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On α-nonexpansive mappings in Banach spaces

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ABSTRACT. In 2011 Aoyama and Kohsaka introduced the α-nonexpansive mappings. Here we present a further study about them and their relationships with other classes of generalized nonexpansive mappings.

1. INTRODUCTION

Nonexpansive mappings are those which have Lipschitz constant equal to 1. For instance, isometries, contractions and resolvents of accretive operators are all nonexpansive. Since the mid sixties of the last century to the present a rich, although still far from being complete, fixed point theory for nonexpansive mappings has been developed. In [5] R. E. Bruck introduced a class of nonexpansive mappings which he called firmly nonexpansive. Given a non empty closed convex subset C of a Banach space $(X, \|\cdot\|)$, a mapping $T : C \to X$ is said to be firmly nonexpansive (FNE in short) if for all $x, y \in C$ and $t \geq 0$ it holds the inequality

$$\|Tx - Ty\| \leq \|t(x-y) + (1-t)(Tx - Ty)\|.$$

Of course, firmly nonexpansive mappings are nonexpansive. It happens that the resolvent of any accretive operator is just FNE. Due, among other things, to this important feature, the concept of FNE mapping has been widely studied and generalized in several ways. For instance, Kohsaka ans Takahashi defined in 2008 nonspreading mappings [10], Takahashi in 2010 hybrid mappings [12], and, in the setting of Hilbert spaces, Aoyama et.al. λ-hybrid mappings [1] again in 2010.

In 2011 Aoyama and Kohsaka [2] introduced, the so called α-nonexpansive mappings, a class of mappings which is wider than the above mentioned, that is, which properly contains the nonexpansive mappings and several of the generalizations of the FNE mappings.

On the other hand, shortly after the publication of the first fixed point results for nonexpansive mappings, several authors aimed to extend these results to wider classes of mappings. For instance, Goebel, Kirk and Shimi [7] in 1973, or Bogin [4] in 1976, gave fixed point theorems for generalized nonexpansive mappings in Banach spaces.

Very recently, in 2011 [11], a new class of nonlinear mappings was introduced. This class, called (L)-type mappings, properly includes the nonexpansive mappings, as well as many of its generalizations. Thus, it seems quite natural to wonder if there are some relationships between α-nonexpansive and (L)-type mappings. The aim of these notes is to analyze these relationships, as well as to study further properties of α-nonexpansive mappings.

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2. PRELIMINARIES

In the following we assume that C is a nonempty subset of a Banach space $(X, \|\cdot\|)$. The closed unit ball of X will be denoted as B_X. Recall that a mapping $T : C \to X$ is nonexpansive if, for all $x, y \in C$, $\|Tx - Ty\| \leq \|x - y\|$.

In the previous section we have reminded the definition of FNE mappings. It should be noted that, in some sense, the class of FNE mapping is quite restrictive. For instance, the identity mapping $Id : B_X \to B_X$ is trivially FNE, but $-Id$ fails to be FNE. Next, we recall the definitions of several classes of nonlinear mappings related to the FNE mappings.

Definition 2.1. A mapping $T : C \to X$ is called:

(1) (See [8], 1993). r-firmly nonexpansive if there exist $r \in (0, 1)$ such that for all $x, y \in C$,
$$\|Tx - Ty\| \leq \|(1 - r)(x - y) + r\,(Tx - Ty)\|.$$
(2) (See [10], 2008). Non-spreading if for all $x, y \in C$,
$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$
(3) (See [12], 2010). Hybrid if for all $x, y \in C$,
$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|x - y\|^2.$$
(4) (See [13], 2011). TJ-1 if for all $x, y \in C$,
$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2.$$
(5) (See [13], 2011). TJ-2 if for all $x, y \in C$,
$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2.$$

Remark 2.1. Several of the above definitions were originally formulated for mappings defined either in Hilbert spaces or in smooth Banach spaces. However, for the shake of simplicity and generality, we shall keep the same terminology for mappings defined in general Banach spaces.

In the setting of Hilbert spaces it is also relevant the following class of mappings.

Definition 2.2. Let C be a nonempty subset of a Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$, and let λ be a real number. A mapping $T : C \to H$ is said to be λ-hybrid if for every $x, y \in C$
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\,\langle x - Tx, y - Ty \rangle.$$

The main class of nonlinear mappings that we will be concerned with is the following.

Definition 2.3 ([2], 2011)**.** For a given real number $\alpha < 1$, a mapping $T : C \to X$ is said to be α-nonexpansive if, for all $x, y \in C$,
$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2. \tag{2.1}$$

It is easy to check that the identity mapping Id is α-nonexpansive for all $\alpha < 1$. In the first place, notice that a mapping $T : C \to X$ is 0-nonexpansive if and only if T is nonexpansive. Secondly, the $\frac{1}{2}$-nonexpansive mappings are just the nonspreading mappings. Finally, we point out that the $\frac{1}{3}$-nonexpansive mappings are the hybrid mappings.

Recall that if $T : C \to X$ is a mapping, a sequence (x_n) in C is called an almost fixed point sequence (a.f.p.s. for short) for T in C whenever $x_n - T(x_n) \to 0_X$.

Definition 2.4 ([6], 2011)**.** A mapping $T : C \to X$ satisfies condition (A) on C whenever there exists an a.f.p.s. for T in each nonempty, closed, convex and T-invariant subset D of C, that is, if $\inf\{\|x - Tx\| : x \in D\} = 0$ for every such subset D.

Our main goal is to study the relationships between α-nonexpansive mappings and the following class of nonlinear mappings.

Definition 2.5 ([11], 2011)**.** A mapping $T : C \to C$, where C is a nonempty closed bounded subset of a Banach space X, satisfies condition (L) (or it is an (L)-type mapping) on C provided that it fulfills the following two conditions

(C_1) T satisfies condition (A) on C.
(C_2) For any a.f.p.s. (x_n) of T in C and each $x \in C$

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\| .$$

Finally we recall here two geometric properties of Banach spaces which are relevant in metric fixed point theory. Given a Banach space $(X, \|\cdot\|)$, its modulus of convexity is the function $\delta_X : [0, 2] \to [0, 1]$ given by

$$\delta_X(\varepsilon) := \inf \left\{1 - \left\|\tfrac{1}{2}(x + y)\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\} .$$

The *characteristic of convexity* of X is the real number $\varepsilon_0(X) := \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}$. The space X is *uniformly convex* whenever $\varepsilon_0(X) = 0$.

