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On the structure of periodic complex Horadam orbits

OVIDIU D. BAGDASAR, PETER J. LARCOMBE and ASHIQ ANJUM

ABSTRACT. Numerous geometric patterns identified in nature, art or science can be generated from recurrent sequences, such as for example certain fractals or Fermat's spiral. Fibonacci numbers in particular have been used to design search techniques, pseudo random-number generators and data structures. Complex Horadam sequences are a natural extension of Fibonacci sequence to complex numbers, involving four parameters (two initial values and two in the defining recursion), therefore successive sequence terms can be visualized in the complex plane. Here, a classification of the periodic orbits is proposed, based on divisibility relations between orders of generators (roots of the characteristic polynomial). Regular star polygons, bipartite graphs and multi-symmetric patterns can be recovered for selected parameter values. Some applications are also suggested.

1. INTRODUCTION

Fibonacci numbers are ubiquitous in nature (flower and fruit patterns), but may also appear in chemistry , music, poetry, electrical networks or...stock exchange [11, Chapter 3]. Geometric patterns related to the Fibonacci numbers were identified as optimal solutions for the layout of mirrors in a concentrated solar power plant [15]. Computing applications include search techniques [12] or the Fibonacci heap data structure [9].

The Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is defined by the recurrence

$$w_{n+2} - pw_{n+1} + qw_n = 0, \quad w_0 = a, w_1 = b, \tag{1.1}$$

where the parameters a, b, p and q are complex numbers. The investigation of this general recursion was initiated by A. F. Horadam [10]. When $(a, b) = (0, 1)$, $(p, q) = (1, 1)$ gives the Lucas, while $(p, q) = (1, -1)$ the Fibonacci sequence. More results on Horadam sequences are listed in the survey paper of Larcombe *et al.* [13].

Periodic Horadam sequences (i.e. the set of points visited by the sequence is finite) were characterized in [2], and arise when zeros of the characteristic equation (generators)

$$x^2 - px + q = 0 \tag{1.2}$$

are roots of unity, denoted by $z_1 = z_1(p, q) = e^{2\pi i p_1/k_1}$ and $z_2 = z_2(p, q) = e^{2\pi i p_2/k_2}$, where p_1, p_2, k_1, k_2 are positive integers. The enumeration of Horadam orbits with fixed period provided a context for the O.E.I.S. sequence no. A102309 [3], while some particular periodic Horadam patterns were presented in [4].

A classification of the orbits based on the properties of the numbers p_1, p_2, k_1, k_2 is proposed, with an analysis of the geometric patterns produced. Among these we recover every regular star polygon in the complex plane, as well as bipartite and multipartite digraphs, whose stable sets are regular polygons (which also have a dual symmetry - the nodes of the cycle can be labeled differently, such that the nodes of same colour are regular polygons). All these diverse patterns are produced by instances of the same formula.

The close relationship with the Fibonacci numbers/patterns suggests that Horadam sequences can be used to generate structures with optimal properties, by appropriate

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Corresponding author: Ovidiu Bagdasar; o.bagdasar@derby.ac.uk

choices of the four sequence parameters: initial conditions a, b and generators z_1, z_2. A first application was the design of a pseudo-random number generator with geometric structure produced by aperiodic Horadam sequences, proposed in [5].

2. Preliminary results

In this section we first present formulas for the general term w_n of the complex Horadam sequence having the characteristic polynomial (1.2). The necessary and sufficient conditions for the periodicity of complex Horadam sequences from [2] are then summarized, followed by definitions of some relevant geometric patterns that can be recovered from the (finite) *orbit* of the periodic sequence $\{w_n\}_{n=0}^{\infty}$ defined as

$$W = \{z \in \mathbb{C} : \exists n \in \mathbb{N} \, s.t. \, w_n = z\} \tag{2.3}$$

One should note that when $\{w_n\}_{n=0}^{\infty}$ is periodic, W is a finite set.

We first present formulas for the general term w_n of the complex Horadam sequence having the characteristic polynomial (1.2). Vieta's relations for the polynomial P give

$$p = z_1 + z_2, \quad q = z_1 z_2, \tag{2.4}$$

showing that the recurrence (1.1) defined for coefficients p, q may alternately be defined through the solutions z_1, z_2 of the characteristic polynomial, referred to as *generators*.

