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Systems of knowledge representation based on stratified graphs. Application in Natural Language Generation

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ABSTRACT. The concept of stratified graph introduces some method of knowledge representation (see [Ţăndăreanu, N., *Knowledge representation by labeled stratified graphs*, Proc. 8th World Multi-Conference on Systemics, Cybernetics and Informatics, **5** (2004), 345–350; Ţăndăreanu, N., *Proving the Existence of Labelled Stratified Graphs*, An. Univ. Craiova Ser. Mat. Inform., **27** (2000), 81–92]) The inference process developed for this method uses the paths of the stratified graphs, an order between the elementary arcs of a path and some results of universal algebras. The order is defined by considering a structured path instead of a regular path. In this paper we define the concept of *system of knowledge representation* as a tuple of the following components: a stratified graph $\mathcal{G}$, a partial algebra Y of real objects, an embedding mapping (an injective mapping that embeds the nodes of $\mathcal{G}$ into objects of Y) and a set of algorithms such that each of them can combine two objects of Y to get some other object of Y. We define also the concept of *inference process* performed by a system of knowledge processing in which the interpretation of the symbolic elements is defined by means of natural language constructions. In this manner we obtained a mechanism for texts generation in a natural language (for this approach, Romanian).

1. INTRODUCTION

The graph theory has many applications in computer science. The interconnection network of a distributed system or of a supercomputer based on large scale parallel processing can be modeled as a graph. Good network topologies can be defined and studies using algebraic models. A number of interconnection network topologies as hypercubes, star-graphs [1], cube-connected cycles, pancake graphs [12], Fibonacci cubes [4, 13, 14, 15] are Cayley graphs and their properties are studies using Cayley groups properties.

The concept of stratified graph provides a method of knowledge representation. This concept was introduced in [8], where the intuitive aspect is presented. The mathematical proof of the existence of this structure was given in [10]. The subject as well as the applications of this concept were developed in a sequence of papers: the algebra of all stratified graphs over a given labeled graph is a join lattice [7], semantics of communication [8], geometrical image generation [8], reconstruction of geometrical image by extracting the semantics of a linguistic spatial description given in a natural language [9], the construction of a stratified graph over an attribute graph in order to find the paths satisfying several restrictions [7], problem solving [5], the use of stratified graphs to model a cooperation between two or more companies [6, 7], etc.

The method of knowledge representation based on stratified graphs uses:

- concepts from graph theory redefined in the new framework (especially the concepts of labeled path);
- elements of universal algebra (Peano algebra, partial algebra, morphism of partial algebras).

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Intuitively, a stratified graph is built over a labeled graph G_0, placing on top a subset of a Peano algebra generated by the label set of G_0.

The main goal is to define the concept of *knowledge processing system* based on stratified graphs and its corresponding *inference process*. The inference is based on the concept of *accepted structured path* and its decomposition into other accepted structured paths. The theoretical results used to define the inference process are completely presented in this paper.

The paper is organized as follows: Section 2 contains the basic concepts of *labeled graph* and *stratified graph*; in Section 3 we define the concept of *structured path* in a labeled graph and the concept of *accepted structured path* in a stratified graph. We establish an useful result concerning the existence of some morphism of universal algebras obtained from the label set of the structured paths to the Peano algebra generated by the elementary labels of the structured graph; We organize the set $ASP(\mathcal{G})$ of the accepted structured paths of $\mathcal{G}$ as a partial algebra with respect a certain partial operation $\oslash$; Section 4 treats two decomposition properties: one for structured paths (with respect to a certain partial binary operation) and the other for accepted structured paths (with respect to $\oslash$), used later to obtain the inference process; in Section 5 we show what we mean by a *system of knowledge representation* based on stratified graphs and we give the formalism of the corresponding *inference process*. We prove that this process is a morphism of partial algebras by means of which for each accepted structured path we can associate an object of a real world. Last section includes conclusions of our study.

2. Basic concepts

We consider a symbol σ of arity 2 and take the sets defined recursively as follows:

$$\begin{cases} B_0 = M_0 \\ B_{n+1} = B_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in B_n \times B_n\}, n \geq 0 \end{cases}$$

where M_0 is a finite set that the description of an element of M_0 does not contain the symbol σ. Thus the set $M_0 = \{\sigma ab, c, d\}$ can not be taken into consideration. The set

$$\mathcal{B} = \bigcup_{n \geq 0} B_n$$

is the Peano σ -algebra ([2]) generated by M_0. We can suppose that $\sigma(x, y)$ is the word σxy over the alphabet $M_0 \cup \{\sigma\}$. Often this algebra is denoted by $\overline{M_0}$.

