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Common fixed points of multivalued F-contractions on metric spaces with a directed graph

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ABSTRACT. In this paper, we establish the existence of common fixed points of multivalued F-contraction mappings on a metric space endowed with a graph. An example is presented to support the results proved herein. Our results unify, generalize and complement various known comparable results in the literature.

1. INTRODUCTION AND PRELIMINARIES

Order oriented fixed point theory is studied in an environment created by a class of partially ordered sets with appropriate mappings satisfying certain order condition like monotonicity, expansivity or order continuity. Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [27], and then by Nieto and Lopez [25]. Further results in this direction under different contractive conditions were proved in [3, 5, 9, 10].

Jachymski [20] introduced a new approach in metric fixed point theory by replacing order structure with a graph structure on a metric space. In this way, the results obtained in ordered metric spaces are generalized (see also [19] and the reference therein); in fact, Gwodzdz-lukawska and Jachymski [18] developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph.

Abbas and Nazir [4] obtained some fixed point results for power graphic contraction pair on a metric space equipped with a graph. Recently, Bojor [16] proved fixed point results for Reich type contractions on such spaces. For more results in this direction, we refer to [8, 15, 17, 26] and reference mentioned therein.

The study of fixed points for multivalued contractions and nonexpansive maps using the Pompeiu-Hausdorff metric was initiated by Markin [23]. Theory of multivalued maps has rich applications in control theory, convex optimization, differential equations and economics. Recently, Wardowski [30] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Abbas et al. [1] obtained common fixed point results introducing the concept of F-contraction mapping with respect to a self mapping on a metric space. Further in this direction, Abbas et al. [2] introduced a notion of generalized F-contraction mapping and employed this concept to obtain a fixed point of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Minak et al. [24] proved some fixed point results for Ciric type generalized F-contractions on complete metric spaces. Recently, Sgroi and Vetro [29] obtained some fixed point results for F-contraction multivalued maps in metric spaces (see also [7]).

The aim of this paper is to prove some common fixed point results for multivalued generalized graphic F-contraction mappings on a metric space endowed with a graph.

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Our results extend and unify various comparable results in the existing literature ([21], [22], [28], and [29]).

In the sequel the letters $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ will denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

Consistent with Jachymski [19], let (X, d) be a metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E(G)$. Let $E^*(G)$ denotes the set of all edges of G that are not loops i.e., $E^*(G) = E(G) - \Delta$. Also assume that the graph G has no parallel edges and, thus one can identify G with the pair $(V(G), E(G))$.

Definition 1.1. [19] An operator $f : X \to X$ is called a Banach G-contraction or simply a G-contraction if

(a) f preserves edges of G; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(f(x), f(y)) \in E(G)$,
(b) f decreases weights of edges of G ; there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

If x and y are vertices of G, then a (directed) path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ ($n \in \{0, 1, 2, ..., k\}$) of vertices such that $x_0 = x$, $x_k = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1, 2, ..., k\}$.

Notice that a graph G is connected if there is a (directed) path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E\left(G\right)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

If $f : X \to X$ is an operator. Set

$$X_f := \{x \in X : (x, f(x)) \in E(G)\}.$$

Jachymski [20] used the following property:

(P) : for any sequence $\{x_n\}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Theorem 1.1. [20] *Let (X, d) be a complete metric space and let G be a directed graph such that $V(G) = X$. Let $E(G)$ and the triplet (X, d, G) has property (P) and $f : X \to X$ a G-contraction. Then the following statements hold:*

(1) f has a fixed point if and only if $X_f \neq \emptyset$;
(2) if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator;
(3) for any $x \in X_f$, $f\mid_{[x]_{\widetilde{G}}}$ is a Picard operator;
(4) if $X_f \subseteq E(G)$, then f is a weakly Picard operator.

For detailed discussion on Picard operators, we refer to Berinde [13, 11, 12, 14].

Let $\digamma$ be the collection of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(F_2) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^h F(\alpha) = 0$.