To avoid trivial cases, we consider distinct generators (for $z_1 = z_2$ one obtains regular polygons) and assume that the recurrence order cannot be reduced ($z_1 z_2 \neq 0$).

The computations and graphs in this paper were realized using Matlab®.

2.1. **Formulas for the general term (distinct generators).**

For distinct roots $z_1 \neq z_2$ of (1.2), the general term of Horadam's sequence $\{w_n\}_{n=0}^{\infty}$ is given by (see [1, Chapter 7])

$$w_n = A z_1^n + B z_2^n, \tag{2.5}$$

where the constants A and B can be obtained from the initial condition, written explicitly as

$$w_n = \frac{1}{z_2 - z_1} \left[(a z_2 - b) z_1^n + (b - a z_1) z_2^n \right]. \tag{2.6}$$

As shown in [2, Theorem 3.2], the only non-degenerated periodic orbits (points or regular polygons generated by a single root) are obtained when both z_1 and z_2 are roots of unity.

2.2. **Star polygons and multipartite graphs.**

Star polygons and multipartite graphs with geometric symmetries can be recovered as periodic orbits of complex Horadam sequences.

Definition 2.1. (Star polygons) For integers k and p the regular star polygon denoted by the Schläfli symbol $\{k/p\}$ can be considered as being constructed by connecting every pth point out of k points regularly spaced in a circular placement (see [8, Chapter 2]).

Definition 2.2. (Multipartite graph) For k a natural number, a k-partite graph W is a graph whose vertex set V is partitioned into k parts, with edges between vertices of different parts only (a 2-partite graph is also called bipartite): $G = (V_0, \ldots, V_{k-1}, E)$ with $E \subset \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. The vertices of $V_i, i = 1 \ldots, k-1$ are called the ith level of G [14, p.4].

3. MAIN RESULTS

As discussed in [2], non-trivial periodic orbits of $\{w_n\}_{n=0}^{\infty}$ are obtained when the two generators $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$ are distinct roots of unity. Here a classification of periodic Horadam orbits is established, based on the orders of z_1 and z_2. It can be assumed that z_1 and z_2 are primitive roots (i.e. $\gcd(p_1, k_1) = \gcd(p_2, k_2) = 1$) and one can show that the period of sequence $\{w_n\}_{n=0}^{\infty}$ is

$$k_1 k_2 / \gcd(k_1, k_2). \tag{3.7}$$

However, the orbit may exhibit a rich variety of patterns, as illustrated below.

First, for $k_2 = 1$ the orbit is the regular star polygon $\{k_1/p_1\}$, while for $k_2 = 2$ the orbit is a bipartite graph. In general, for integers k_1, k_2 satisfying $d = \gcd(k_1, k_2)$, one obtains a multipartite graph whose vertices can be divided into either k_2 regular k_1/d-gons, or into k_1 regular k_2/d-gons. This shows that multipartite graphs with bivalent symmetry can be obtained from periodic orbits of complex Horadam sequences.

3.1. Regular star polygons: $z_2 = 1$ $(ord(z_2) = 1)$.

Theorem 3.1. *If $z_1 = e^{2\pi i p/k}$ is a primitive kth root $(k \geq 2)$ and $z_2 = 1$, the orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is the regular star polygon $\{k/p\}$. The property is illustrated in Fig. 1.*

Proof. In this case, the general formula (2.5) gives

$$w_n = A z_1^n + B, \tag{3.8}$$

where A and B are constant expressions of z_1. The sequence $\{w_n\}_{n=0}^{\infty}$ can be obtained from $\{z_1^n\}_{n=0}^{\infty}$ by rotating with $\arg(A)$, scaling with $|A|$ and translating with B, therefore the shape of the orbit is a regular k-gon, similar to the orbit of $\{z_1^n\}_{n=0}^{\infty}$ ([2, Lemma 2.2]).