For a nonempty bounded subset C of a Banach space X, we will denote

$$\operatorname{diam}(C) := \sup\{\|x - y\| : x, y \in C\} \qquad \text{and} \qquad \operatorname{rad}(C) := \inf_{x\in C} \sup\{\|x - y\| : y \in C\}.$$

A Banach space X is said to have *normal structure* if every bounded closed convex subset C of X with $\operatorname{diam}(C) > 0$, satisfies that $\operatorname{rad}(C) < \operatorname{diam}(C)$, that is, if the set C has a point x_0 such that

$$\sup\left\{\|x_0 - x\| : x \in C\right\} < \operatorname{diam}(C).$$

3. α-NONEXPANSIVE MAPPINGS: STRAIGHTFORWARD CONSEQUENCES OF THE DEFINITION

Although α-nonexpansive mappings are defined for any real number $\alpha < 1$, we first point out that this concept is trivial for $\alpha < 0$. From now on, C denotes a nonempty, closed, bounded and convex subset of X.

Remark 3.2. *For $\alpha < 0$, the unique α-nonexpansive mapping is the identity $Id : C \to C$.* Indeed, taking $y = x \in C$ in inequality (2.1) we obtain, for every $x \in C$, that

$$0 \leq 2\alpha \|Tx - x\|^2 .$$

Bearing in mind that $\alpha < 0$, it follows that $Tx = x$ for all $x \in C$.

Proposition 3.1. *Every constant mapping $T : C \to C$, that is, every mapping defined as $Tx = x_0 \in C$ for all $x \in C$, is α-nonexpansive provided that $0 \leq \alpha \leq \frac{2}{3}$.*

Proof. Since T is constant, the left hand side of (2.1) is just 0. The right hand side of this inequality is nonnegative whenever $0 \leq \alpha \leq \frac{1}{2}$. Hence, we only need to show that (2.1) holds when $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. In this case, $3\alpha \leq 2$ which implies that

$$\frac{2(2\alpha - 1)}{\alpha} \leq 1. \tag{3.2}$$

From the convexity of the real function $t \mapsto t^2$, for every $x, y \in X$, we have

$$\|x - y\|^2 \leq \big(\|x\| + \|y\|\big)^2 = 4\left(\frac{\|x\| + \|y\|}{2}\right)^2 \leq 4\,\frac{\|x\|^2 + \|y\|^2}{2} = 2\big(\|x\|^2 + \|y\|^2\big).$$

It follows from (3.2) that

$$\frac{2\alpha-1}{\alpha}\,\|x-y\|^2 \le \frac{2(2\alpha-1)}{\alpha}\left(\,\|x\|^2+\|y\|^2\,\right) \le \|x\|^2+\|y\|^2\,.$$

Then, if $u,v\in C$, taking $x=u-x_0$ and $y=v-x_0$ in this inequality, we obtain that

$$\frac{2\alpha-1}{\alpha}\,\|u-v\|^2 \le \|u-x_0\|^2+\|v-x_0\|^2\,.$$

It follows that

$$\|Tu-Tv\|^2 = 0 \le \alpha\left(\,\|u-x_0\|^2+\|v-x_0\|^2\,\right)+(1-2\alpha)\,\|u-v\|^2\,.$$

□

The above proposition is sharp in the sense that there are constant mappings failing to be α-nonexpansive for $\alpha>\frac{2}{3}$.

Example 3.1. Let B_X be the closed unit ball of a Banach space $(X,\|\cdot\|)$, and let $T:B_X\to B_X$ be the constant mapping given by $Tx=0_X$. Suppose that T is α-nonexpansive for some $\alpha\in(\frac{2}{3},1)$. Then, for any $v\in S_X\subset B_X$, taking $x=v$, $y=-v$ in Definition 2.3 we obtain that

$$0\le 2\alpha+4(1-2\alpha)=4-6\alpha,$$

that is, $\alpha\le\frac{2}{3}$ which is a contradiction.

Proposition 3.2. *Every TJ1 mapping $T:C\to X$ is $\frac{1}{4}$-nonexpansive.*

Proof. For every $x,y\in C$,

$$2\|Tx-Ty\|^2\le\|x-y\|^2+\|Tx-y\|^2$$

and hence

$$2\|Tx-Ty\|^2\le\|x-y\|^2+\|Ty-x\|^2.$$

Thus

$$4\|Tx-Ty\|^2\le 2\|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2.$$

It follows that

$$\|Tx-Ty\|^2\le\frac{1}{4}\|Tx-y\|^2+\frac{1}{4}\|Ty-x\|^2+(1-2\cdot\frac{1}{4})\|x-y\|^2.$$

□

In the same way, on can prove the following result.

Proposition 3.3. *Every TJ2 mapping $T:C\to X$ is nonspreading and hence $\frac{1}{2}$-nonexpansive.*

In the setting of Hilbert spaces, it may be checked that both TJ-1 and TJ2 mappings are λ-hybrid.

3.1. **Relative to 2-periodic points.** According to the following propositions, the subclass of 0-nonexpansive mappings is, in some sense, very singular in the class of α-nonexpansive mappings. For instance, if a mapping T admits any non fixed 2-periodic point, then T cannot be α-nonexpansive, except, perhaps, for $\alpha=0$. More precisely, one has the following result.

Proposition 3.4. *Let $T:C\to C$ be a mapping. If there exists $x\in C$ such that $T^2x=x$ then either*

(a) x is a fixed point for T, or

(b) for any $\alpha\in(0,1)$, T is not α-nonexpansive.

Proof. If T is α-nonexpansive for $\alpha \in (0,1)$ then

$$\begin{aligned}\|x - Tx\|^2 = \|T^2x - Tx\|^2 &\leq \alpha\|T^2x - x\|^2 + \alpha\|Tx - Tx\|^2 + (1-2\alpha)\|Tx - x\|^2 \\ &= (1-2\alpha)\|Tx - x\|^2.\end{aligned}$$

From this we obtain that $0 \leq -\alpha\|Tx - x\|^2$. If x is not a fixed point for T we get a contradiction, because $\alpha > 0$. □

Of course, there are examples of nonexpansive mappings admitting non fixed 2-periodic points, as for instance the mapping $T(x) = -x$ defined on B_X. In the same way than the above result, the following proposition is an easy test to discard that a given mapping is α-nonexpansive for $\alpha \in (0,1)$.

Proposition 3.5. *Let $T : C \to C$ be a mapping. Assume that there exists a point $x \in C$ such that $\|x - Tx\| = \|Tx - T^2x\| = \|T^2x - x\|$. Then either*

(a) x is a fixed point of T, or
(b) For any $\alpha \in (0,1)$, T is not α-nonexpansive.