Latif and Beg [22] introduced a notion of K-multivalued mapping as an extension of Kannan mapping to multivalued mappings. Rus [28] coined the term R-multivalued mapping which is a generalization of a K-multivalued mapping. Abbas and Rhoades [6] introduced the notion of a generalized R-multivalued mappings, which in turn generalize R-multivalued mappings, and obtained common fixed point results for such mappings. Let (X, d) be a metric space. Denote by $P(X)$ the family of all nonempty subsets of X, by $P_{cl}(X)$ the family of all nonempty closed subset of X.

A point x in X is a fixed point of a multivalued mapping $T : X \to P(X)$ iff $x \in Tx$. The set of all fixed points of multivalued mapping T is denoted by $Fix(T)$.

Suppose that $T_1, T_2 : X \to P_{cl}(X)$. Set

$$X_{T_1,T_2} := \{x \in X : (x, u_x) \in E(G) \text{ where } u_x \in T_1(x) \cap T_2(x)\}.$$

Now we give the following definition:

Definition 1.2. Let $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued mappings. Suppose that for every vertex x in G and for every $u_x \in T_i(x), i \in \{1,2\}$ we have $(x, u_x) \in E(G)$. A pair (T_1, T_2) is said to form:

(I). a graphic F_1-contraction if for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and

$$\tau + F(d(u_x, u_y)) \leq F(M_1(x, y; u_x, u_y)), \tag{1.1}$$

hold, where τ is a positive real number and

$$M_1(x, y; u_x, u_y) = \max\left\{d(x,y), d(x,u_x), d(y,u_y), \frac{d(x,u_y) + d(y,u_x)}{2}\right\}.$$

(II). a graphic F_2-contraction if for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1,2\}$ with $i \neq j$ such that $(u_x, u_y) \in E^*(G)$ and we have

$$\tau + F(d(u_x, u_y)) \leq F(M_2(x, y; u_x, u_y)), \tag{1.2}$$

where τ is a positive real number and

$$M_2(x, y; u_x, u_y) = \alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y) + \delta_1 d(x,u_y) + \delta_2 d(y,u_x),$$

and $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Note that for different choices of mappings F, one obtains different contractivity conditions.

Recall that, a map $T : X \to P_{cl}(X)$ is said to be upper semicontinuous, if for $x_n \in X$ and $y_n \in Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in Tx_0$.

A clique in an undirected graph $G = (V, E)$ is a subset of the vertex set $W \subset V$, such that for every two vertices in W, there exists an edge connecting the two. This is equivalent to saying that the subgraph induced by W is complete i.e., for every $x, y \in W(G)$, we have $(x, y) \in E(G)$.

2. Common fixed point results

In this section, we obtain several common fixed point results for two multivalued mappings on a metric space endowed with a directed graph. We start with the following result.

Theorem 2.2. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If mappings $T_1, T_2 : X \to P_{cl}(X)$ form a graphic F_1-contraction pair, then following statement hold:*

(i) $Fix(T_1) \neq \emptyset$ *or* $Fix(T_2) \neq \emptyset$ *if and only if* $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$.

(ii) $X_{T_1,T_2} \neq \emptyset$ *provided that* $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$.

(iii) If $X_{T_1,T_2} \neq \emptyset$ *and G is weakly connected, then* $Fix\,(T_1) = Fix\,(T_2) \neq \emptyset$ *provided that either (a) either T_1 or T_2 is upper semicontinuous, or (b) F is continuous, either T_1 or T_2 is bounded and G has property (P).*

(iv) $Fix\,(T_1) \cap Fix\,(T_2)$ *is a clique of* $\widetilde{G}$ *if and only if* $Fix\,(T_1) \cap Fix\,(T_2)$ *is a singleton.*

Proof. To prove (i), let $x^* \in T_1(x^*)$. Assume $x^* \notin T_2\,(x^*)$, then since (T_1, T_2) form a graphic F_1-contraction pair, there exists an $x \in T_2\,(x^*)$ with $(x^*, x) \in E^*\,(G)$ such that

$$\tau + F\,(d(x^*, x)) \leq F(M_1(x^*, x^*; x^*, x)),$$

where

$$M_1(x^*, x^*; x^*, x) = \max\left\{d(x^*, x^*), d(x^*, x^*), d(x, x^*), \frac{d(x^*, x) + d(x^*, x^*)}{2}\right\} = d(x, x^*).$$