There is a close relation between these orbits and regular star polygons. As from w_n to w_{n+1} there is a jump of p adjacent vertices in the k-gon, the directed orbit of $\{w_n\}_{n=0}^{\infty}$ for $\gcd(p, k) = 1$ generates the regular star polygon $\{k/p\}$. In Fig. 1 are depicted the star polygons (a) $\{7/2\}$ and (b) $\{7/3\}$. □

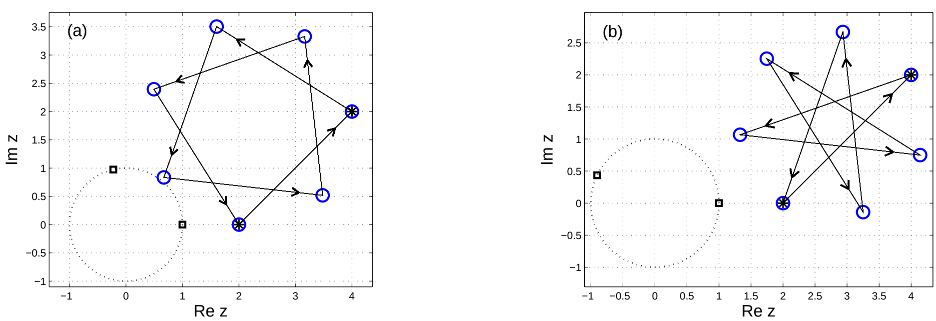

FIGURE 1. First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.6), for $z_2 = 1$ and (a) $k = 7$, $z_1 = e^{2\pi i \frac{2}{7}}$; (b) $k = 7$, $z_1 = e^{2\pi i \frac{3}{7}}$; when $a = 2$ and $b = 4 + 2i$. Arrows indicate the direction of the orbit visiting w_0, w_1 (star), $w_2, \ldots, w_N$ (circle). The dotted line represents the unit circle.

3.2. **Bipartite graphs:** $z_2 = -1$ $(ord(z_2) = 2)$**.** In this section we characterize the bipartite periodic orbits of $\{w_n\}_{n=0}^{\infty}$ obtained for $z_1 = e^{2\pi i p/k}$ and $z_2 = -1$.

Theorem 3.2. *(k is odd) Let $k \geq 2$ be an odd number, z_1 a primitive kth root and $z_2 = -1$. The orbit of sequence $\{w_n\}_{n=0}^{\infty}$ is a $2k$gon, whose nodes can be divided into two regular k-gons representing a bipartite graph. The property is illustrated in Fig. 2 (a).*

Proof. In this case, the general formula (2.5) gives

$$w_n = Az_1^n + (-1)^n B, \tag{3.9}$$

where A and B are constants. As z_1 is a primitive kth root and $\gcd(k, 2) = 1$, the period of sequence $\{w_n\}_{n=0}^{\infty}$ computed from formula (3.7) is $2k$ and the orbit is

$$W = \{w_0, w_1, \ldots, w_{2k-1}\}.$$

The set W can be partitioned into the disjoint ordered sets W_0 and W_1

$$\begin{aligned} W_0 &= \{A + B, Az_1^2 + B, \ldots, Az_1^{k-1} + B, Az_1^{k+1} + B, \ldots, Az_1^{2k-2} + B\}, \\ W_1 &= \{Az_1 - B, Az_1^3 - B, \ldots, Az_1^{k-2} - B, Az_1^k - B, \ldots, Az_1^{2k-1} - B\}, \end{aligned} \tag{3.10}$$

containing the sequence terms of even and odd index, respectively. As $z_1^k = 1$, we have

$$\begin{aligned} W_0 &= \{A + B, Az_1^2 + B, \ldots, Az_1^{k-1} + B, Az_1^1 + B, \ldots, Az_1^{k-2} + B\}, \\ W_1 &= \{Az_1 - B, Az_1^3 - B, \ldots, Az_1^{k-2} - B, Az_1^0 - B, \ldots, Az_1^{k-1} - B\}. \end{aligned} \tag{3.11}$$

Finally, the sets W_0 and W_1 contain the points

$$\{Az_1^j + B, j = 0, \ldots, k-1\}, \quad \{Az_1^j - B, j = 0, \ldots, k-1\}, \tag{3.12}$$

so W_0 and W_1 represent the vertices of two regular k-gons, visited alternatively by the sequence $\{w_n\}_{n=0}^{\infty}$. Moreover, (3.11) shows that w_n jumps over a number of $2p$ adjacent vertices with each consecutive visit in any of the k-gons W_0 or W_1. □