Proof. If T is α-nonexpansive with $\alpha \in (0,1)$, then

$$\begin{aligned}\|Tx - x\|^2 = \|T^2x - Tx\|^2 &\leq \alpha\|T^2x - x\|^2 + \alpha\|Tx - Tx\|^2 + (1-2\alpha)\|Tx - x\|^2 \\ &\leq \alpha\|Tx - x\|^2 + (1-2\alpha)\|Tx - x\|^2.\end{aligned}$$

Therefore, $0 \leq -\alpha\|Tx - x\|^2$. If x is not a fixed point we have a contradiction with the fact that $\alpha \in (0,1)$. □

4. A FIXED POINT FREE NONSPREADING MAP

There are well known fixed point free 0-nonexpansive mappings on closed convex and bounded subsets of (nonreflexive) Banach spaces. But, for $\alpha \in (0,1)$ it is unclear whether there exist such fixed point free α-nonexpansive mappings. In [2] it is proved the following theorem.

Theorem 4.1. *Assume that X is a uniformly convex Banach space, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be an α-nonexpansive mapping for some real number $\alpha \in (-\infty, 1)$. Then $Fix(T)$ is nonempty if and only if there exists $x \in C$ such that $(T^n x)$ is bounded.*

The aim of this section is to prove that the fixed point theory for α-nonexpansive mappings does have a sense, even for $\alpha \in (0,1)$. In order to see this, we will give a family of fixed-point free $\frac{1}{2}$-nonexpansive mappings.

Example 4.2. Let $\mathcal{C}[0,1]$ be the Banach space of continuous functions $x : [0,1] \to \mathbb{R}$ endowed with the sup norm. Consider the subset C defined as

$$C := \left\{x \in \mathcal{C}[0,1] : x(0) = 0 \leq x(t) \leq x(1) = 1\right\}.$$

Let $f : [0,1] \to [0,1]$ be a continuous function with $f(1) = 1$, such that $f(t) \neq 1$ for t close enough to 1. Then, the mapping $T_f : C \to C$ given by $T_f x(t) = f(t)\,x(t)$ is $\frac{1}{2}$-nonexpansive with no fixed points.

Indeed, it is obvious that T_f does not have fixed points in C. Moreover, for $t \in [0,1]$ and $x, y \in C$, one has

$$\begin{aligned} f(t)\,|x(t)-y(t)| &\le \frac{1}{2}(1+f(t))\ |x(t)-y(t)| \\ &\le \frac{1}{2}\,|x(t)-T_f y(t)| + \frac{1}{2}\,|T_f x(t)-y(t)| \\ &\le \frac{1}{2}\,\|x-T_f y\|_\infty + \frac{1}{2}\,\|T_f x - y\|_\infty \end{aligned}$$

and therefore

$$\|T_f x - T_f y\|_\infty \le \frac{1}{2}\|x - T_f y\|_\infty + \frac{1}{2}\|T_f x - y\|_\infty.$$

Bearing in mind the convexity of the function $t \mapsto t^2$, we obtain that

$$\|T_f x - T_f y\|_\infty^2 \le \frac{1}{2}\|x - T_f y\|_\infty^2 + \frac{1}{2}\|T_f x - y\|_\infty^2,$$

that is, T_f is $\frac{1}{2}$-nonexpansive.

5. Some further properties of α-nonexpansive mappings

In this section we will try to get a better knowledge of the α-nonexpansive mappings. In order to do this, the following notation will be useful. For $\alpha < 1$ let

$$\mathcal{N}_\alpha(C) := \big\{T : C \to X \mid T \text{ is } \alpha\text{-nonexpansive}\big\}.$$

5.1. **An interpolation-type property.** It is known that a given mapping $T : C \to X$ can belong to more than one of the subfamilies $\mathcal{N}_\alpha$. For instance, in [2] it was shown that every firmly nonexpansive selfmapping of C belongs to $\mathcal{N}_\alpha(C)$ whenever $0 \le \alpha \le \frac{1}{2}$. Moreover, all the examples of α-nonexpansive mappings given in [2] are, in fact, in $\mathcal{N}_\alpha(C)$ for $0 \le \alpha \le \frac{1}{2}$. Below we shall give an example of a mapping in $\mathcal{N}_\alpha(C)$ for $\frac{1}{2} < \alpha \le 1$. We start with the following result. It follows directly from the definition.

Proposition 5.6. *Assume that $T \in \mathcal{N}_{\alpha_1}(C) \cap \mathcal{N}_{\alpha_2}(C)$, with $\alpha_1 \neq \alpha_2$, then $T \in \mathcal{N}_{\frac{\alpha_1+\alpha_2}{2}}(C)$.*

Corollary 5.1. *Let $T \in \mathcal{N}_{\alpha_1}(C) \cap \mathcal{N}_{\alpha_2}(C)$ with $\alpha_1 < \alpha_2$ then $T \in \mathcal{N}_\alpha(C)$ for every $\alpha \in [\alpha_1, \alpha_2]$.*

Proof. If $T \in \mathcal{N}_{\alpha_1}(C) \cap \mathcal{N}_{\alpha_2}(C)$ with $\alpha_1 < \alpha_2$ then, from the above proposition, by induction we get that T belongs to $\mathcal{N}_{\frac{(2^n-k)\alpha_1 + k\alpha_2}{2^n}}(C)$ for all $n \in \mathbb{N}$ and $k = 0, \dots, 2^n$. It is well known that the set S, defined as

$$S = \left\{ \frac{(2^n-k)\alpha_1 + k\alpha_2}{2^n} : n \in \mathbb{N}, k = 0, \dots, 2^n \right\},$$

is dense in $[\alpha_1, \alpha_2]$. Given $\alpha \in [\alpha_1, \alpha_2]$ there is a sequence (s_n) in S such that $\lim_n s_n = \alpha$. Fix $x, y \in C$. For every s_n we have

$$\|Tx - Ty\|^2 \le s_n\|Tx - y\|^2 + s_n\|Ty - x\|^2 + (1-2s_n)\|x-y\|^2.$$

Letting $n \to \infty$,

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1-2\alpha)\|x-y\|^2.$$

□

The following example shows that the class $\mathcal{N}_{\frac{1}{2}}$ is not stable under arithmetic means.