Thus we have

$$\tau + F\,(d(x^*, x)) \leq F(d(x^*, x)),$$

a contradiction as $\tau > 0$. Hence $x^* \in T_2\,(x^*)$ and so $Fix(T_1) \subseteq Fix(T_2)$. Similarly, $Fix(T_2) \subseteq Fix(T_1)$ and therefore $Fix(T_1) = Fix(T_2)$. Also, if $x^* \in T_2(x^*)$, then we have $x^* \in T_1(x^*)$. The converse is straightforward.

To prove (ii), let $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$. Then there exists $x \in X$ such that $x \in T_1(x) \cap T_2(x)$. As $\Delta \subseteq E(G)$, we conclude that $X_{T_1,T_2} \neq \emptyset$.

To prove (iii), Suppose that x_0 is an arbitrary point of X. If $x_0 \in T_1\,(x_0)$ or $x_0 \in T_2\,(x_0)$, then the proof is finished. So we assume that $x_0 \notin T_i\,(x_0)$ for $i \in \{1, 2\}$. Now for $i, j \in \{1, 2\}$ with $i \neq j$, if $x_1 \in T_i(x_0)$, then there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E^*(G)$ such that

$$\tau + F\,(d(x_1, x_2)) \leq F(M_1(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M_1(x_0, x_1; x_1, x_2) &= \max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

If $M_1(x_0, x_1; x_1, x_2) = d(x_1, x_2)$, then

$$\tau + F\,(d(x_1, x_2)) \leq F(d(x_1, x_2)),$$

gives a contradiction as $\tau > 0$. Therefore $M_1(x_0, x_1; x_1, x_2) = d(x_0, x_1)$ and we have

$$\tau + F\,(d(x_1, x_2)) \leq F\,(d(x_0, x_1))\,.$$

Similarly, for the point x_2 in $T_j(x_1)$, there exists $x_3 \in T_i(x_2)$ with $(x_2, x_3) \in E^*(G)$ such that

$$\tau + F(d(x_2, x_3)) \leq F(M_1(x_1, x_2; x_2, x_3)),$$

where

$$\begin{aligned} M_1(x_1, x_2; x_2, x_3) &= \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned}$$

In case $M_1(x_1, x_2; x_2, x_3) = d(x_2, x_3)$, then

$$\tau + F(d(x_2, x_3)) \leq F(d(x_2, x_3)),$$

gives a contradiction as $\tau > 0$. Therefore $M_1(x_1, x_2; x_2, x_3) = d(x_1, x_2)$ and we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

Continuing this way, for $x_{2n} \in T_j(x_{2n-1})$, there exist $x_{2n+1} \in T_i(x_{2n})$ with $(x_{2n}, x_{2n+1}) \in E^*(G)$ such that

$$\tau + F(d(x_{2n}, x_{2n+1})) \leq F(M_1(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})),$$

that is,

$$\tau + F(d(x_{2n}, x_{2n+1})) \leq F(d(x_{2n-1}, x_{2n})).$$

In a similar manner, for $x_{2n+1} \in T_j(x_{2n})$, there exist $x_{2n+2} \in T_i(x_{2n+1})$ such that for $(x_{2n+1}, x_{2n+2}) \in E^*(G)$ implies

$$\tau + F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_j(x_{n-1})$, there exist $x_{n+1} \in T_i(x_n)$ with $(x_n, x_{n+1}) \in E^*(G)$ and it satisfies

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)).$$

Therefore

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\leq \ldots \leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{2.3}$$

From (2.3), we obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ that together with (F_2) gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now by (F_3), there exists $h \in (0, 1)$ such that

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^h F(d(x_n, x_{n+1})) = 0.$$

From (2.3) we have

$$\begin{aligned} &[d(x_n, x_{n+1})]^h F(d(x_n, x_{n+1})) - [d(x_n, x_{n+1})]^h F(d(x_0, x_{n+1})) \\ \leq\; & -n\tau[d(x_n, x_{n+1})]^h \leq 0. \end{aligned}$$