By $Initial(\overline{M_0})$ we denote the collection of all subsets of B satisfying the following conditions: $K \in Initial(\overline{M_0})$ if

- $M_0 \subseteq K \subseteq B$
- if $\sigma(u, v) \in K$, $u \in \overline{M_0}$, $v \in \overline{M_0}$ then $u \in K$ and $v \in K$

We consider a finite set S and denote by $2^{S \times S}$ the collection of all subsets of $S \times S$. We define the mapping $prod_S : dom(prod_S) \longrightarrow 2^{S \times S}$ as follows:

$$dom(prod_S) = \{(\rho_1, \rho_2) \in 2^{S \times S} \times 2^{S \times S} \mid \rho_1 \circ \rho_2 \neq \emptyset\}$$
$$prod_S(\rho_1, \rho_2) = \rho_1 \circ \rho_2$$

where $\circ$ is the usual operation between the binary relations:

$$\rho_1 \circ \rho_2 = \{(x, y) \in S \times S \mid \exists z \in S : (x, z) \in \rho_1, (z, y) \in \rho_2\}$$

We denote by $R(prod_S)$ the set of all the restrictions of the mapping $prod_S$:

$$R(prod_S) = \{u \mid u \prec prod_S\}$$

where $u \prec prod_S$ means that $dom(u) \subseteq prod_S$ and $u(\rho_1, \rho_2) = prod_S(\rho_1, \rho_2)$ for $(\rho_1, \rho_2) \in dom(u)$.

Let us consider a nonempty set $T_0 \subseteq S \times S$. If u is an element of $R(prod_S)$ then we denote by $Cl_u(T_0)$ the *closure* of T_0 in the partial algebra $(2^{S\times S}, \{u\})$. This is the smallest subset Q of $2^{S\times S}$ such that $T_0 \subseteq Q$ and Q is closed under u. It is known that this is the union $\bigcup_{n\geq 0} X_n$, where

$$\begin{cases} X_0 = T_0 \\ X_{n+1} = X_n \cup \{u(\rho_1, \rho_2) \mid (\rho_1, \rho_2) \in dom(u) \cap (X_n \times X_n)\}, n \geq 0 \end{cases}$$

If $L \in Initial(\overline{M_0})$ then the pair $(L, \{\sigma_L\})$, where

- $dom(\sigma_L) = \{(x, y) \in L \times L \mid \sigma(x, y) \in L\}$
- $\sigma_L(x, y) = \sigma(x, y)$ for every $(x, y) \in dom(\sigma_L)$

is a partial algebra.

In the remainder of this section we give a short presentation of two following concepts: labeled graph and stratified graph. In what follows we summarize these concepts.

By a*labeled graph* we understand a tuple $G = (S_0, L_0, T_0, f_0)$, where S_0 is a finite set of nodes, L_0 is a set of elements named *labels*, T_0 is a set of binary relations on S_0 and $f_0 : L_0 \longrightarrow T_0$ is a surjective function. Because the empty set is not used in knowledge representation we suppose that the empty set is not an element of T_0. Such a structure admits a graphical representation. Each element of S_0 is represented by a rectangle specifying the corresponding node. We draw an arc from $x_1 \in S_0$ to $x_2 \in S_0$ and this arc is labeled by $a \in L_0$ if $(x_1, x_2) \in f_0(a)$. This case is shown in Figure 1.

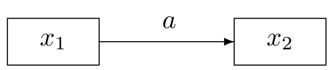

FIGURE 1. A labeled arc

Consider a labeled graph $G_0 = (S, L_0, T_0, f_0)$. A *stratified graph* ([10]) $\mathcal{G}$ over G_0 is a tuple (G_0, L, T, u, f) where

- $L \in Initial(\overline{L_0})$
- $u \in R(prod_S)$ and $T = Cl_u(T_0)$
- $f : (L, \{\sigma_L\}) \longrightarrow (2^{S\times S}, \{u\})$ is a morphism of partial algebras such that $f_0 \prec f$, $f(L) = T$ and if $(f(x), f(y)) \in dom(u)$ then $(x, y) \in dom(\sigma_L)$

The existence of this structure, as well as the uniqueness is proved in [10]:

Proposition 2.1. *For every labeled graph $G_0 = (S_0, L_0, T_0, f_0)$ and every $u \in R(prod_S)$ there is just one stratified graph (G_0, L, T, u, f) over G_0.*

3. ACCEPTED STRUCTURED PATHS

We consider a labeled graph $G_0 = (S, L_0, T_0, f_0)$. A *regular path* over G_0 is a pair $([x_1, \ldots, x_{n+1}], [a_1, \ldots, a_n])$ such that $(x_i, x_{i+1}) \in f_0(a_i)$ for every $i \in \{1, \ldots, n\}$.

Definition 3.1. We denote by $STR(G_0)$ the smallest set satisfying the following conditions:

- For every $a \in L_0$ and $(x, y) \in f_0(a)$ we have $([x, y], a) \in STR(G_0)$.
- If $([x_1, \ldots, x_k], u) \in STR(G_0)$ and $([x_k, \ldots, x_n], v) \in STR(G_0)$ then $([x_1, \ldots, x_k, \ldots, x_n], [u, v]) \in STR(G_0)$.

An element of the set $STR(G_0)$ is a **structured path** of G_0.