Example 5.3. Consider the mapping $T : [0,3] \to [0,3]$ given by $Tx = 2\chi_{\{3\}}(x)$. When $x, y \neq 3$ we have

$$|Tx - Ty|^2 = 0 \leq \frac{1}{2}|x - Ty|^2 + \frac{1}{2}|y - Tx|^2.$$

If $x, y \neq 3$ or $x = y = 3$, then

$$2|Tx - Ty|^2 = 8 \leq 9 + |y - 2|^2 = |x - Ty|^2 + |y - Tx|^2.$$

Hence we deduce that $T \in \mathcal{N}_{\frac{1}{2}}$. However if we define $\tilde{T} := \frac{1}{2}T + \frac{1}{2}Id$, i.e.

$$\tilde{T}x = \begin{cases} \frac{5}{2} & \text{if } x = 3, \\ \frac{x}{2} & \text{otherwise,} \end{cases}$$

this mapping is not $\frac{1}{2}$-nonexpansive. Indeed, taking $x = 3$ and $y = 2$ we have

$$2|\tilde{T}(3) - \tilde{T}(2)|^2 = \frac{9}{2} \qquad \text{and} \qquad |3 - \tilde{T}(2)|^2 + |2 - \tilde{T}(3)|^2 = 4 + \frac{1}{4} = \frac{17}{4}.$$

Notice that it can be easily checked that $\tilde{T}$ is $\frac{3}{4}$-nonexpansive.

We finish this section studying the set $\bigcap_{\alpha\in[0,1)} \mathcal{N}_\alpha(C)$. We know that $Id \in \bigcap_{\alpha\in[0,1)} \mathcal{N}_\alpha(C)$. It is quite natural to wonder if this identity mapping is the only element in this set. We will give a negative answer, (see Example 5.4 below), with the help of the following obvious result.

Proposition 5.7. *For each $i = 1, 2$, let C_i be a nonempty subset of a normed space $(X_i, \|\cdot\|_i)$ and $\alpha \in [0,1)$. Assume that, for each $i = 1, 2$, $T_i : C_i \to C_i$ is an α-nonexpansive mapping with respect to the norm $\|\cdot\|_i$. Then, the mapping $T : C_1 \times C_2 \to C_1 \times C_2$, defined by*

$$T(x_1, x_2) := \big(T_1(x_1), T_2(x_2)\big),$$

is α-nonexpansive with respect to the product norm $\|(x_1, x_2)\| := \Big[\|x_1\|_1^2 + \|x_2\|_2^2\Big]^{\frac{1}{2}}$.

Example 5.4. In the real line, consider $C_1 = C_2 = [0,1]$. Let $T_1 = Id : C_1 \to C_1$. Let $T_2 : C_2 \to C_2$ be the null mapping. Then it is obvious that T_1 is α-nonexpansive for every $\alpha \in [0,1)$. Moreover, T_2 is also α-nonexpansive for every $\alpha \in [0,1)$. Indeed, for $x, y \in [0,1]$, we have that

$$\begin{aligned} \alpha|x - T_2y|^2 + \alpha|y - T_2x|^2 + (1-2\alpha)|x - y|^2 &= \alpha(x^2 + y^2) + (1-2\alpha)(x-y)^2 \\ &= (1-\alpha)(x^2 + y^2) - 2(1-2\alpha)xy \\ &\geq 2(1-\alpha)xy - 2(1-2\alpha)xy \\ &= 2\alpha xy \geq |T_2x - T_2y|^2. \end{aligned}$$

From the previous proposition, for all $\alpha \in [0,1)$, the mapping $T : [0,1] \times [0,1] \to [0,1] \times [0,1]$, defined by

$$T(x_1, x_2) := \big(T_1x_1, T_2x_2\big),$$

is α-nonexpansive with respect to the standard Euclidean norm, and, of course, it is not the identity mapping.

6. Relationships with other classes of mappings

This section is devoted to the study of the relationships between the α-nonexpansive mappings and other classes of nonlinear mappings which are relevant in metric fixed point theory.

6.1. **Continuous mappings.** It is obvious that every 0-nonexpansive mapping is continuous. However, for $\alpha > 0$ there exists no relationship between α-nonexpansiveness and continuity. In [2] it is proved that given $\alpha \in \left(0, \frac{1}{2}\right]$ there exists a discontinuous α-nonexpansive mapping. The next example shows that there exist discontinuous α-nonexpansive mappings when $\alpha \in \left(\frac{1}{2}, 1\right)$ even in the real line.

Example 6.5. Consider $X = \mathbb{R}$ with its usual norm and $C = [0,1]$. Let $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\varepsilon \in \left(0, 2 - \frac{1}{\alpha}\right)$. Then, the mapping $T_\varepsilon : C \to C$ given by

$$T_\varepsilon x = \begin{cases} 0 & \text{if } x \neq 1 \\ \varepsilon & \text{if } x = 1 \end{cases}$$

is α-nonexpansive. Take $x, y \in [0,1]$. Indeed, if x, y are both different to 1, in the same way that T_2 in Example 5.4, inequality (2.1) can be deduced. This equality is trivial when $x = y = 1$. Without loss of generality we can assume that $0 \leq x < 1$ and $y = 1$. Note that (2.1) is equivalent to the following inequality

$$0 \leq \alpha(x-\varepsilon)^2 + \alpha + (1-2\alpha)(1-x)^2 - \varepsilon^2,$$

which is true because the function $f_{\alpha,\varepsilon} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_{\alpha,\varepsilon}(t) = \alpha(t-\varepsilon)^2 + \alpha + (1-2\alpha)(1-t)^2 - \varepsilon^2$$

is increasing on $[0,1]$ and $f_{\alpha,\varepsilon}(0) \geq 0$.

Remark 6.3. Notice that the previous mappings are also λ-hybrid for $\lambda < 0$.

6.2. **λ-firmly nonexpansive mappings.** It is well known that the class of λ-firmly nonexpansive mappings is wider than the class of firmly nonexpansive mappings. Proposition 2.3 in [2] established that a firmly nonexpansive is α-nonexpansive for $\alpha \in [0, \frac{1}{2}]$. The following proposition generalizes this result.