On taking limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^h = 0.$$

Hence $\lim_{n\to\infty} n^{\frac{1}{h}} d(x_n, x_{n+1}) = 0$ and there exists $n_1 \in \mathbb{N}$ such that $n^{\frac{1}{h}} d(x_n, x_{n+1}) \leq 1$ for all $n \geq n_1$. So we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/h}}$$

for all $n \geq n_1$. Now consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous, then as $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ implies that $x^* \in T_i(x^*)$. Using (i), we get $x^* \in T_i(x^*) = T_j(x^*)$. Similarly the result hold when T_j is upper semicontinuous.

Suppose that F is continuous. Since x_{2n} converges to x^* as $n \to \infty$ and $(x_{2n}, x_{2n+1}) \in E(G)$, we have $(x_{2n}, x^*) \in E(G)$. For $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that $(x_{2n}, u_n) \in E^*(G)$. As $\{u_n\}$ is bounded, $\limsup\limits_{n\to\infty} u_n = u^*$, and $\liminf\limits_{n\to\infty} u_n = u_*$ both exist. Assume that $u^* \neq x^*$. Since (T_1, T_2) form a graphic F_1-contraction,

$$\tau + F(d(x_{2n}, u_n)) \leq F(M_1(x_{2n-1}, x^*; x_{2n}, u_n)),$$

where

$$\begin{aligned} M_1(x_{2n-1}, x^*; x_{2n}, u_n) = \max \Big\{ & d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, u_n), \\ & \frac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2} \Big\}. \end{aligned}$$

On taking $\limsup$ implies

$$\tau + F(d(x^*, u^*)) \leq F(d(x^*, u^*)),$$

a contradiction. Hence $u^* = x^*$. Similarly, taking the $\liminf$ gives $u_* = x^*$. Since $u_n \in T_i(x^*)$ for all $n \geq 1$ and $T_i(x^*)$ is a closed set, it follows that $x^* \in T_i(x^*)$. Now from (i), we get $x^* \in T_i(x^*)$ and hence $Fix(T_1) = Fix(T_2)$.

Finally to prove (iv), suppose the set $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. We are to show that $Fix(T_1) \cap Fix(T_2)$ is singleton. Assume on contrary that there exist u and v such that $u, v \in Fix(T_1) \cap Fix(T_2)$ but $u \neq v$. As $(u, v) \in E^*(G)$ and (T_1, T_2) form a graphic F_1-contraction, so for $(u_x, v_y) \in E^*(G)$ implies

$$\begin{aligned} \tau + F(d(u, v)) &\leq F(M_1(u, v; u, v)) \\ &= F\left(\left\{\max\{d(u, v), d(u, u), d(v, v), \frac{d(u, v) + d(v, u)}{2}\right\}\right) \\ &= F(d(u, v)), \end{aligned}$$

a contradiction as $\tau > 0$. Hence $u = v$. Conversely, if $Fix(T_1) \cap Fix(T_2)$ is singleton, then it follows that $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$.

□

Example 2.1. Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\} = V(G)$,

$$\begin{aligned} E(G) &= \{(x, y) : x \leq y \text{ where } x, y \in V(G)\} \text{ and} \\ E^*(G) &= \{(x, y) : x < y \text{ where } x, y \in V(G)\}. \end{aligned}$$

Let $V(G)$ be endowed with usual metric. Define $T_1, T_2 : X \to P_{cl}(X)$ as follows:

$$T_1(x) = \{x_1\} \text{ for } x \in X, \text{ and}$$
$$T_2(x) = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_{n-1}\} & , \quad x = x_n, \text{ for } n > 1. \end{cases}$$

Take $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ and $\tau = 1$. For $(u_x, u_y) \in E^*(G)$, we consider the following cases:

(1) If $x = x_1, y = x_m$, for $m > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{m-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} \\ &= \frac{m^2 - m - 2}{2}e^{-m} \\ &< \frac{m^2 + m - 2}{2}e^{-1} \\ &= e^{-1}d(x, y) \\ &\leq e^{-1}M_1(x, y; u_x, u_y). \end{aligned}$$