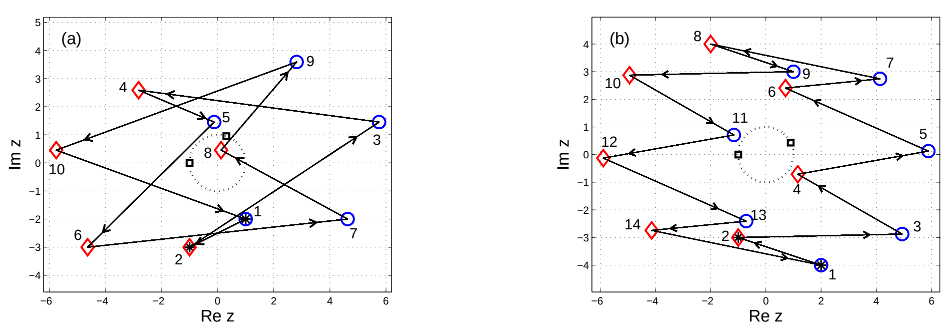

FIGURE 2. First $N = 100$ orbit terms of $\{w_n\}_{n=0}^{N}$ obtained from (2.6), for $z_2 = -1$ and (a) $k = 5$ (odd), $z_1 = e^{2\pi i \frac{1}{5}}$ and $a = 2 - 4i$, $b = -1 - 3i$; (b) $k = 14$ (even), $z_1 = e^{2\pi i \frac{1}{14}}$ and $a = 1 - 2i$, $b = -1 - 3i$. Arrows indicate the direction of the orbit visiting w_0, w_1 (star), $w_2, \ldots, w_N$. The sets W_0 and W_1 defined in (3.10) and (3.13) are represented by circles and diamonds respectively. The dotted line represents the unit circle.

Using a similar idea, one can prove the result for k even.

Theorem 3.3. *(k is even) Let $k \geq 2$ be an even number, z_1 a primitive kth root and $z_2 = -1$. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is a k-gon, whose nodes can be divided into two regular $k/2$-gons representing a bipartite graph. The property is illustrated in Fig. 2 (b).*

Proof. When k is even, z_2 is also a kth root and the period of $\{w_n\}_{n=0}^{\infty}$ from (3.7) is k. As in Theorem 3.2, the orbit W can be partitioned into disjoint ordered sets W_0 and W_1

$$W_0 = \{A + B, Az_1^2 + B, \dots, Az_1^{k-2} + B\},$$
$$W_1 = \{Az_1 - B, Az_1^3 - B, \dots, Az_1^{k-1} - B\}, \tag{3.13}$$

containing the sequence terms of even and odd index, respectively. The number $Z = z_1^2$ is a primitive $k/2$th root, therefore the set $\{1, Z, \dots, Z^{k/2-1}\}$ represents the vertices of a $k/2$-gon. The sets W_0 and W_1 can be written in terms of Z as

$$W_0 = \{AZ^j + B, j = 0, \dots, k/2 - 1\},$$
$$W_1 = \{Az_1 Z^j - B, j = 0, \dots, k/2 - 1\}, \tag{3.14}$$

so W_0 and W_1 represent the vertices of two regular $k/2$-gons, which are visited alternatively by the sequence $\{w_n\}_{n=0}^{\infty}$. Moreover, (3.13) shows that w_n jumps over p adjacent vertices each time it visits W_0 or W_1 (as $z_1^{n+2} = z_1^n e^{2\pi i \frac{2p}{k}} = z_1^n e^{2\pi i \frac{p}{k/2}}$). □

3.3. **Multipartite graphs.** In this section we characterize the multipartite periodic orbits of $\{w_n\}_{n=0}^{\infty}$ obtained for $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$.