Proposition 6.8. *Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$, and $\lambda \in [0,1)$. If $T : C \to X$ is λ-firmly nonexpansive, then T is an α-nonexpansive mapping with $\alpha = \frac{\lambda}{1+\lambda}$.*

Proof. Let $x, y \in C$. Since T is λ-firmly nonexpansive, we have that

$$\begin{aligned}
\|Tx - Ty\| &\leq \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\| \\
&= \left\|\lambda(1-\lambda)(x - Ty + Tx - y) + (1-\lambda)^2(x-y) + \lambda^2(Tx-Ty)\right\| \\
&\leq \lambda(1-\lambda)\Big[\, \|x - Ty\| + \|Tx - y\| \,\Big] + (1-\lambda)^2 \|x-y\| + \lambda^2 \|Tx - Ty\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|Tx - Ty\| &\leq \frac{\lambda(1-\lambda)}{1-\lambda^2}\Big[\, \|x - Ty\| + \|Tx - y\| \,\Big] + \frac{(1-\lambda)^2}{1-\lambda^2}\|x-y\| \\
&= \alpha\Big[\, \|x - Ty\| + \|Tx - y\| \,\Big] + (1-2\alpha)\|x-y\|,
\end{aligned}$$

where $\alpha = \frac{\lambda}{1+\lambda}$. Bearing in mind that $1 - 2\alpha > 0$ and the convexity of the function $t \mapsto t^2$, we deduce that

$$\|Tx - Ty\|^2 \leq \alpha\left[\|x - Ty\|^2 + \|y - Tx\|^2\right] + (1-2\alpha)\|x-y\|^2.$$

□

Remark 6.4. Notice that if T is λ-firmly nonexpansive, with $0 \leq \lambda \leq \frac{1}{2}$, we also have that T is α-nonexpansive with $\alpha = \lambda$. Indeed, for any $x, y \in C$ we have that

$$\begin{aligned} \|Tx - Ty\| &\leq \|(1-\lambda)(x-y) + \lambda\,(Tx - Ty)\| \\ &= \|\lambda(x - Ty + Tx - y) + (1-2\lambda)(x-y)\| \\ &\leq \lambda\Big[\,\|x - Ty\| + \|Tx - y\|\,\Big] + (1-2\lambda)\,\|x-y\|\,. \end{aligned}$$

Using the convexity of the function $t \mapsto t^2$, we obtain that

$$\|Tx - Ty\|^2 \leq \lambda\left[\|x - Ty\|^2 + \|y - Tx\|^2\right] + (1-2\lambda)\,\|x-y\|^2\,.$$

It is known that every λ-firmly nonexpansive mapping is nonexpansive. Then, using this fact along with Corollary 5.1, we deduce the following result.

Corollary 6.2. *Let C be a nonempty subset of a normed space X, and $\lambda \in [0,1)$. If $T : C \to X$ is λ-firmly nonexpansive mapping, then $T \in \mathcal{N}_\alpha$ for all $\alpha \in [0, \hat{\lambda}]$, where*

$$\hat{\lambda} := \begin{cases} \lambda & \textit{if} \quad 0 \leq \lambda \leq \frac{1}{2}, \\ \frac{\lambda}{1+\lambda} & \textit{if} \quad \frac{1}{2} < \lambda < 1. \end{cases}$$

Remark 6.5. The proofs of Proposition 6.8 and Remark 6.4 allow us to claim that every λ-firmly nonexpansive mapping is in fact a generalized nonexpansive mapping in the sense of [4]. (See Section 6.4 below).

6.3. **Contractive mappings.**

Proposition 6.9. *Let C be a nonempty subset of a Banach space X. If $T : C \to X$ is k-contractive for $k \in (\frac{1}{3}, 1)$, then T is α-nonexpansive for every $\alpha \in \left[0, \frac{1-k}{1+k}\right]$.*

Proof. For $x, y \in C$,

$$\begin{aligned} \|Tx - Ty\| \leq k\|x-y\| &= \left\| k(x-y) + \frac{1-k}{2}(Tx - Ty) - \frac{1-k}{2}(Tx - Ty)\right\| \\ &\leq \left\| k(x-y) + \frac{1-k}{2}(Tx - Ty)\right\| + \frac{1-k}{2}\|Tx - Ty\|. \end{aligned}$$

Then

$$\frac{1+k}{2}\|Tx - Ty\| \leq \left\| k(x-y) + \frac{1-k}{2}(Tx - Ty)\right\|,$$

from which it follows immediately

$$\|Tx - Ty\| \leq \left\| \frac{2k}{1+k}(x-y) + \left(\frac{1-k}{1+k}\right)(Tx - Ty)\right\|.$$

Thus, the mapping T is $\frac{1-k}{1+k}$-firmly nonexpansive. From Corollary 6.2 we obtain the desired result. □

The following example shows that the range of values of α given in the above result might not be sharp in the setting of Hilbert spaces.

Example 6.6. Let H be a Hilbert space. The mapping $T : B_H \to B_H$ given by $Tx = \frac{1}{2}x$ is α-nonexpansive for every $\alpha \in \left[0, \frac{3}{4}\right]$. For $x, y \in B_H$,

$$\begin{aligned}\|Tx - y\|^2 + \|Ty - x\|^2 &= \frac{1}{4}\left(\|x - 2y\|^2 + \|y - 2x\|^2\right)\\ &= \frac{5}{4}(\|x\|^2 + \|y\|^2) - 2\langle x, y\rangle\\ &= \|x - y\|^2 + \frac{1}{4}\left(\|x\|^2 + \|y\|^2\right).\end{aligned}$$

Then, for $\alpha \in \left[0, \frac{3}{4}\right]$, we have that

$$\begin{aligned}(1 - 2\alpha)\|x - y\|^2 + \alpha\left(\|Tx - y\|^2 + \|Ty - x\|^2\right) &= (1 - \alpha)\|x - y\|^2 + \frac{\alpha}{4}\left(\|x\|^2 + \|y\|^2\right)\\ &\geq \frac{\|x - y\|^2}{4} + \frac{\alpha}{4}\left(\|x\|^2 + \|y\|^2\right)\\ &\geq \|Tx - Ty\|^2.\end{aligned}$$

Proposition 6.10. *Let C be a nonempty subset of a Banach space X. If $T : C \to X$ is k-contractive, with $k \in \left[0, \frac{1}{3}\right]$, then T is α-nonexpansive for every $\alpha \in \left[0, \frac{1}{2}\right]$.*

Proof. For $x, y \in C$,

$$\|Tx - Ty\| \leq k\|x - y\| \leq k\left[\|x - Ty\| + \|y - Tx\| + \|Tx - Ty\|\right].$$

Then

$$\|Tx - Ty\| \leq \frac{k}{1 - k}\left[\|x - Ty\| + \|y - Tx\|\right] \leq \frac{1}{2}\left[\|x - Ty\| + \|y - Tx\|\right].$$