(2) If $x = x_n, y = x_{n+1}$ with $n > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{n-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-[\frac{d(x,u_y)+d(y,u_x)}{2}]} \\ &= \frac{n^2 - n - 2}{2}e^{\frac{-3n-2}{2}} \\ &< \frac{n^2 + 4n}{2}e^{-1} \\ &= e^{-1}\left[\frac{d(x, u_y) + d(y, u_x)}{2}\right] \\ &\leq e^{-1}M_1(x, y; u_x, u_y). \end{aligned}$$

(3) When $x = x_n$, $y = x_m$ with $m > n > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{n-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(x,u_x)} \\ &= \frac{n^2 - n - 2}{2}e^{-n} \\ &< \frac{n^2 + n - 2}{2}e^{-1} \\ &= e^{-1}d(x, u_x) \\ &\leq e^{-1}M_1(x, y; u_x, u_y). \end{aligned}$$

Now we show that for $x, y \in X$, $u_x \in T_2(x)$; there exists $u_y \in T_1(y)$ such that $(u_x, u_y) \in E^*(G)$ and (1.1) is satisfied. For this, we consider the following cases:

(1) If $x = x_n$, $y = x_1$ with $n > 1$, we have for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_1(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} \\ &= \frac{n^2-n-2}{2}e^{-n} \\ &< \frac{n^2+n-2}{2}e^{-1} \\ &= e^{-1}d(x,y) \\ &\leq e^{-1}M_1(x,y;u_x,u_y). \end{aligned}$$

(2) In case $x = x_n$, $y = x_m$ with $m > n > 1$, then for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(y,u_y)} \\ &= \frac{n^2-n-2}{2}e^{n^2-n-m^2-m} \\ &< \frac{m^2+m-2}{2}e^{-1} \\ &= e^{-1}d(y,u_y) \\ &\leq e^{-1}M_1(x,y;u_x,u_y). \end{aligned}$$

Thus for all x, y in $V(G)$, (1.1) is satisfied. Hence all the conditions of Theorem 2.1 are satisfied. Moreover, $x_1 = 1$ is the common fixed point of T_1 and T_2 with $Fix(T_1) = Fix(T_2)$.

The following results generalizes Theorem 3.4 in [28].

Theorem 2.3. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If $T_1, T_2 : X \to P_{cl}(X)$ form a graphic F_2-contraction pair, then following statements hold:*

(i) $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.

(ii) $X_{T_1,T_2} \neq \emptyset$ provided that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.

(iii) If $X_{T_1,T_2} \neq \emptyset$ and G is weakly connected, then $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that either (a) either T_1 or T_2 is upper semicontinuous, or (b) F is continuous, either T_1 or T_2 is bounded and G has property (P).

(iv) $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton.

Proof. To prove (i), let $x^* \in T_1(x^*)$. Assume $x^* \notin T_2(x^*)$, then since (T_1, T_2) form a graphic F_2-contraction pair, there exists an $x \in T_2(x^*)$ with $(x^*, x) \in E^*(G)$ such that

$$\tau + F(d(x^*, x)) \leq F(M_2(x^*, x^*; x^*, x)),$$

where

$$\begin{aligned} M_2(x^*, x^*; x^*, x) &= \alpha d(x^*, x^*) + \beta d(x^*, x^*) + \gamma d(x, x^*) + \delta_1 d(x^*, x) + \delta_2 d(x^*, x^*) \\ &= (\gamma + \delta_1)d(x, x^*), \end{aligned}$$

Thus we have

$$\begin{aligned} \tau + F(d(x^*, x)) &\leq F((\gamma + \delta_1)d(x^*, x)) \\ &\leq F(d(x^*, x)), \end{aligned}$$

a contradiction as $\tau > 0$. Hence $x^* \in T_2(x^*)$ and so $Fix(T_1) \subseteq Fix(T_2)$. Similarly, $Fix(T_2) \subseteq Fix(T_1)$ and therefore $Fix(T_1) = Fix(T_2)$. Also, if $x^* \in T_2(x^*)$, then we

have $x^* \in T_1(x^*)$. The converse is straightforward.
To prove (ii), let $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$. Then there exists $x \in X$ such that $x \in T_1(x) \cap T_2(x)$. Since $\Delta \subseteq E(G)$, we conclude that $X_{T_1,T_2} \neq \emptyset$.