The concept of structured path introduces some order between the arcs taken into consideration for a regular path. A structured path can be represented as in Figure 2.

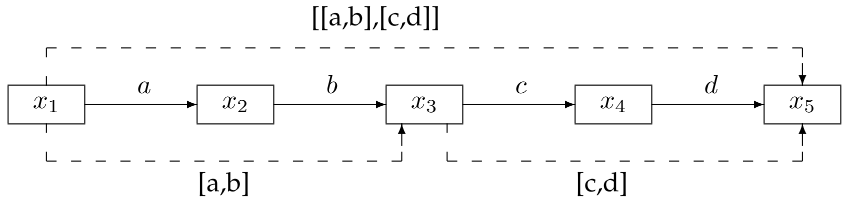

FIGURE 2. Intuitive representation of structured paths

Thus in Figure 2 we represented three structured paths:

$([x_1, x_2, x_3], [a, b])$
$([x_3, x_4, x_5], [c, d])$
$([x_1, x_2, x_3, x_4, x_5], [[a, b], [c, d]])$

Let us consider the set

$$\mathcal{L}(X) = \{[x_1, \ldots, x_n] \mid n \geq 1, x_i \in X, i = 1, \ldots, n\}$$

This is the set of all nonempty lists over X. We denote $first([x_1, \ldots, x_n]) = x_1$ and $last([x_1, \ldots, x_n]) = x_n$.

We define the mapping

$$\otimes : STR(G_0) \times STR(G_0) \longrightarrow STR(G_0)$$

as follows:

- $dom(\otimes) = \{((\alpha_1, u_1), (\alpha_2, u_2)) \mid (\alpha_1, u_1) \in STR(G_0), (\alpha_2, u_2) \in STR(G_0),$ $last(\alpha_1) = first(\alpha_2)\}$
- If $([x_1, \ldots, x_k], u) \in STR(G_0)$ and $([x_k, \ldots, x_n], v) \in STR(G_0)$ then

$$([x_1, \ldots, x_k], u) \otimes ([x_k, \ldots, x_n], v) = ([x_1, \ldots, x_n], [u, v])$$

Proposition 3.2. *Consider a labeled graph $G_0 = (S, L_0, T_0, f_0)$ and the set*

$$K(G_0) = \{([x, y], a) \mid a \in L_0, (x, y) \in f_0(a)\} \tag{3.1}$$

The set $STR(G_0)$ is the $\otimes$-Peano algebra generated by $K(G_0)$.

Proof. From Definition 3.1 we deduce that $STR(G_0)$ is the smallest set containing $K(G_0)$ and closed under $\otimes$ operation. It follows that $STR(G_0)$ is the $\otimes$-Peano algebra generated by $K(G_0)$. □

We define

$$STR_2(G_0) = \{w \mid \exists(\alpha, w) \in STR(G_0)\}$$

In fact, $STR_2(G_0)$ represents the projection of the set $STR(G_0)$ on the second axis: in a classical notation we write $STR_2(G_0) = pr_2(STR(G_0))$.

Proposition 3.3.

$$pr_2 K(G_0) = L_0$$

Proof. Take $a \in pr_2K(G_0)$. There is $([x,y],a) \in K(G_0)$. From (3.1) we have $a \in L_0$. Thus we have $pr_2K(G_0) \subseteq L_0$. Conversely, take an arbitrary element $a \in L_0$. We have $f_0(a) \in T_0$ and $\emptyset \notin T_0$. Take $(x,y) \in f_0(a)$. From (refp01) we have $([x,y],a) \in K(G_0)$, therefore $a \in pr_2K(G_0)$. It follows that $L_0 \subseteq pr_2K(G_0)$. □

We define the mapping $* : STR_2(G_0) \times STR_2(G_0) \longrightarrow STR_2(G_0)$ as follows:

- $dom(*) = \{(\beta_1,\beta_2) \mid \exists \alpha_1,\alpha_2 : (\alpha_1,\beta_1) \in STR(G_0), (\alpha_2,\beta_2) \in STR(G_0), last(\alpha_1) = first(\alpha_2)\}$
- If $(\beta_1,\beta_2) \in dom(*)$ then $\beta_1 * \beta_2 = [\beta_1,\beta_2]$

Remark 3.1. The pair $(STR_2(G_0),*)$ becomes a partial algebra.

