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A generalization of Nadler fixed point theorem

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ABSTRACT. Jleli and Samet gave a new generalization of the Banach contraction principle in the setting of Branciari metric spaces [Jleli, M. and Samet, B., *A new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014**:38 (2014)]. The purpose of this paper is to study the existence of fixed points for multivalued mappings, under a similar contractive condition, in the setting of complete metric spaces. Some examples are provided to illustrate the new theory.

1. INTRODUCTION AND PRELIMINARIES

The following theorem of Nadler [13] was the first successful attempt to combine the concepts of multivalued and contraction mappings for obtaining an existence result of fixed point.

Theorem 1.1 ([13]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued mapping satisfying $H(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, where k is a constant such that $k \in]0, 1[$ and $CB(X)$ denotes the family of nonempty, closed and bounded subsets of X. Then T has a fixed point, that is, there exists a point $u \in X$ such that $u \in Tu$.*

It is well-known that multivalued mappings play a crucial role in different branches of mathematics; in particular, in view of their applications to optimal control problems. As a matter of fact, optimization has undergone enormous theoretical and practical developments over the last decades. Thus, this field of research has increased its significance in mathematical modelling of real processes and phenomena arising in many scientific disciplines such as physics, biology and economics. Moreover, metric spaces and their generalizations furnish an useful tool for the study of multivalued mappings and fixed point theory. Finally, fixed point theorems provide useful tools to solve practical nonlinear problems, expressed as fixed point problems [14, 15, 16, 17, 18, 19]. Consequently, many generalizations, extensions and applications of Nadler's theorem have appeared in the literature, see for instance [1, 3, 4, 5, 6, 7, 8, 9, 12, 20, 21]. In [2], we have a generalization of Nadler fixed point theorem in the non self case.

Here, we start from looking at the paper of Jleli and Samet [10], who introduced a new concept of contraction. Then, we establish some results of fixed point for multivalued mappings, under a new contractive condition, in the setting of complete metric spaces. Clearly, the presented theorems extend well-known results of the existing literature on metric spaces. Some examples are provided to illustrate the new theory.

We collect some basic definitions, lemmas and notations, which will be used throughout the paper. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and $\mathbb{N}$ denote the set of positive integers. Let (X, d) be a metric space. For $A, B \in CB(X)$, define the function

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$H\colon CB(X)\times CB(X)\to\mathbb{R}^+$ by

$$H(A,B)=\max\{\delta(A,B),\delta(B,A)\},$$

where

$$\delta(A,B)=\sup\{d(a,B),a\in A\},\qquad \delta(B,A)=\sup\{d(b,A),b\in B\}$$

Whit

$$d(a,C)=\inf\{d(a,x),x\in C\}.$$

Note that H is called the Pompeiu-Hausdorff metric induced by the metric d, see [6]. Also, we denote by , $CL(X)$ be the family of nonempty and closed subsets of X and $K(X)$ be the family of nonempty and compact subsets of X.

Remark 1.1. The function $H : CL(X)\times CL(X)\to [0,+\infty[$ is a generalized Pompeiu-Hausdorff metric, that is, $H(A,B)=+\infty$ if $\max\{\delta(A,B),\delta(B,A)\}$ does not exist.

By definition of generalized Pompeiu-Hausdorff metric, one deduces easily the following lemma, see also [11].

Lemma 1.1. *Let (X,d) be a metric space and $A,B\in CL(X)$ with $H(A,B)>0$. Then, for each $h>1$ and for each $a\in A$ there exists $b=b(a)\in B$ such that $d(a,b)<h\,H(A,B)$.*

By the properties of closed sets, one deduces the following lemma.

Lemma 1.2. *Let (X,d) be a metric space. For $A\in CL(X)$ and $x\in X$, we have*

$$d(x,A)=0\Longleftrightarrow x\in A.$$

2. Fixed point for weak F-contractions

We study the existence of fixed points for multivalued mappings in a metric setting, by adapting the ideas in [10]. First, we give the following definitions.

Definition 2.1. We denote by $\mathcal{F}$ the family of all functions $F:]0,+\infty[\to]1,+\infty[$ with the following properties:

- (F1) F is non-decreasing;
- (F2) for each sequence $\{t_n\}$ of positive numbers $\lim_{n\to+\infty}t_n=0$ if and only if $\lim_{n\to+\infty}F(t_n)=1$;
- (F3) for each sequence $\{t_n\}$ of positive numbers with $\lim_{n\to+\infty}t_n=0$ there exist $\alpha\in]0,1[$ and $\lambda\in]0,+\infty]$ such that $\lim_{n\to+\infty}\frac{F(t_n)-1}{(t_n)^\alpha}=\lambda$.