To prove (iii), suppose that x_0 is an arbitrary point of X. For $i, j \in \{1, 2\}$, with $i \neq j$, take $x_1 \in T_i(x_0)$, there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E^*(G)$ such that

$$\tau + F\left(d(x_1, x_2)\right) \leq F(M_2(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M_2(x_0, x_1; x_1, x_2) &= \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) \\ &\quad + \delta_1 d(x_0, x_2) + \delta_2 d(x_1, x_1) \\ &\leq (\alpha + \beta + \delta_1) d(x_0, x_1) + (\gamma + \delta_1) d(x_1, x_2). \end{aligned}$$

If $d(x_0, x_1) \leq d(x_1, x_2)$, then we have

$$\begin{aligned} \tau + F\left(d(x_1, x_2)\right) &\leq F((\alpha + \beta + \gamma + 2\delta_1) d(x_1, x_2)) \\ &\leq F(d(x_1, x_2)), \end{aligned}$$

gives a contradiction as $\tau > 0$. Therefore

$$\tau + F\left(d(x_1, x_2)\right) \leq F\left(d(x_0, x_1)\right).$$

Continuing this process, for $x_{2n} \in T_j(x_{2n-1})$, there exists $x_{2n+1} \in T_i\left(x_{2n}\right)$ such that for $(x_{2n}, x_{2n+1}) \in E^*\left(G\right)$, we have

$$\tau + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})\right),$$

where

$$\begin{aligned} M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) &= \alpha d(x_{2n-1}, x_{2n}) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x_{2n}, x_{2n+1}) \\ &\quad + \delta_1 d(x_{2n-1}, x_{2n+1}) + \delta_2 d(x_{2n}, x_{2n}) \\ &\leq (\alpha + \beta + \delta_1) d(x_{2n-1}, x_{2n}) + (\gamma + \delta_1) d(x_{2n}, x_{2n+1}). \end{aligned}$$

If $d(x_{2n-1}, x_{2n}) \leq d(x_{2n}, x_{2n+1})$, then

$$\begin{aligned} \tau + F\left(d(x_{2n}, x_{2n+1})\right) &\leq F\left((\alpha + \beta + \gamma + 2\delta_1) d(x_{2n}, x_{2n+1})\right) \\ &\leq F\left(d(x_{2n}, x_{2n+1})\right), \end{aligned}$$

gives a contradiction as $\tau > 0$. Therefore

$$\tau + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(d(x_{2n-1}, x_{2n})\right).$$

In a similar way, for $x_{2n+1} \in T_j(x_{2n})$, there exists $x_{2n+2} \in T_i\left(x_{2n+1}\right)$ with $(x_{2n+1}, x_{2n+2}) \in E^*\left(G\right)$ such that

$$\tau + F\left(d(x_{2n+1}, x_{2n+2})\right) \leq F\left(d(x_{2n}, x_{2n+1})\right).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_j(x_{n-1})$, there exists $x_{n+1} \in T_i\left(x_n\right)$ with $(x_n, x_{n+1}) \in E^*\left(G\right)$ such that

$$\tau + F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-1}, x_n)\right).$$

Therefore

$$\begin{aligned} F\left(d(x_n, x_{n+1})\right) &\leq F\left(d(x_{n-1}, x_n)\right) - \tau \leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau \\ &\leq \ldots \leq F\left(d(x_0, x_1)\right) - n\tau. \end{aligned}$$

Thus, $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$ together with (F_2) gives $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Following arguments similar to those in proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous, then as $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ implies that $x^* \in T_i(x^*)$. Using (i), we get $x^* \in T_i(x^*) = T_j(x^*)$. Similarly the result hold when T_j is upper semicontinuous.

