of Wisconsin in which he discussed a number of important unsolved problems. In next year, Hyers [7] first give some partial answer of Ulam's question for Banach spaces and then this type of stability is called the Ulam-Hyers stability. This opened an avenue for further study and development of analysis in this field. Subsequently, several mathematicians have been studied and extended Ulam-Hyers stability in many ways, for example, Bota-Boriceanu and Petruşel [2], Brzdek et al. [3], Cădariu and Radu [4, 5], Rus [11, 12, 13, 14], Tişe-Tişe [18] and references therein. In particular, there are a number of results studied and extended Ulam-Hyers stability for fixed point problems such as Bota et al. [1], Kutbi and Sintunavarat [8], Rus and Şerban [15], Sintunavarat [17]. On the other hand, the notion of well-posedness and limit shadowing property of a fixed point problem have evoked much interest to many researchers, for example, De Blassi and Myjak [6], Lahiri and Das [9], Popa [10].

Here, we give the definitions of generalized Ulam-Hyers stability, well-posedness and limit shadowing property in sense of a fixed point problem.

Definition 3.4. Let (X, d) be a metric space and $T : X \to X$ be a mapping. The fixed point problem

$$Tx = x \tag{3.4}$$

is called generalized Ulam-Hyers stable if and only if there exists the function $\xi : [0, \infty) \to [0, \infty)$ which is increasing, continuous at 0 and $\xi(0) = 0$ such that for each $\varepsilon > 0$ and for each $w^* \in X$ which is an ε-solution of the fixed point equation (3.4), i.e. w^* satisfies the inequality

$$d(w^*, Tw^*) \leq \varepsilon, \tag{3.5}$$

there exists a solution $x^* \in X$ of the equation (3.4) such that

$$d(x^*, w^*) \leq \xi(\varepsilon).$$

Remark 3.3. If the function ξ define by $\xi(t) = ct$ for all $t \geq 0$, where $c > 0$, then the fixed point equation (3.4) is said to be Ulam-Hyers stable.

Definition 3.5 ([6])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. The fixed point problem of T is said to be well-posed if satisfies the following conditions:

- T has a unique fixed point x^* in X;
- for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have $\lim_{n\to\infty} d(x_n, x^*) = 0$.

Definition 3.6. Let (X, d) be a metric space and $T : X \to X$ be a mapping. We say that the fixed point problem of T has the limit shadowing property in X if for any sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} d(T^n z, x_n) = 0$.

Concerning the generalized Ulam-Hyers stable, well-posedness and limit shadowing property of the fixed point problem for a self-map of a complete metric space satisfying the conditions of Theorem 2.3, we have the following results.

Theorem 3.4. *Let (X, d) be a complete metric space. Suppose that all the hypotheses of Theorem 2.3 hold and additionally that the function $\xi : [0, \infty) \to [0, \infty)$ which is define by $\xi(t) := t - \psi(t)$ is a strictly increasing and onto. Then the following assertions holds:*

(a) *if $\alpha(a, b) \geq 1$ for all a, b which are an ε-solution of the fixed point equation (3.4), then the fixed point problem of T is generalized Ulam-Hyers stable.*

(b) *if ψ is continuous function and $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$, where $x_n \in X$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and x^* is a fixed point of T, then the fixed point problem of T is well-posed.*

(c) *if* ψ *is continuous function and* $\alpha(x_n, x^*) \geq 1$ *for all* $n \in \mathbb{N}$, *where* $x_n \in X$ *with* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* x^* *is a fixed point of* T, *then the fixed point problem of* T *has the limit shadowing property in* X.

Proof. From the proof of Theorem 2.3, we obtain that T has a unique fixed point and so let x^* is a unique fixed point of T.

From the hypothesis in (a), we claim that the fixed point problem of T is generalized Ulam-Hyers stable. Let $\varepsilon > 0$ and $w^* \in X$ be a solution of (3.5), i.e,

$$d(w^*, Tw^*) \leq \varepsilon.$$

It is obvious that the fixed point x^* of T satisfies inequality (3.5). From hypothesis in (a), we get $\alpha(x^*, w^*) \geq 1$. Now we have

$$\begin{aligned} d(x^*, w^*) &= d(Tx^*, w^*) \\ &\leq d(Tx^*, Tw^*) + d(Tw^*, w^*) \\ &\leq \alpha(x^*, w^*)d(Tx^*, Tw^*) + d(Tw^*, w^*) \\ &\leq \psi(d(x^*, w^*)) + \varepsilon. \end{aligned}$$

This implies that $d(x^*, w^*) - \psi(d(x^*, w^*)) \leq \varepsilon$, that is, $\xi(d(x^*, w^*)) \leq \varepsilon$ Therefore,

$$d(x^*, w^*) \leq \xi^{-1}(\varepsilon).$$

Since ξ^{-1} is increasing, continuous at 0 and $\xi^{-1}(0) = 0$, the fixed point problem of T is generalized Ulam-Hyers stable.