Example 2.1. Let $F:]0,+\infty[\to]1,+\infty[$ be defined by $F(t)=e^{\sqrt{t}}$ or $F(t)=e^{\sqrt{te^t}}$. Clearly, F satisfies $(F1)-(F3)$.

Definition 2.2. Let (X,d) be a metric space. A multivalued mapping $T:X\to CL(X)$ is called a weak F-contraction if there exist a function $F\in\mathcal{F}$ and a positive number $k\in]0,1[$ such that

$$F(H(Tx,Ty))\le[F(d(x,y))]^k \tag{2.1}$$

for all $x,y\in X$ with $H(Tx,Ty)\neq 0$.

Definition 2.3. Let (X,d) be a metric space. A multivalued mapping $T:X\to CL(X)$ is called a generalized weak F-contraction if there exist a function $F\in\mathcal{F}$ and a positive number $k\in]0,1[$ such that

$$F(H(Tx,Ty))\le[F(M(x,y))]^k$$

for all $x, y \in X$ with $H(Tx, Ty) \neq 0$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}.$$

Remark 2.2. Let (X,d) be a metric space and let $T : X \to CL(X)$ be a weak F-contraction. From (2.1), we obtain

$$\ln F(H(Tx,Ty)) \leq k \ln[F(d(x,y))] < \ln F(d(x,y)).$$

As F is non-decreasing, we deduce

$$H(Tx,Ty) < d(x,y), \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

This implies that every weak F-contraction is a nonexpansive multivalued mapping, that is, $H(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$.

Example 2.2. Let (X,d) be a metric space and let $T : X \to CL(X)$ be a weak F-contraction with respect to the function $F(t) = e^{\sqrt{t}}$ (see Example 2.1). Then

$$e^{\sqrt{H(Tx,Ty)}} \leq \left[e^{\sqrt{d(x,y)}}\right]^k, \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

This implies

$$H(Tx,Ty) \leq k^2 d(x,y), \quad \text{for all } x,y \in X. \tag{2.2}$$

From (2.2), we get that every Nadler multivalued mapping $T : X \to CL(X)$, that is, a multivalued mapping such that $H(Tx,Ty) \leq k\, d(x,y)$ for all $x,y \in X$ for some $k \in]0,1[$, is a weak F-contraction.

Example 2.3. Let (X,d) be a metric space and let $T : X \to CL(X)$ be a weak F-contraction with respect to the function $F(t) = e^{\sqrt{te^t}}$. Then

$$e^{\sqrt{H(Tx,Ty)e^{H(Tx,Ty)}}} \leq \left[e^{\sqrt{d(x,y)e^{d(x,y)}}}\right]^k, \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

This implies

$$\frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty) - d(x,y)} \leq k^2, \text{ for all } x,y \in X,\ Tx \neq Ty. \tag{2.3}$$

First, we give a result of existence of a fixed point for multivalued weak F-contractions with compact values.

Theorem 2.2. *Let (X,d) be a complete metric space and let $T : X \to K(X)$ be a weak F-contraction. Then T has a fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Tx_0$. Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$ and hence $d(x_1, Tx_1) > 0$. Since Tx_1 is a compact set there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1)$. Now, from (2.1), we deduce

$$\begin{aligned} 1 < F(d(x_1,x_2)) &= F(d(x_1,Tx_1)) \\ &\leq F(H(Tx_0,Tx_1)) \leq [F(d(x_0,x_1))]^k. \end{aligned}$$

If $x_2 \in Tx_2$, we deduce that x_2 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_2 \notin Tx_2$ and hence $d(x_2, Tx_2) > 0$. Since Tx_2 is a compact set there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) = d(x_2, Tx_2)$. Next, from (2.1), we deduce

$$\begin{aligned} 1 < F(d(x_2, x_3)) &= F(d(x_2, Tx_2)) \\ &\leq F(H(Tx_1, Tx_2)) \leq [F(d(x_1, x_2))]^k. \end{aligned}$$