Theorem 3.4. *Let $k_1, k_2, d \geq 2$ be natural numbers s.t. $gcd(k_1, k_2) = d$ and z_1, z_2 be k_1th and k_2th primitive roots, respectively. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is then a k_1k_2/d-gon, whose nodes can be divided into k_1 regular k_2/d-gons representing a multipartite graph. By duality, the nodes of the orbit can also be divided into k_2 regular k_1/d-gons.*

Proof. As $\gcd(k_1, k_2) = d$ the period of $\{w_n\}_{n=0}^{\infty}$ from (3.7) is k_1k_2/d, and the orbit consists of the first k_1k_2/d terms

$$W = \{w_0, w_1, \dots, w_{k_1k_2/d-1}\}. \tag{3.15}$$

As in the previous theorem,W can be partitioned into k_2 disjoint sets $W_0, \dots, W_{k_2-1}$

$$W_0 = \{A + B, Az_1^{k_2} + Bz_2^{k_2}, \dots, Az_1^{(k_1/d-1)k_2} + Bz_2^{(k_1/d-1)k_2}\}, \tag{3.16}$$
$$W_1 = \{Az_1 + Bz_2, Az_1^{k_2+1} + Bz_2^{k_2+1}, \dots, Az_1^{(k_1/d-1)k_2+1} + Bz_2^{(k_1/d-1)k_2+1}\},$$
$$\dots$$
$$W_{k_2-1} = \{Az_1^{k_2-1} + Bz_2^{k_2-1}, Az_1^{2k_2-1} + Bz_2^{2k_2-1}, \dots, Az_1^{(k_1/d)k_2-1} + Bz_2^{(k_1/d)k_2-1}\}.$$

As $z_2^{k_2} = 1$ the above sets can be simplified to

$$W_0 = \{A + B, Az_1^{k_2} + B, \dots, Az_1^{(k_1/d-1)k_2} + B\}, \tag{3.17}$$
$$W_1 = \{Az_1 + Bz_2, Az_1^{k_2+1} + Bz_2, \dots, Az_1^{(k_1/d-1)k_2+1} + Bz_2\},$$
$$\dots$$
$$W_{k_2-1} = \{Az_1^{k_2-1} + Bz_2^{k_2-1}, Az_1^{2k_2-1} + Bz_2^{k_2-1}, \dots, Az_1^{(k_1/d)k_2-1} + Bz_2^{k_2-1}\}.$$

It can be checked that no two terms in any of the sets $W_0, \dots, W_{k_2-1}$ are equal. Should this be the case, we would have the coefficients j, r and q s.t.

$$Az_1^{rk_2+j} + Bz_2^j = Az_1^{qk_2+j} + Bz_2^j, \quad j \in \{0 \dots, k_2 - 1\}, \quad r, q \in \{0, \dots, (k_1/d) - 1\}.$$

For $A \neq 0$ this is only possible when $z_1^{(r-q)k_2} = 1$, which only happens for $r = q$, as $\gcd(k_2, k_1) = d$ and z_1 is a primitive k_1th root. This shows that

$$W_j = \{Az_1^r + Bz_2^j, r = 0, \dots, (k_1/d) - 1\}, \quad j = 0, \dots, k_2 - 1, \tag{3.18}$$

so $W_0, \dots, W_{k_2-1}$ represent the vertices of k_2 regular k_1/d-gons, which are visited alternatively by the sequence $\{w_n\}_{n=0}^{\infty}$. Moreover, (3.17) shows that w_n jumps over a number of $k_2 p_1$ adjacent vertices with each consecutive visit in any of the k_1/d-gons $W_0, \dots, W_{k_2-1}$.

By duality, the vertices (3.15) can also be divided as

$$\begin{aligned} (3.19) \quad W_1' &= \{A + B, Az_1^{k_1} + Bz_2^{k_1}, \dots, Az_1^{(k_2/d-1)k_1} + Bz_2^{(k_2/d-1)k_1}\}, \\ W_2' &= \{Az_1 + Bz_2, Az_1^{k_1+1} + Bz_2^{k_1+1}, \dots, Az_1^{(k_2/d-1)k_1+1} + Bz_2^{(k_2/d-1)k_1+1}\}, \\ &\dots \\ W_{k_1-1}' &= \{Az_1^{k_1-1} + Bz_2^{k_1-1}, Az_1^{2k_1-1} + Bz_2^{2k_1-1}, \dots, Az_1^{(k_2/d)k_1-1} + Bz_2^{(k_2/d)k_1-1}\}, \end{aligned}$$

and using the same argument as above $W_0', \dots, W_{k_1-1}'$ represent the vertices of k_1 regular k_2/d-gons, which are visited alternatively by the sequence $\{w_n\}_{n=0}^{\infty}$. Moreover, (3.19) shows that w_n jumps over a number of $k_1 p_2$ adjacent vertices with each consecutive visit in any of the k_2/d-gons $W_0', \dots, W_{k_1-1}'$. □

This result has a number of direct corollaries which present interest on their own.