Suppose that F is continuous. Since x_{2n} converges to x^* as $n \to \infty$ and $(x_{2n}, x_{2n+1}) \in E(G)$, we have $(x_{2n}, x^*) \in E(G)$. For $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that $(x_{2n}, u_n) \in E^*(G)$. As $\{u_n\}$ is bounded, $\limsup\limits_{n\to\infty} u_n = u^*$, and $\liminf\limits_{n\to\infty} u_n = u_*$ both exist. Assume that $u^* \neq x^*$. Since (T_1, T_2) form a graphic F_2-contraction,

$$\tau + F(d(x_{2n}, u_n)) \le F(M_2(x_{2n-1}, x^*; x_{2n}, u_n)),$$

where

$$\begin{aligned} M_2(x_{2n-1}, x^*; x_{2n}, u_n) &= \alpha d(x_{2n-1}, x^*) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x^*, u_n) \\ &\quad + \delta_1 d(x_{2n-1}, u_n) + \delta_2 d(x^*, x_{2n}). \end{aligned}$$

On taking $\limsup$ implies

$$\begin{aligned} \tau + F(d(x^*, u^*)) &\le F((\gamma + \delta_1) d(x^*, u^*)) \\ &\le F(d(x^*, u^*)), \end{aligned}$$

a contradiction. Hence $u^* = x^*$. Similarly, taking the $\liminf$ gives $u_* = x^*$. Since $u_n \in T_i(x^*)$ for all $n \ge 1$ and $T_i(x^*)$ is a closed set, it follows that $x^* \in T_i(x^*)$. Now from (i), we get $x^* \in T_i(x^*)$ and hence $Fix(T_1) = Fix(T_2)$.

Finally to prove (iv), suppose the set $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. We are to show that $Fix(T_1) \cap Fix(T_2)$ is singleton. Assume on contrary that there exist u and v such that $u, v \in Fix(T_1) \cap Fix(T_2)$ but $u \neq v$. As $(u, v) \in E^*(G)$ and (T_1, T_2) form a graphic F_2-contraction, so for $(u_x, v_y) \in E^*(G)$, we have

$$\begin{aligned} \tau + F(d(u, v)) &\le F(M_2(u, v; u, v)) \\ &= F(\alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) + \delta_1 d(u, v) + \delta_2 d(v, u)\}) \\ &= F((\alpha + \delta_1 + \delta_2) d(u, v)) \\ &\le F(d(u, v)), \end{aligned}$$

a contradiction as $\tau > 0$. Hence $u = v$. Conversely, if $Fix(T_1) \cap Fix(T_2)$ is singleton, then it follows that $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$. □

Remark 2.1. Let (X, d) be a complete metric space endowed with a directed graph G. If we replace (1.2) by either of the following three conditions:

$$\tau + F(d(u_x, u_y)) \le F(\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)), \tag{2.4}$$

where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma \le 1$, or

$$\tau + F(d(u_x, u_y)) \le F(h[d(x, u_x) + d(y, u_y)]), \tag{2.5}$$

where $h \in [0, \frac{1}{2}]$, or

$$\tau + F(d(u_x, u_y)) \le F(d(x, y)). \tag{2.6}$$

Then the conclusions obtained in Theorem 2.2 remain true.

Remark 2.2. (1) If $E(G) := X \times X$, then clearly G is connected and our Theorem 2.1 improves and generalizes (i) Theorem 1.9 in [6], (ii) Theorem 4.1 in [22], (iii) Theorem 3.4 of [28], and (iv) Theorem 3.1 of [29].

(2) If $E(G) := X \times X$, then Theorem 2.3 improves and extends Theorem 3.4 in [28], and Theorem 3.4 in [29].
(3) If $E(G) := X \times X$, then our Remark 2.4 extends and generalizes (i) Theorem 3.4 in [28] and (ii) Theorem 4.1 of [22].
(4) If $E(G) := X \times X$, then our Remark 2.4 improves and generalizes Theorem 4.1 in [22].
(5) If we take $T_1 = T_2$ in graphic F_1-contraction pair and graphic F_2-contraction pair, then we obtain the fixed point results for graphic F_1-contraction and graphic F_2-contraction of a single map.