Assume that $x_3 \notin Tx_3$. Iterating this procedure, we construct a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$1 < F(d(x_n, x_{n+1})) = F(d(x_n, Tx_n)) \leq F(H(Tx_{n-1}, Tx_n)) \leq [F(d(x_{n-1}, x_n))]^k. \tag{2.4}$$

Now, from (2.4), we get

$$1 < F(d(x_n, x_{n+1})) \leq [F(d(x_0, x_1))]^{k^n}. \tag{2.5}$$

This implies

$$\lim_{n \to +\infty} F(d(x_n, x_{n+1})) = 1$$

and by (F-2) we have

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

We claim that $\{x_n\}$ is a Cauchy sequence; for this, we use condition (F-3). Let $t_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. From (F-3), there exist $\alpha \in]0, 1[$ and $\lambda \in]0, +\infty]$ such that

$$\lim_{n \to +\infty} \frac{F(t_n) - 1}{(t_n)^\alpha} = \lambda.$$

Let $\beta \in]0, \lambda[$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[d(x_n, x_{n+1})]^\alpha = [t_n]^\alpha \leq \beta^{-1}[F(t_n) - 1] = \beta^{-1}[F(d(x_n, x_{n+1})) - 1], \text{ for all } n > n_0.$$

Using (2.5) and the above inequality, we obtain

$$n[d(x_n, x_{n+1})]^\alpha \leq \beta^{-1} n([F(d(x_0, x_1))]^{k^n} - 1), \text{ for all } n > n_0. \tag{2.6}$$

Letting $n \to +\infty$ in (2.6), we get

$$\lim_{n \to +\infty} n[d(x_n, x_{n+1})]^\alpha = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$, $n_1 \geq n_0$, such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/\alpha}}, \text{ for all } n > n_1.$$

Let $m > n > n_1$. Then

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \frac{1}{k^{1/\alpha}}$$

and so $\{x_n\}$ is a Cauchy sequence in X. Hence, there exists $z \in X$ such that $x_n \to z$, as $n \to +\infty$.

As T is nonexpansive, we deduce

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\ &\leq d(z, x_{n+1}) + H(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(x_n, z). \end{aligned}$$

Passing to limit as $n \to +\infty$, we get

$$d(z, Tz) \leq 0$$

which implies $d(z, Tz) = 0$. Finally, since Tz is closed we obtain that $z \in Tz$, that is, z is a fixed point of T. □

Proceeding as in the proof of Theorem 2.2, we obtain the following result for multivalued mappings that satisfy a F-contractive condition of Ćirić type.

Theorem 2.3. *Let (X, d) be a complete metric space and let $T : X \to K(X)$ be a generalized weak F-contraction with respect to a continuous $F \in \mathcal{F}$. Then T has a fixed point.*

For a result of existence of fixed point for weak F-contraction with closed bounded values, it is necessary a supplementary condition of regularity for the function F.

Theorem 2.4. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exist a right continuous function $F \in \mathcal{F}$ and a positive number $k \in]0, 1[$ such that*

$$F(H(Tx, Ty)) < [F(d(x, y))]^k \tag{2.7}$$

for all $x, y \in X$ with $H(Tx, Ty) \neq 0$. Then T has a fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Tx_0$. Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$ and hence $d(x_1, Tx_1) > 0$.

Then, from (2.7), we deduce

$$1 < F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)) < [F(d(x_0, x_1))]^k.$$

Since F is right continuous there exists a number $q_1 > 1$ such that

$$F(q_1 H(Tx_0, Tx_1)) \leq [F(d(x_0, x_1))]^k. \tag{2.8}$$

From

$$d(x_1, Tx_1) \leq H(Tx_0, Tx_1) < q_1 H(Tx_0, Tx_1),$$

by Lemma 1.1, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq q_1 H(Tx_0, Tx_1)$. Using (F-1) and (2.8), from this inequality, we obtain

$$1 < F(d(x_1, x_2)) \leq F(q_1 H(Tx_0, Tx_1)) \leq [F(d(x_0, x_1))]^k.$$

If $x_2 \in Tx_2$, we deduce that x_2 is a fixed point of T and so we can conclude the proof. Assume that $x_2 \notin Tx_2$. Since F is right continuous there exists a number $q_2 > 1$ such that

$$F(q_2 H(Tx_1, Tx_2)) \leq [F(d(x_1, x_2))]^k. \tag{2.9}$$

Next, from

$$d(x_2, Tx_2) \leq H(Tx_1, Tx_2) < q_2 H(Tx_1, Tx_2),$$

by Lemma 1.1, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq q_2 H(Tx_1, Tx_2)$. Using (F-1) and (2.9), from this inequality, we obtain

$$1 < F(d(x_2, x_3)) \leq F(q_2 H(Tx_1, Tx_2)) \leq [F(d(x_1, x_2))]^k \leq [F(d(x_0, x_1))]^{k^2}.$$