Proposition 3.4. *$STR_2(G_0)$ is the $*$-Peano algebra generated by L_0.*

Proof. The set $STR(G_0)$ is the $\otimes$-Peano algebra generated by $K(G_0)$. This means that $STR(G_0) = \bigcup_{n\geq 0} M_n$, where

$$\begin{cases} M_0 = K(G_0) \\ M_{n+1} = M_n \cup \{\gamma \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), \gamma = \alpha \otimes \beta\} \end{cases} \tag{3.2}$$

It follows that

$$STR_2(G_0) = pr_2STR(G_0) = pr_2(\bigcup_{n\geq 0} M_n) = \bigcup_{n\geq 0} pr_2M_n =$$

$$pr_2M_0 \cup \bigcup_{n\geq 0} pr_2M_{n+1} = pr_2K(G_0) \cup \bigcup_{n\geq 0} pr_2M_{n+1}$$

therefore by Proposition 3.3 we have

$$STR_2(G_0) = L_0 \cup \bigcup_{n\geq 0} pr_2M_{n+1} \tag{3.3}$$

Based on (3.2) we obtain

$$pr_2M_{n+1} = pr_2M_n \cup pr_2X_n \tag{3.4}$$

where $X_n = \{\gamma \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), \gamma = \alpha \otimes \beta\}$.
From (3.4) we find that

$$pr_2M_{n+1} = pr_2M_n \cup \{pr_2\gamma \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), \gamma = \alpha \otimes \beta\} \tag{3.5}$$

We intend to evaluate the item $pr_2\gamma$ for $\gamma \in X_n$. Consider an arbitrary element $\gamma \in X_n$. There are $(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n)$ such that $\gamma = \alpha \otimes \beta$. This means that $\alpha = ([x_1,\ldots,x_k],u_1)$, $\beta = ([x_k,\ldots,x_m],v_1)$ and $\gamma = ([x_1,\ldots,x_k,\ldots,x_m], [u_1,v_1])$. It follows that $pr_2\gamma = [u_1,v_1]$ and by the definition of the operation $*$ we have $[u_1,v_1] = u_1 * v_1$. Thus, if $\gamma = \alpha \otimes \beta$, where $(\alpha,\beta) \in M_n \times M_n$ then $pr_2\gamma = pr_2\alpha * pr_2\beta$. This property allows to rewrite (3.5) as follows

$$pr_2M_{n+1} = pr_2M_n \cup \{w \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), w = pr_2\alpha * pr_2\beta\} \tag{3.6}$$

Let us denote $Y_n = pr_2M_n$ for every $n \geq 0$. We prove now the following property

$$\begin{aligned} \{w \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), w = pr_2\alpha * pr_2\beta\} &= \\ \{\omega \mid \exists(u,v) \in (Y_n \times Y_n) \cap dom(*) : \omega = u * v\} \end{aligned} \tag{3.7}$$

Take $w = pr_2\alpha * pr_2\beta$ for some $(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n)$. Consider $u = pr_2\alpha$ and $v = pr_2\beta$. Obviously $u,v \in Y_n$ and $w = u * v$. Thus we proved the inclusion

$$\begin{aligned} \{w \mid \exists(\alpha,\beta) \in dom(\otimes) \cap (M_n \times M_n), w = pr_2\alpha * pr_2\beta\} &\subseteq \\ \{\omega \mid \exists(u,v) \in (Y_n \times Y_n) \cap dom(*) : \omega = u * v\} \end{aligned} \tag{3.8}$$

$$\begin{tikzcd} STR_2(G_0) \times STR_2(G_0) \arrow[r, "*"] \arrow[d, "h \times h"'] & STR_2(G_0) \arrow[d, "h"] \\ (\overline{L_0})_\sigma \times (\overline{L_0})_\sigma \arrow[r, "\sigma"'] & (\overline{L_0})_\sigma \end{tikzcd}$$

FIGURE 3. Commutative diagram

We prove now the converse inclusion. To prove this property we consider an element $\omega = u * v$ for some $(u, v) \in (Y_n \times Y_n) \cap dom(*)$. But $Y_n = pr_2 M_n$ and $u \in Y_n$. It follows that there is $\alpha = ([x_1, \ldots, x_k], u) \in M_n$ and $\beta \in ([y_1, \ldots, y_m], v) \in M_n$ such that $x_k = y_1$. We deduce that $(\alpha, \beta) \in dom(\otimes) \cap (M_n \times M_n)$ such that $\omega = pr_2\alpha * pr_2\beta$. This shows that

$$\begin{aligned} &\{w \mid \exists(\alpha, \beta) \in dom(\otimes) \cap (M_n \times M_n), w = pr_2\alpha * pr_2\beta\} \supseteq \\ &\{\omega \mid \exists(u, v) \in (Y_n \times Y_n) \cap dom(*) : \omega = u * v\} \end{aligned} \tag{3.9}$$

Now, from (3.8) and (3.9) we obtain (3.7).
From (3.6) and (3.7) we obtain

$$pr_2 M_{n+1} = pr_2 M_n \cup \{\omega \mid \exists(u, v) \in (Y_n \times Y_n) \cap dom(*) : \omega = u * v\}$$

Equivalently we can write that

$$Y_{n+1} = Y_n \cup \{\omega \mid \exists(u, v) \in (Y_n \times Y_n) \cap dom(*) : \omega = u * v\} \tag{3.10}$$

From $Y_0 = L_0$ and (3.10) we obtain that $\bigcup_{n \geq 0} Y_n = \overline{L_0}$, where $\overline{L_0}$ is taken under operation $*$. From (3.3) we obtain $STR_2(G_0) = \bigcup_{n \geq 0} Y_n$, therefore $STR_2(G_0) = (\overline{L_0})_*$ and the proposition is proved. $\square$

Proposition 3.5. *The mapping $h : (STR_2(G_0), *) \longrightarrow ((\overline{L_0})_\sigma, \sigma)$ defined by*

$$h(p) = \begin{cases} p \text{ if } p \in L_0 \\ \\ \sigma(h(u), h(v)) \text{ if } p = [u, v], u \in STR_2(G_0), v \in STR_2(G_0) \end{cases}$$

is a morphism of partial algebras. In other words, the diagram from Figure 3 is commutative.