Using again the convexity of the function $t \mapsto t^2$, we have that

$$\|Tx - Ty\|^2 \leq \frac{1}{2}\left[\|x - Ty\|^2 + \|y - Tx\|^2\right],$$

that is, T is non-spreading on C. Since T is also 0-nonexpansive, from Corollary 5.1 we obtain that T is α-nonexpansive for every $\alpha \in \left[0, \frac{1}{2}\right]$. □

6.4. **Generalized nonexpansive mappings.** Recall that a mapping $T : C \to X$ is said to be generalized nonexpansive if there exist five nonnegative constants a_i ($i = 1, \ldots, 5$) with $\sum_{i=1}^{5} a_i \leq 1$ and such that, for every $x, y \in C$,

$$\|Tx - Ty\| \leq a_1\|x - y\| + a_2\|x - Tx\| + a_3\|y - Ty\| + a_4\|x - Ty\| + a_5\|y - Tx\|.$$

Since the distance function is symmetric we can replace a_2, a_3 with $(a_2 + a_3)/2$ and a_4, a_5 with $(a_4 + a_5)/2$. Thus, the above definition is equivalent to the existence of nonnegative constants a, b, c satisfying that $a + 2b + 2c \leq 1$ such that for $x, y \in C$

$$\|Tx - Ty\| \leq a\|x - y\| + b\big(\|x - Tx\| + \|y - Ty\|\big) + c\big(\|x - Ty\| + \|y - Tx\|\big).$$

Proposition 6.11. *Let $T : C \to X$ be a generalized nonexpansive mapping with $b = 0$. Then T is c-nonexpansive.*

Proof. If $b = 0$ then $a \leq 1 - 2c$. For $x, y \in C$, from the Jensen inequality one has,

$$\begin{aligned}\|Tx - Ty\|^2 &\leq \Big(a\,\|x - y\| + c\left(\|x - Ty\| + \|y - Tx\|\right)\Big)^2\\ &\leq a\,\|x - y\|^2 + c\,\|x - Ty\|^2 + c\,\|y - Tx\|^2\\ &\leq c\,\|x - Ty\|^2 + c\,\|y - Tx\|^2 + (1 - 2c)\,\|x - y\|^2.\end{aligned}$$

□

Example 6.7. Let us consider the mapping $T : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ given by $Tx = x^2$.

For every fixed $\alpha \in (\frac{5}{6}, 1)$, put $h(t) := t + t^2$, $g(t) := 2\alpha t$. It is easy to check that $h(t) \leq g(t)$ for all $t \in [0, \frac{2}{3}]$.

It follows that for every $(x, y) \in [0, \frac{2}{3}] \times [0, \frac{2}{3}]$,

$$a(x, y) := g(x) - h(x) + (1 - \alpha)h(y) \geq 0, \text{ and } b(x, y) := g(y) - h(y) + (1 - \alpha)h(x) \geq 0.$$

Therefore,

$$(y - y^2)a(x, y) + (x - x^2)b(x, y) \geq 0,$$

that is,

$$\alpha((x - y^2)^2 + (y - x^2)^2) + (1 - 2\alpha)(x - y)^2 - (x^2 - y^2)^2 \geq 0,$$

or, in other words, for $\frac{5}{6} \leq \alpha < 1$, the mapping T is α-nonexpansive.

However, in [11, Example 3.7] it is showed that T fails to be generalized nonexpansive on $[0, \frac{2}{3}]$.

6.5. **Mean nonexpansive mappings.** In 2007 Goebel and Japón Pineda [9] introduced a new class of mappings called a-mean nonexpansive mappings, which is wider than the class of the nonexpansive mappings. Recall that $T : C \to C$ is a-mean nonexpansive if for all $x, y \in C$

$$\sum_{i=1}^{n} a_i \left\| T^i x - T^i y \right\| \leq \|x - y\|,$$

where the multi-index $a = (a_1, a_2, \ldots, a_n)$ satisfies $a_i \geq 0$, for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$. For technical reasons, it is always assumed that the first coefficient a_1 and the last a_n are nonzero, $a_1 > 0$ and $a_n > 0$. In what follows, for the sake of simplicity, we will be concerned with the particular case of multi-indices of length 2, that is, with the form $(a, 1 - a)$ with $a \neq 0$.

Definition 6.6. *Let $a \in (0, 1]$ and C a nonempty subset of a normed space $(X, \|\cdot\|)$. We say that $T : C \to C$ is a-mean nonexpansive if*

$$a \|Tx - Ty\| + (1 - a) \left\|T^2 x - T^2 y\right\| \leq \|x - y\| \qquad \text{for all } x, y \in C. \tag{6.3}$$

Remark 6.6. Notice that every a-mean nonexpansive mapping is continuous, because $a > 0$. Then, by the considerations included in Section 6.1, we deduce that, for any $0 < \alpha < 1$, there exist mappings in $\mathcal{N}_\alpha(C)$ which are not a-mean nonexpansive for any $0 < a \leq 1$.

The following example, along with the previous remark, shows that none of the classes a-mean nonexpansive mappings and α-nonexpansive mappings is included in the other one.

Example 6.8 (Example 2 in [9])**.** Consider as C the unit ball in the space $\mathbb{R}^4$ endowed with the ℓ_1-norm, that is $\|x\|_1 = \sum_{n=1}^{4} |x_i|$. Let $\tau : \mathbb{R} \to [-\frac{1}{3}, \frac{1}{3}]$ be the function that truncates the argument on the levels $-\frac{1}{3}$ and $\frac{1}{3}$, that is,

$$\tau(t) = \max\left\{-\frac{1}{3}, \min\left\{\tfrac{1}{3}, t\right\}\right\}.$$

Define the mapping $T : C \to C$ by

$$Tx = T(x_1, x_2, x_3, x_4) = \left(\tau(\tfrac{2}{3}x_4), \tau(2x_1), 0, \tau(\tfrac{6}{5}x_3)\right).$$

It is claimed in [9] that, for all $a \in (0, 1]$, the mapping T is a-mean nonexpansive. However, for any $\alpha \in (0, 1)$, T is not α-nonexpansive. Indeed, for $x = (\frac{1}{6}, \frac{1}{6}, 0, 0)$, we have that $Tx = (0, \frac{1}{3}, 0, 0)$ and $T^2x = (0, 0, 0, 0)$. Since $\|Tx - x\|_1 = \left\|Tx - T^2x\right\|_1 = \left\|x - T^2x\right\|_1 = \frac{1}{3}$, and x is not a fixed point of T, from Proposition 3.5 we deduce that, for any $0 < \alpha < 1$, T is not α-nonexpansive.