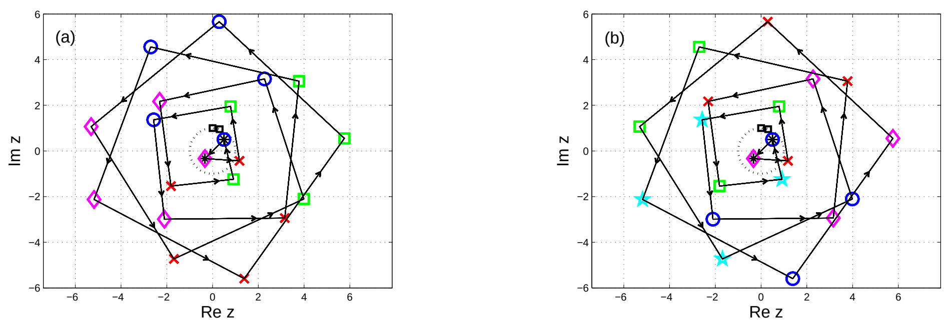

FIGURE 3. First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^{N}$ obtained from (2.6), for $k_1 = 4$, $k_2 = 5$. We compute $w_0, \dots, w_N$ for $z_1 = e^{2\pi i \frac{1}{5}}$, $z_2 = e^{2\pi i \frac{1}{4}}$ and initial conditions $a = (1+i)/2$, $b = -(1+i)/3$. The orbits are partitioned into (a) four regular pentagons; (b) five squares; Arrows indicate the direction of the orbit from one term to the next. The dotted line is the unit circle.

Corollary 3.1. *Let $k_1, k_2 \geq 2$ be s.t. gcd$(k_1, k_2) = 1$ and z_1, z_2 be k_1th and k_2th primitive roots, respectively. The orbit of the sequence $\{w_n\}_{n=0}^{\infty}$ is then a $k_1 k_2$-gon, whose nodes can be divided into k_1 regular k_2-gons representing a multipartite graph. By duality, the orbit can also be divided into k_2 regular k_1-gons. The property is illustrated in Fig. 3, where the twenty points of the orbit can be decomposed in either four regular pentagons, or five squares.*

Corollary 3.2. *If $k_2 | k_1$ the orbit is a k_1-gon whose nodes can be divided into k_2 regular k_1/k_2-gons. The asymmetry in Fig. 4 illustrates this, as the only regular polygons that can be identified in the periodic orbit of $\{w_n\}_{n=0}^{\infty}$ are 1- and 2-gons.*

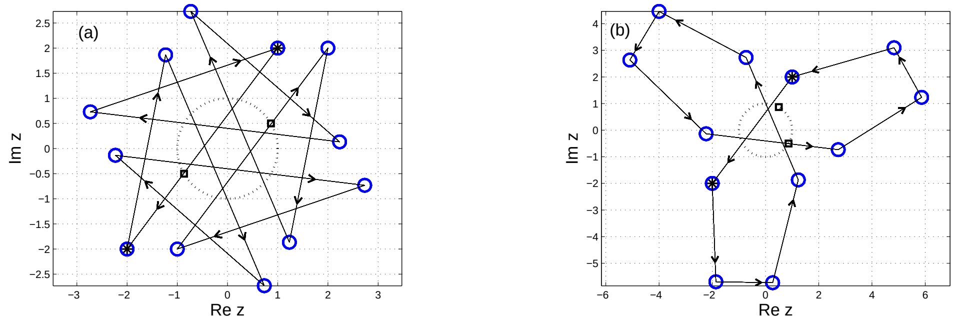

FIGURE 4. First $N = 100$ orbit terms of sequence $\{w_n\}_{n=0}^N$ obtained from (2.6), for initial conditions are $a = 1 + 2i$ and $b = -2 - 2i$. We compute $w_0, \dots, w_N$ for (a) $k_1 = k_2 = 12$, $z_1 = e^{2\pi i \frac{1}{12}}$, $z_2 = e^{2\pi i \frac{7}{12}}$; (b) $k_1 = 2k_2 = 12$, $z_1 = e^{2\pi i \frac{1}{6}}$, $z_2 = e^{2\pi i \frac{11}{12}}$. Arrows indicate the direction of the orbit from one term to the next. The dotted line is the unit circle.