Proof. Consider $(u, v) \in dom(*)$. There are $([x_1, \ldots, x_k], u) \in STR(G_0)$ and $([x_k, \ldots, x_n], v) \in STR(G_0)$. If this is the case then $u*v = [u, v] \in STR_2(G_0)$ and $h([u, v]) = \sigma(h(u), h(v))$. Thus the diagram is commutative. $\square$

Definition 3.2. We define the set $ASP(\mathcal{G})$ as follows: $([x_1, \ldots, x_{n+1}], c) \in ASP(\mathcal{G})$ if and only if $([x_1, \ldots, x_{n+1}], c) \in STR(G_0)$ and $h(c) \in L$.
An element of $ASP(\mathcal{G})$ is named **accepted structured path** over $\mathcal{G}$.

We remark that we can consider the binary operation $\oslash$ for the case of accepted structured paths, defined on the subset $ASP(\mathcal{G}) \subseteq STR(G_0)$

$$\oslash : ASP(\mathcal{G}) \times ASP(\mathcal{G}) \longrightarrow ASP(\mathcal{G})$$

as follows:

- $dom(\oslash) = \{((p_1, u_1), (p_2, u_2)) \mid (p_1, u_1) \in ASP(\mathcal{G}), (p_2, u_2) \in ASP(\mathcal{G}),$ $last(p_1) = first(p_2), h([u_1, u_2]) \in L\}$

- If $([x_1,\ldots,x_k],u)\in ASP(\mathcal{G})$ and $([x_k,\ldots,x_n],v)\in ASP(\mathcal{G})$ then

$$([x_1,\ldots,x_k],u)\oslash([x_k,\ldots,x_n],v)=([x_1,\ldots,x_n],[u,v])$$

Remark 3.2. Suppose that $d_1=([x_1,\ldots,x_k],u)\in ASP(\mathcal{G})$, $d_2=([y_1,\ldots,y_r],v)\in ASP(\mathcal{G})$ and $(d_1,d_2)\in dom(\otimes)$. By the above definition we have $x_k=y_1$ and $h([u,v])\in L$. We obtain $d_1\oslash d_2=([x_1,\ldots,x_k,y_2,\ldots,y_r],[u,v])\in ASP(\mathcal{G})$ because $h([u,v])\in L$. Thus the mapping $\oslash$ is well defined.

Proposition 3.6. *The set $ASP(\mathcal{G})$ is the $\oslash$-Peano algebra generated by $K(G_0)$.*

Proof. Consider the sets

$$\left\{\begin{array}{l} M_0=K(G_0)\\ M_{n+1}=M_n\cup\{d\mid\exists d_1,d_2\in M_n: d=d_1\oslash d_2\}\end{array}\right.\tag{3.11}$$

Let us prove the following property:

$$ASP(\mathcal{G})=\bigcup_{n\geq 0}M_n\tag{3.12}$$

By induction on $n\geq 0$ we can verify that $M_n\subseteq ASP(\mathcal{G})$, therefore

$$\bigcup_{n\geq 0}M_n\subseteq ASP(\mathcal{G})\tag{3.13}$$

We prove now by induction on $n\geq 2$ that for every $([x_1,\ldots,x_n],c)\in ASP(\mathcal{G}$ there is $k\geq 0$ such that $([x_1,\ldots,x_n],c)\in M_k$.
For $n=2$ we have $([x_1,x_2],c)\in ASP(\mathcal{G})$, therefore $([x_1,x_2],c)\in STR(G_0)$ and $h(c)\in L$. But $length(c)=1$ and thus $h(c)=c\in L_0$. It follows that $([x_1,x_2],c)\in K(G_0)=M_0$ and the property is true for $n=2$.
Suppose that the property is true for every $k\leq m$ and take an element $([x_1,\ldots,x_{m+1}],c)\in ASP(\mathcal{G})$. By Proposition 4.8 we deduce that there is $s\in\{2,\ldots,m\}$ and there are $u,v\in STR_2(G_0)$ such that $([x_1,\ldots,x_s],u)\in ASP(\mathcal{G})$ and $([x_s,\ldots,x_{m+1}],v)\in ASP(\mathcal{G})$. Applying the inductive assumption we deduce that there are r,q such that $([x_1,\ldots,x_s],u)\in M_r$ and $([x_s,\ldots,x_{m+1}],v)\in M_q$. It follows that $([x_1,\ldots,x_{m+1}],c)\in M_{max\{r,q\}+1}$.
It follows that

$$ASP(\mathcal{G})\subseteq\bigcup_{n\geq 0}M_n\tag{3.14}$$

Now from (3.13) and (3.14) we obtain (3.12) and the proposition is proved. □

4. SPLITTING PROPERTIES

In this section we obtain two splitting properties: one of them refers to the decomposition of a structured path; the other gives the decomposition of an accepted structured path. The first splitting property is used to prove the second property.