6.6. **Relation with (L)-type mappings.** Very recently in [11] the authors introduced a class of (single-valued) nonexpansive generalized mappings, which they called (L)-type mappings. Such class properly contains several other classes of mappings which in turn are more general than the class of nonexpansive mappings.

We will check that there are some relations between α-nonexpansive mappings and (L)-type mappings. In order to do this, we need the following result. Recall first that if ℓ_∞ denote the Banach space of bounded real sequences with the supremum norm, it is known that there exists a bounded linear functional μ on ℓ_∞ such that the following three conditions hold:

(1) If $(t_n) \in \ell_\infty$ with $t_n \geq 0$ for every positive integer n, then $\mu(t_n) \geq 0$;
(2) If $t_n \equiv 1$ then $\mu(t_n) = 1$;
(3) For every $(t_n) \in \ell_\infty$, $\mu(t_n) = \mu(t_{n+1})$

Such a functional μ is called a Banach limit. It is well known that if μ is a Banach limit, for every $(t_n) \in \ell_\infty$

$$\liminf_n t_n \leq \mu(t_n) \leq \limsup_n t_n.$$

Lemma 6.1. *Let $T : C \to X$ be an α-nonexpansive mapping. Assume that (x_n) is a bounded a.f.p.s. for T in C. Then, for all $x \in C$, every Banach limit μ satisfies*

$$\mu\left(\|x_n - Tx\|^2\right) \leq \mu\left(\|x_n - x\|^2\right).$$

Proof. We have that for $x \in C$

$$\begin{aligned}\|x_n - Tx\|^2 &\leq (\|x_n - Tx_n\| + \|Tx_n - Tx\|)^2 \\ &= \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\|\|Tx_n - Tx\| + \|Tx_n - Tx\|^2.\end{aligned}$$

Taking Banach limits in both sides, using the fact that $\|x_n - Tx_n\| \to 0$ and the α-nonexpansiveness of T we obtain

$$\begin{aligned}\mu\|x_n - Tx\|^2 &\leq \mu\|Tx_n - Tx\|^2 \\ &\leq \mu\left(\alpha\|Tx_n - x\|^2 + \alpha\|Tx - x_n\|^2 + (1-2\alpha)\|x_n - x\|^2\right).\end{aligned}$$

Then,

$$(1-\alpha)\mu\|x_n - Tx\|^2 \leq \alpha\mu\|Tx_n - x\|^2 + (1-2\alpha)\mu\|x_n - x\|^2. \tag{6.4}$$

Bearing in mind that

$$(\|x_n - x\| - \|Tx_n - x_n\|)^2 \leq \|Tx_n - x\|^2 \leq (\|Tx_n - x_n\| + \|x_n - x\|)^2,$$

we deduce that

$$\mu\|x_n - x\|^2 = \mu\|Tx_n - x\|^2.$$

Replacing this equality in (6.4) we get

$$(1-\alpha)\mu\|x_n - Tx\|^2 \leq (1-\alpha)\mu\|x_n - x\|^2.$$

Since $1 - \alpha > 0$ we obtain the desired result. □

Theorem 6.2. *Let $T : C \to C$ be an α-nonexpansive mapping for some $0 \leq \alpha < 1$, where C is a nonempty closed bounded subset of X. If T satisfies condition (A) on C, then it satisfies condition (L).*

Proof. It is enough to see that the mapping T satisfies condition (C_2) of Definition 2.5. Let (x_n) be an a.f.p.s. for T on C. There exists a subsequence (x_{n_k}) of (x_n) such that $\limsup_n \|x_n - Tx\|^2 = \lim_k \|x_{n_k} - Tx\|^2$. Then, by Lemma 6.1, for any Banach limit μ,

$$\begin{aligned}\left(\limsup_{n\to\infty} \|x_n - Tx\|\right)^2 &= \limsup_{n\to\infty} \|x_n - Tx\|^2 = \lim_{k\to\infty} \|x_{n_k} - Tx\|^2 = \mu\|x_{n_k} - Tx\|^2 \\ &\le \mu\|x_{n_k} - x\|^2 \le \limsup_{k\to\infty} \|x_{n_k} - x\|^2 \le \limsup_{n\to\infty} \|x_n - x\|^2 \\ &= \left(\limsup_{n\to\infty} \|x_n - x\|\right)^2\end{aligned}$$

which concludes the proof. □

According Theorem 4.2. and Theorem 4.4. in [11], an L-type mapping $T : C \to C$ has a fixed point provided that either the set C or the Banach space X have suitable properties. Hence we obtain the following fixed point results for α-nonexpansive mappings.

Corollary 6.3. *Let C be a nonempty compact convex subset of a Banach space X. Let $T : C \to C$ be a mapping such that:*

(i) T is α-nonexpansive for some $0 \le \alpha < 1$, and
(ii) T satisfies condition (A) on C.

Then, T has a fixed point.

Corollary 6.4. *Let C be a nonempty weakly compact convex subset of a Banach space X with normal structure. Let $T : C \to C$ be a mapping such that:*

(i) T is α-nonexpansive for some $0 \le \alpha < 1$, and
(ii) T satisfies condition (A) on C.

Then, T has a fixed point.

Remark 6.7. The main fixed point result for α-nonexpansive mappings in [2], (see Theorem 1.1.), contains the assumption of uniform convexity of the Banach space under consideration. Notice that normal structure is considerably weaker than uniform convexity. However, in Corollary (6.4) Assumption (ii), which is not present in [2, Theorem 1.1.], appears. It is unclear whether every α-nonexpansive selfmapping defined on a closed, convex and bounded subset C of a Banach space satisfies condition (A) on C. In other words, we do not know whether assumption (ii) of the above corollaries is essential for the fixed point result. Of course for $\alpha = 0$, property (A) is automatically fulfilled. The following result shows that this fact holds also for $\alpha = 1/2$. Its proof is based on Lemma 2.2 in [3].