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REFERENCES

[1] Abbas, M., Ali, B. and Romaguera, S., *Fixed and periodic points of generalized contractions in metric spaces*, Fixed Point Theory Appl, **2013**, 2013:243, 11 pp.
[2] Abbas, M., Ali, B. and Romaguera, S., *Generalized contraction and invariant approximation results on nonconvex subsets of normed spaces*, Abstr. Appl. Anal., **2014**, Art. ID 391952, 5 pp.
[3] Abbas, M., Khamsi, M. A. and Khan, A. R., *Common fixed point and invariant approximation in hyperbolic ordered metric spaces*, Fixed Point Theory Appl., **2011**, 2011:25, 14 pp.
[4] Abbas, M. and Nazir, T., *Common fixed point of a power graphic contraction pair in partial metric spaces endowed with a graph*, Fixed Point Theory Appl., **2013**, 2013:20, 8 pp.
[5] Abbas, M., Nazir, T. and Radenović, S., *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett., **24** (2011), 1520–1526
[6] Abbas, M., and Rhoades, B. E., *Fixed point theorems for two new classes of multivalued mappings*, Appl. Math. Lett., **22** (2009), 1364–1368
[7] Acar, O., Durmaz, G. and Minak, G., *Generalized multivalued F-contractions on complete metric spaces*, Bull. Iranian Math. Soc., **40** (2014), No. 6, 1469–1478
[8] Aleomraninejad, S. M. A., Rezapoura, Sh. and Shahzad, N., *Some fixed point results on a metric space with a graph*, Topology Appl., **159** (2012), 659–663
[9] Altun, I. and Simsek, H., *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl., **2010**, Art. ID 621469, 17 pp.
[10] Beg, I. and Butt, A. R., *Fixed point of set-valued graph contractive mappings*, J. Inequal. Appl., **52** (2013), 7 pp.
[11] Berinde, V., *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, **9** (2004), No. 1, 43–53
[12] Berinde, V., *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin–Heidelberg, 2007
[13] Berinde, M. and Berinde, V., *On a general class of multivalued weakly Picard mappings*, J. Math. Anal. Appl., **326** (2007), 772–782
[14] Berinde, V., *General constructive fixed point theorems for Ciric -type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), No. 2, 10–19
[15] Bojor, F., *On Jachymski's theorem*, An. Univ. Craiova Ser. Mat. Inform., An. Univ. Craiova Ser. Mat. Inform., **40** (2013), No. 1, 23–28
[16] Bojor, F., *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal., **75** (2012), 3895–3901
[17] Chifu, C. I. and Petrusel, G. R., *Generalized contractions in metric spaces endowed with a graph*, Fixed Point Theory Appl., **2012**, 2012:161, 9 pp.
[18] Gwozdz-Lukawska, G. and Jachymski, *J., IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl., **356** (2009), 453–463
[19] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373
[20] Jachymski, J. and Jozwik, I., *Nonlinear contractive conditions: a comparison and related problems*, Banach Center Publ., **77** (2007), 123–146
[21] Kannan, R., *Some results on fixed points*, Bull. Calcutta. Math. Soc., **60** (1968), 71–76
[22] Latif, A. and Beg, I., *Geometric fixed points for single and multivalued mappings*, Demonstratio Math., **30** (1997), No. 4, 791–800
[23] Markin, J. T., *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc., **38** (1973), 545–547

[24] Minak, G., Helvasi, A. and Altun, I., *Ciric type generalized F-contractions on complete metric spaces and fixed point results*, Filomat **28** (2014), No. 6, 1143–1151

[25] Nieto, J. J. and Lopez, R. R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239

[26] Nicolae, A., O'Regan, D. and Petrusel, A., *Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph*, J. Georgian Math. Soc., **18** (2011), 307–327

[27] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443

[28] Rus, I. A., Petrusel, A., and Sintamarian, A., *Data dependence of fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., **52** (2003) 1944–1959

[29] Sgroi, M. and Vetro, C., *Multi-valued F-Contractions and the Solution of certain functional and integral equations*, Filomat, **27** (2013), No. 7, 1259–1268

[30] Wardowski, D., *Fixed points of new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012**, 2012:94, 6 pp.

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