4. Summary and future work

In this paper the orbits of periodic Horadam sequences were classified, based on their geometric patterns. Regular and star polygons in the complex plane, as well as bipartite and multipartite digraphs, with regular geometric patterns were recovered.

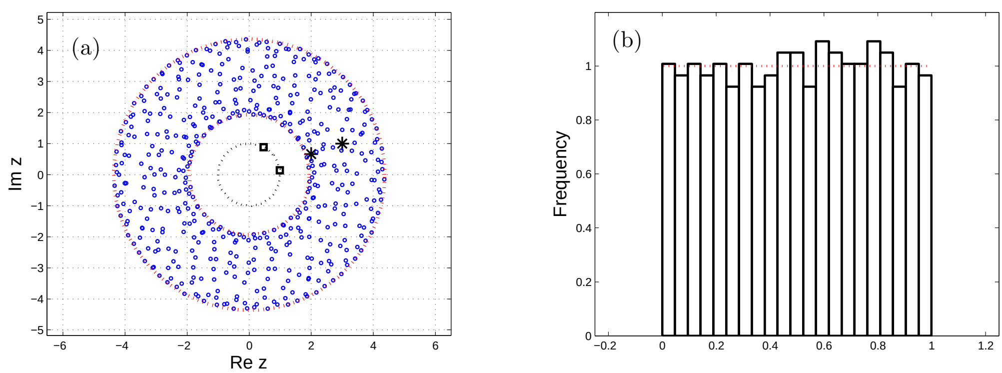

FIGURE 5. (a) First $N = 1001$ terms of $\{w_n\}_{n=0}^\infty$ given by (2.6) when $z_1 = e^{2\pi i \frac{4}{23}}$, $z_2 = e^{2\pi i \frac{2}{89}}$ (squares) and $a = 2 + 2/3i$, $b = 3 + i$ (stars). The solid line represents the unit circle. Boundaries of annulus $U(0, \big| |A| - |B| \big|, |A| + |B|)$ from (4.20) are also represented; (b) Histogram of arguments $\text{Arg}(w_n)$, $n = 0, \dots, 1000$.

From formula (2.5), any periodic Horadam orbit is located inside the annulus

$$\{z \in \mathbb{C} : \big| |A| - |B| \big| \le |z| \le |A| + |B|\}, \text{ where } A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}. \tag{4.20}$$

This remains true when the generators satisfy $|z_1| = |z_2| = 1$, but are not roots of unity.

A Horadam-based pseudo-random number generator based on non-periodic orbits was proposed in [5] and evaluated using Monte Carlo simulations, comparing well with Lagged Fibonacci and Mersenne Twister routines. Despite exhibiting a number of good features, auto-correlation was poor, while the sequence was aperiodic. Geometric patterns based on $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$ for k_1, k_2 large prime numbers, will have long periodic orbits, addressing the machine floating point precision. In Fig. 5 (a), a periodic orbit producing a "good" cover for the annulus is shown, along with the distribution of arguments in Fig. 5 (b) which seems quasi-uniform. Further research is necessary to find an optimal cover for the annulus of radii $0 < r_1 < r_2$, with a fixed number of points.

The periodicity conditions formulated in [6] for generalized Horadam sequences, can be used to establish the number of fixed-period sequences and their geometric structure.

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DEPARTMENT OF COMPUTING AND MATHEMATICS
UNIVERSITY OF DERBY
KEDLESTON ROAD, DE22 1GB, DERBY, UNITED KINGDOM
E-mail address: o.bagdasar@derby.ac.uk
E-mail address: p.j.larcombe@derby.ac.uk
E-mail address: a.anjum@derby.ac.uk