Proposition 4.7. *(splitting property I)*
If $([x_1,\ldots,x_{n+1}],c)\in STR(G_0)$ and $n\geq 2$ then there are $u,v\in STR_2(G_0)$ and $k\in\{2,\ldots,n\}$, uniquely determined, such that
$c=[u,v]$
$([x_1,\ldots,x_k],u)\in STR(G_0)$
$([x_k,\ldots,x_{n+1}],v)\in STR(G_0)$

Proof. We denote by $(\overline{L_0})_*$ the *-Peano algebra generated by L_0. By Proposition 3.4 we have $STR_2(G_0) = (\overline{L_0})_*$. In a similar manner we consider the $\otimes$-Peano algebra generated by $K(G_0)$, denoted by $(\overline{K(G_0)})_\otimes$. By Proposition 3.2 we have $STR(G_0) = (\overline{K(G_0)})_\otimes$.
Take $([x_1, \ldots, x_{n+1}], c) \in STR(G_0)$, $n \geq 2$. This implies that $c \in STR_2(G_0) = (\overline{L_0})_*$, therefore there are $u, v \in STR_2(G_0)$, uniquely determined, such that $c = [u, v]$. Thus $([x_1, \ldots, x_{n+1}], [u, v]) \in STR(G_0) = (\overline{K(G_0)})_\otimes$. It follows that there are the elements, uniquely determined, $d_1 = ([y_1, \ldots, y_s], \gamma_1) \in STR(G_0)$, $d_2 = ([z_1, \ldots, z_p], \gamma_2) \in STR(G_0)$ such that $(d_1, d_2) \in dom(\otimes)$ and

$$([x_1, \ldots, x_{n+1}], [u, v]) = d_1 \otimes d_2 \tag{4.15}$$

Take $k = s$. Obviously k is uniquely determined.
From $(d_1, d_2) \in dom(\otimes)$ we deduce that $y_s = z_1$ and

$$d_1 \otimes d_2 = ([y_1, \ldots, y_s, z_2, \ldots, z_p], [\gamma_1, \gamma_2]) \tag{4.16}$$

From (4.15) and (4.16) we deduce that

$$[x_1, \ldots, x_{n+1}] = [y_1, \ldots, y_s, z_2, \ldots, z_p] \tag{4.17}$$

$$[u, v] = [\gamma_1, \gamma_2]$$

We have $u, v, \gamma_1, \gamma_2 \in STR_2(G_0))$, $STR_2(G_0)$ is a $*$ -Peano algebra and from $[u, v] = [\gamma_1, \gamma_2]$ we deduce $u = \gamma_1$ and $v = \gamma_2$. From (4.17) we deduce that $n+1 = s+p-1$ and $x_1 = y_1$, …, $x_s = y_s$, $x_{s+1} = z_2$, …, $x_{n+1} = z_p$. It follows that $d_1 = ([x_1, \ldots, x_s], u)$ and $d_2 = ([x_s, \ldots, x_{n+1}], v)$. But $d_1 \in STR(G_0)$ and $d_2 \in STR(G_0)$. Thus we have $([x_1, \ldots, x_k], u) \in STR(G_0)$ and $([x_k, \ldots, x_{n+1}], v) \in STR(G_0)$. The proposition is proved. $\square$

Proposition 4.8. *(splitting property II)*
If $([x_1, \ldots, x_{n+1}], c) \in ASP(\mathcal{G}) \setminus K(G_0)$ *then there are* $u, v \in STR_2(G_0)$ *and* $k \in \{2, \ldots, n\}$, *uniquely determined, such that*

$c = [u, v]$
$([x_1, \ldots, x_k], u) \in ASP(\mathcal{G})$
$([x_k, \ldots, x_{n+1}], v) \in ASP(\mathcal{G})$

Proof. Consider $([x_1, \ldots, x_{n+1}], c) \in ASP(\mathcal{G})$ and $n \geq 2$. Because $ASP(\mathcal{G}) \subseteq STR(G_0)$ we can apply Proposition 4.7. Thus, there are $u, v \in STR_2(G_0)$ and $k \in \{2, \ldots, n\}$, uniquely determined, such that

$c = [u, v]$
$([x_1, \ldots, x_k], u) \in STR(G_0)$
$([x_k, \ldots, x_{n+1}], v) \in STR(G_0)$