Theorem 6.3. *Let C be a convex closed bounded subset of a Banach space X. Assume that $T : C \to C$ is an $\frac{1}{2}$-nonexpansive mapping. Then for every $x \in C$ the sequence $(T^n x)$ is an a.f.p.s. for T.*

Proof. Let $x \in C$. Replacing y by Tx in Definition 2.3 we have that

$$\|T^2x - Tx\|^2 \le \frac{1}{2}\|T^2x - x\|^2. \tag{6.5}$$

Replacing again y by T^2x in Definition 2.3 and using this inequality we obtain that

$$\begin{aligned}\|T^3x - Tx\|^2 &\le \frac{1}{2}\|T^3x - x\|^2 + \frac{1}{2}\|Tx - T^2x\|^2 \\ &\le \frac{1}{2}\|T^3x - x\|^2 + \frac{1}{2^2}\|T^2x - x\|^2.\end{aligned}$$

Assume that for some $n \geq 3$,

$$\|T^n x - Tx\|^2 \leq \frac{1}{2}\|T^n x - x\|^2 + \frac{1}{2^2}\|T^{n-1}x - x\|^2 + \cdots + \frac{1}{2^{n-1}}\|T^2 x - x\|^2.$$

Then, using that T is $\frac{1}{2}$-nonexpansive along with this assumption, we obtain

$$\begin{aligned}\|T^{n+1}x - Tx\|^2 &\leq \frac{1}{2}\|T^{n+1}x - x\|^2 + \frac{1}{2}\|Tx - T^n x\|^2 \\ &\leq \frac{1}{2}\|T^{n+1}x - x\|^2 + \frac{1}{2^2}\|T^n x - x\| + \cdots + \frac{1}{2^n}\|T^2 x - x\|.\end{aligned}$$

Therefore for all $n \geq 3$ and for all $x \in C$

$$\|T^n x - Tx\|^2 \leq \frac{1}{2}\|T^n x - x\|^2 + \frac{1}{2^2}\|T^{n-1}x - x\|^2 + \cdots + \frac{1}{2^{n-1}}\|T^2 x - x\|^2. \tag{6.6}$$

Define the constants $c_{n,k}$, with $1 \leq k \leq n$, by the following recurrence relation.

$$c_{1,1} = c_{2,1} = c_{2,2} = 1,$$

for $n \geq 3$, $c_{n,1} := 1, c_{n,n} := c_{n,n-1}$, and

$$c_{n,k} := c_{n-1,1} + c_{n-1,2} + \cdots + c_{n-1,k}$$

whenever $k = 2, \ldots, n-1$.

Now we claim that every $x \in C$ and every $n \geq 1$,

$$\|T^{n+1}x - T^n x\|^2 \leq \frac{c_{n,1}}{2^n}\|T^{n+1}x - x\|^2 + \frac{c_{n,2}}{2^{n+1}}\|T^n x - x\|^2 + \cdots + \frac{c_{n,n}}{2^{2n-1}}\|T^2 x - x\|^2. \tag{6.7}$$

Of course (6.5) is just our claim for $n = 1$. We assume that (6.7) holds. Then, using Tx instead of x in (6.7) and from (6.6), we have

$$\begin{aligned}\|T^{n+2}x - T^{n+1}x\|^2 &\leq \frac{c_{n,1}}{2^n}\|T^{n+2}x - Tx\|^2 + \frac{c_{n,2}}{2^{n+1}}\|T^{n+1}x - Tx\|^2 + \cdots \\ &\qquad + \frac{c_{n,n-1}}{2^{2n-2}}\|T^4 x - Tx\|^2 + \frac{c_{n,n}}{2^{2n-1}}\|T^3 x - Tx\|^2 \\ &\leq \frac{c_{n,1}}{2^n}\left(\frac{1}{2}\|T^{n+2}x - x\|^2 + \cdots + \frac{1}{2^{n+1}}\|T^2 x - x\|^2\right) \\ &\quad + \frac{c_{n,2}}{2^{n+1}}\left(\frac{1}{2}\|T^{n+1}x - x\|^2 + \cdots + \frac{1}{2^n}\|T^2 x - x\|^2\right) + \cdots \\ &\quad + \frac{c_{n,n-1}}{2^{n+1}}\left(\frac{1}{2}\|T^4 x - x\|^2 + \frac{1}{2^2}\|T^3 x - x\| + \frac{1}{2^3}\|T^2 x - x\|^2\right) \\ &\quad + \frac{c_{n,n}}{2^{2n-1}}\left(\frac{1}{2}\|T^3 x - x\|^2 + \frac{1}{2^2}\|T^2 x - x\|^2\right) \\ &= \frac{c_{n,1}}{2^{n+1}}\|T^{n+2}x - x\|^2 + \frac{c_{n,1} + c_{n,2}}{2^{n+2}}\|T^{n+1}x - x\|^2 \\ &\quad + \frac{c_{n,1} + c_{n,2} + c_{n,3}}{2^{n+3}}\|T^n x - x\|^2 \\ &\quad + \cdots + \frac{c_{n,1} + c_{n,2} + \cdots + c_{n,n-1} + c_{n,n}}{2^{2n}}\|T^3 x - x\|^2 \\ &\quad + \frac{c_{n,1} + c_{n,2} + \cdots + c_{n,n-1} + c_{n,n}}{2^{2n+1}} + \|T^2 x - x\|^2 \\ &= \frac{c_{n+1,1}}{2^{n+1}}\|T^{n+2}x - x\|^2 + \frac{c_{n+1,2}}{2^{n+2}}\|T^{n+1}x - x\|^2 + \frac{c_{n+1,3}}{2^{n+3}}\|T^n x - x\|^2 \\ &\quad + \cdots + \frac{c_{n+1,n}}{2^{2n}}\|T^3 x - x\|^2 + \frac{c_{n+1,n+1}}{2^{2n+1}}\|T^2 x - x\|^2.\end{aligned}$$

Thus, by induction, our claim is proven.

It is well know that

$$\frac{c_{n,1}}{2^n} + \frac{c_{n,2}}{2^{n+1}} + \cdots + \frac{c_{n,n}}{2^{2n-1}} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^{2n} s ds.$$

Then,

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\|^2 \leq \lim_{n\to\infty} \frac{2\text{diam}^2(C)}{\pi} \int_0^{\frac{\pi}{2}} \sin^{2n} t \, dt = 0.$$

Thus, $(T^n x)$ is an a.f.p.s. as we claimed. □

Corollary 6.5. *Let C be a nonempty weakly compact convex subset of a Banach space X with normal structure. Let $T : C \to C$ be a $\frac{1}{2}$-nonexpansive mapping. Then, T has a fixed point.*

Proof. It follows immediately from the above theorem that T has an a.f.p.s. in every T-invariant closed convex subset C' of C, that is, that T satisfies property (A) on C. Then the result is a direct consequence of Corollary 6.4. □

Remark 6.8. According to [10, Theorem 4.1], if C is a nonempty closed convex and bounded subset of a smooth strictly convex Banach space X, and $T : C \to C$ is a nonspreading mapping, then, T has a fixed point. The above result does not require the assumptions on smoothness and strict convexity for the set C in presence of normal structure. Notice that the definition of nonspreading mappings which we have considered is sligthly different than in [10].

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