Assume that $x_3 \notin Tx_3$. Iterating this procedure, we construct a sequence $\{x_n\} \subset X$ and a sequence $\{q_n\} \subset]1, +\infty[$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$1 < F(d(x_n, x_{n+1})) \leq F(q_n H(Tx_{n-1}, Tx_n)) \leq [F(d(x_{n-1}, x_n))]^k, \text{ for all } n \in \mathbb{N}.$$

Then

$$1 < F(d(x_n, x_{n+1})) \leq [F(d(x_0, x_1))]^{k^n}, \text{ for all } n \in \mathbb{N}.$$

This implies

$$\lim_{n\to+\infty} F(d(x_n, x_{n+1})) = 1$$

and by (F-2) we have

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

Proceeding as in the proof of Theorem 2.2, we prove that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $z \in X$ such that $x_n \to z$.

As T is nonexpansive, then

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\ &\leq d(z, x_{n+1}) + H(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(x_n, z). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get

$$d(z, Tz) \leq 0$$

which implies $d(z, Tz) = 0$. Finally, since Tz is closed we obtain that $z \in Tz$, that is, z is a fixed point of T. □

Proceeding as in the proof of Theorem 2.4, we obtain the following result for multivalued mappings that satisfy a F-contractive condition of Ćirić type.

Theorem 2.5. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a generalized weak F-contraction with respect to a continuous function $F \in \mathcal{F}$ such that*

$$F(H(Tx, Ty)) < [F(M(x, y))]^k$$

for all $x, y \in X$ with $H(Tx, Ty) \neq 0$, where $k \in]0, 1[$. Then T has a fixed point.

Example 2.4. Let X be the set defined by $X = \{x_n : n \in \mathbb{N}\}$, where $x_n = 2^{-1}n(n+1)$ for all $n \in \mathbb{N}$. Let $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Clearly, (X, d) is a complete metric space. Define the mapping $T : X \to K(X)$ by

$$Tx = \begin{cases} \{x_1\} & \text{if } x = x_1, \\ \{x_1, \ldots, x_{n-1}\} & \text{if } x = x_n \text{ and } n > 1 \end{cases}$$

and let $F(t) = e^{\sqrt{te^t}} \in \mathcal{F}$ for all $t > 0$. We claim that T is a multivalued weak F-contraction with respect to above function F. At such end we check that (2.3) holds. First, we note that $H(Tx_m, Tx_n) > 0$ if and only if ($n = 1$ and $m > 2$) or ($m > n > 1$). Then we consider the two cases.

Case 1. For $m > 2$ and $n = 1$, we have

$$\begin{aligned} & \frac{H(Tx_m, Tx_1)}{d(x_m, x_1)} e^{H(Tx_m, Tx_1) - d(x_m, x_1)} \\ = & \frac{x_{m-1} - x_1}{x_m - x_1} e^{x_{m-1} - x_m} \\ = & \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned}$$

Case 2. For $m > n > 1$ we have

$$\begin{aligned} & \frac{H(Tx_m, Tx_n)}{d(x_m, x_n)} e^{H(Tx_m,Tx_n)-d(x_m,x_n)} \\ = & \frac{x_{m-1} - x_{n-1}}{x_m - x_n} e^{x_{m-1}-x_{n-1}-x_m+x_n} \\ = & \frac{m+n-1}{m+n+1} e^{n-m} < e^{n-m} \leq e^{-1}. \end{aligned}$$

This shows that (2.3) is satisfied with $k = e^{-1/2}$. Thus T is a multivalued weak F-contraction, therefore, all conditions of Theorem 2.2 are satisfied and so T has a fixed point in X. On the other hand, since

$$\lim_{n\to+\infty} \frac{H(Tx_n, Tx_1)}{d(x_n, x_1)} = \lim_{n\to+\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1,$$

then T is not a Nadler multivalued mapping.

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