But $h(c) \in L$, therefore from the definition of the mapping h we deduce that $\sigma(h(u), h(v)) \in L$. We have $h(u) \in (\overline{L_0})_\sigma$ and $h(v) \in (\overline{L_0})_\sigma$. From $L \in Initial((\overline{L_0})_\sigma)$ we deduce that $h(u) \in L$ and $h(v) \in L$. This shows that $([x_1, \ldots, x_k], u) \in ASP(\mathcal{G})$ and $([x_k, \ldots, x_{n+1}], v) \in STR(G_0)$. $\square$

Corollary 4.1. *For every* $d \in ASP(\mathcal{G})$ *one and only one of the following conditions is satisfied:*

(1) $d \in K(G_0)$
(2) there are $d_1 \in ASP(\mathcal{G})$ *and* $d_2 \in ASP(\mathcal{G})$, *uniquely determined, such that* $d = d_1 \oslash d_2$.

Proof. Suppose that $d = ([x_1, \ldots, x_{n+1}], c) \in ASP(\mathcal{G}) \setminus K(G_0)$. We apply Proposition 4.8. There are $u, v \in STR_2(G_0)$ and $k \in \{2, \ldots, n\}$, uniquely determined, such that $c = [u, v]$, $([x_1, \ldots, x_k], u) \in ASP(\mathcal{G})$ and $([x_k, \ldots, x_{n+1}], v) \in ASP(\mathcal{G})$. From $d \in ASP(\mathcal{G})$ we know that $h(c) \in L$. We have

$$h([u, v]) \in L \tag{4.18}$$

$$\begin{tikzcd} K(G_0) \arrow[r, "id"] \arrow[dr, "T_0"'] & \mathcal{A}SP(\mathcal{G}) \arrow[d, "IP_{SKP}"] \\ & Y \end{tikzcd}$$

FIGURE 4. The extension IP_{SKP} of T_0

$$d_1 = ([x_1, \ldots, x_k], u) \in ASP(\mathcal{G}) \tag{4.19}$$

$$d_2 = ([x_k, \ldots, x_{n+1}], v) \in ASP(\mathcal{G}) \tag{4.20}$$

From (4.18), (4.19) and (4.20) we deduce that $(d_1, d_2) \in dom(\oslash)$. Moreover, $d = d_1 \oslash d_2$. Let us suppose that $d = ([y_1, \ldots, y_s], u_1) \oslash ([z_1, \ldots, z_r], u_2)$, $u_1 \in STR_2(G_0)$ and $u_2 \in STR_2(u_2)$. Denote $p_1 = ([y_1, \ldots, y_s], u_1)$ and $p_2 = ([z_1, \ldots, z_r], u_2)$. We have $y_s = z_1$ and $h([u_1, u_2]) \in L$. It follows that $d = ([y_1, \ldots, y_s, z_2, \ldots, z_r], [u_1, u_2])$. But $d = ([x_1, \ldots, x_{n+1}], c)$, therefore $s+r-1 = n+1$, $y_1 = x_1$, ..., $y_s = x_s = z_1$, $z_2 = x_{s+1}$, ..., $z_r = x_{n+1}$. In conclusion we have

$$p_1 = ([x_1, \ldots, x_s], u_1) \tag{4.21}$$

$$p_2 = ([x_s, \ldots, x_{n+1}], u_2) \tag{4.22}$$

From $(d_1, d_2) \in dom(\oslash)$ we deduce also $h([u_1, u_2]) \in L$. But $h([u_1, u_2]) = \sigma(h(u_1), h(u_2))$ and $h(u_1) \in (\overline{L_0})_\sigma$, $h(u_2) \in (\overline{L_0})_\sigma$. We have $L \in Initial((\overline{L_0})_\sigma)$ and $\sigma(h(u_1), h(u_2)) \in L$, therefore $h(u_1) \in L$ and $h(u_2) \in L$. If we use now (4.21) and (4.22) then we deduce that $p_1 \in ASP(\mathcal{G})$ and $p_2 \in ASP(\mathcal{G})$. □

5. INFERENCE PROCESS BASED ON ACCEPTED STRUCTURED PATHS

We consider a stratified graph $\mathcal{G} = (G_0, L, T, u, f)$ over $G_0 = (S_0, L_0, T_0, f_0)$. Let $\mathcal{Y} = (Y, \odot)$ be a binary algebra and an injective mapping $ob : S \longrightarrow Y$. We suppose that for each $u \in L$ we have an algorithm $Alg_u : Y \times Y \longrightarrow Y$. This means that Alg_u is a partial mapping. In other words $dom(Alg_u) \subseteq Y \times Y$ and for every pair $(x, y) \in dom(Alg_u)$ given as input for Alg_u this algorithm gives as output some element of Y.