Next, we prove that the fixed point problem of T is well-posed under the assumption in (b). Let $\{x_n\}$ be sequence in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From assumption, we get $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. Now, we obtain that

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x_n, Tx_n) + \alpha(x_n, x^*)d(Tx_n, Tx^*) \\ &\leq d(x_n, Tx_n) + \psi(d(x_n, x^*)) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in above inequality, we get $\lim_{n\to\infty} d(x_n, x^*) = 0$ and so the fixed point problem of T is well-posed.

Finally, we prove that T has a limit shadowing under the assumption (c). Let $\{x_n\}$ be sequence in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Similarly to case (b), we get $\lim_{n\to\infty} d(x_n, x^*) = 0$. Since x^* is a fixed point of T, we have $\lim_{n\to\infty} d(x_n, T^n x^*) = \lim_{n\to\infty} d(x_n, x^*) = 0$. Therefore, T has the limit shadowing property. □

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References

[1] Bota, M. F., Karapınar, E. and Mleşniţe, O., *Ulam-Hyers stability results for fixed point problems via* α-ψ-*contractive mapping in* (b)-*metric space*, Abstr. Appl. Anal., Volume **2013** (2013), Article ID 825293, 6 pp.

[2] Bota-Boriceanu, M. F. and Petruşel, A., *Ulam-Hyers stability for operatorial equations*, Analel Univ. Al. I. Cuza, Iaşi, **57** (2011), 65–74

[3] Brzdek, J., Chudziak, J. and Pales, Z., *A fixed point approach to stability of functional equations*, Nonlinear Anal., **74** (2011), 6728–6732

[4] Cădariu, L. and Radu, V., *The fixed points method for the stability of some functional equations*, Carpathian J. Math., **23** (2007), No. 1-2, 63–72

[5] Cădariu, L. and Radu, V., *Fixed points and the stability of some classes of functional equations*, Carpathian J. Math., **24** (2008), No. 3, 299–309
[6] De Blassi, F. S. and Myjak, J., *Sur la porosite des contractions sans point fixe*, C. R. Math. Acad. Sci. Paris, **308** (1989), 51–54
[7] Hyers, D. H. , *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences of the United States of America, **27** (1941), No. 4, 222–224
[8] Kutbi, M. A. and Sintunavarat, W., *Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contraction mapping in metric spaces*, Abstr. Appl. Anal., Volume 2014 (2014), Article ID 268230, 6 pp.
[9] Lahiri, B. K. and Das, P., *Well-posednes and porosity of certain classes of operators*, Demonstratio Math., **38** (2005), 170-176
[10] Popa, V., *Well posedness of fixed point problem in compact metric spaces*, Bul. Univ. Petrol-Gaze Ploies. Ser. Mat. Inf. Fiz., **60** (2008), No 1, 1-4
[11] Rus, I. A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10** (2009), No. 2, 305–320
[12] Rus, I. A., *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math. **26** (2010), No. 1, 103–107
[13] Rus, I. A., *Ulam stability of the operatorial equations, Functional equations in mathematical analysis*, 287–305, Springer Optim. Appl., **52**, Springer, New York, 2012
[14] Rus, I. A., *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation*, Nonlinear analysis and variational problems, 147–152, Springer Optim. Appl., **35**, Springer, New York, 2010
[15] Rus, I. A. and Şerban, M. A., *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., **29** (2013), No. 2, 239–258
[16] Samet, B., Vetro, C. and Vetro, P., *Fixed-point theorems for α-ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165
[17] Sintunavarat, W., *Generalized Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for α-β-contraction mapping in metric spaces*, The Scientific World Journal, Volume 2014 (2014), Article ID 569174, 7 pp.
[18] Tişe, F. A. and Tişe, I. C., *Ulam-Hyers-Rassias stability for set integral equations*, Fixed Point Theory, **13** (2012), No. 2, 659–668

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