Definition 5.3. A **system of knowledge** based on stratified graphs is a tuple

$$SKP = (\mathcal{G}, (Y, \odot), ob, \{Alg_u\}_{u \in L})$$

where

- $\mathcal{G} = (G_0, L, T, u, f)$ is a stratified graph over $G_0 = (S, L_0, T_0, f_0)$;
- $(Y, \odot)$ is a binary partial algebra;
- $ob : S \longrightarrow Y$ is an injective mapping;
- For each $u \in L$ the entity Alg_u is an algorithm that defines a mapping

$$Alg_u : dom(Alg_u) \longrightarrow Y$$

where $dom(Alg_u) \subseteq Y \times Y$.

$$\begin{tikzcd}
ASP(\mathcal{G}) \times ASP(\mathcal{G}) \arrow[r, "\oslash"] \arrow[d, "IP_{SKP} \times IP_{SKP}"'] & ASP(\mathcal{G}) \arrow[d, "IP_{SKP}"] \\
Y \times Y \arrow[r, "\odot"'] & Y
\end{tikzcd}$$

FIGURE 5. Commutative diagram

For a knowledge processing system based on stratified graphs we can define the inference process as in the next definition.

Definition 5.4. The **inference process** IP_{SKP} generated by the system of knowledge representation SKP is the mapping

$$IP_{SKP} : ASP(\mathcal{G}) \longrightarrow Y$$

defined as follows:

$$d \in K(G_0) \Longrightarrow IP_{SKP}(d) = T_0(d)$$

$$d \in M_{n+1} \Longrightarrow IP_{SKP}(d) = \begin{cases} IP_{SKP}(d) \text{ if } d \in M_n \\ \\ IP_{SKP}(d_1) \odot IP_{SKP}(d_2) \text{ if } d_1, d_2 \in M_n, d = d_1 \oslash d_2 \end{cases}$$

where the sequence $\{M_n\}_{n\geq 0}$ is defined in (3.11).

Proposition 5.9. *The mapping IP_{SKP} is well defined.*

Proof. Really, every element $d \in ASP(G)$ containing at least three nodes can be uniquely broken into two accepted structured paths d_1 and d_2, that is $d = d_1 \otimes d_2$. □

Proposition 5.10. $dom(IP_{SKP}) = ASP(\mathcal{G})$

Proof. Really, $dom(IP_{SKP}) = \bigcup_{n\geq 0} M_n = ASP(\mathcal{G})$. □

Proposition 5.11. *The inference process $IP_{SKP} : (ASP(\mathcal{G}), \oslash) \longrightarrow (Y, \odot)$ defined by the system of knowledge processing SKP is a morphism of partial algebras.*

Proof. The classical proof of such property in the domain of the partial algebra is graphically represented in Figure 4 and Figure 5, where id is the identity mapping. In fact, we have the following properties:

(1) If $(d_1, d_2) \in dom(\oslash)$ then $d_1 = ([x_1, \ldots, x_k], u)$, $d_2 = ([y_1, \ldots, y_r], v)$, $x_k = y_1$ and $h([u, v]) \in L$.

(2) Using the previous notations based on the fact that $d_1, d_2 \in ASP(\mathcal{G}) = \bigcup_{n\geq 0} M_n$ we deduce that there are $p \geq 0$ and $q \geq 0$ such that $d_1 \in M_p$ and $d_2 \in M_q$. If $r = max\{p, q\}$ then $d = d_1 \oslash d_2 \in M_{r+1}$ and from Definition 5.4 we obtain $IP_{SKP}(d) = IP_{SKP}(d_1) \odot IP_{SKP}(d_2)$. In other words, we have

$$IP_{SKP}(d_1 \oslash d_2) = IP_{SKP}(d_1) \odot IP_{SKP}(d_2)$$

This property shows that the diagram from Figure 5 is commutative.

□

Based on the previous concepts and results we can propose the following algorithm of the inference process.

Inference algorithm

Input:

$SKP = (\mathcal{G}, (Y, \odot), ob, \{Alg_u\}_{u \in L})$;

$(x, y) \in S \times S$

Method:

Compute $C_{(x,y)} = \{d \in ASP(\mathcal{G}) \mid first(d) = x, last(d) = y\}$;

Output:

$IP_{SKP}(C_{(x,y)}) = \{w \in Y \mid \exists d \in C_{(x,y)} : IP_{SKP}(d) = w\}$

End of algorithm

6. Text Generation with Accepted Structured Paths. A case study

In order to demonstrate the role the structured paths can have in a Natural Language Generation mechanism based on labeled graph representations, we consider the labeled stratified graph given in [3] and shown in Figure 6:

$$G_0 = (S, L_0, T_0, f_0)$$

where $S = \{x_1, x_2, x_3, x_4\}$, $L_0 = \{CAT\ Det, CAT\ Noun, CAT\ Adj\}$, $T_0 = \{\rho_1, \rho_2\}$ with $\rho_1 = \{(x_1, x_2)\}$, $\rho_2 = \{(x_2, x_3)\}$ and $\rho_3 = \{(x_3, x_4)\}$, $f_0(CAT\ Det) = \rho_1$, $f_0(CAT\ Noun) = \rho_2$ and $f_0(CAT\ Adj) = \rho_3$.

Consider the mapping $u \in R(prod_S)$. The set $Cl_u(T_0)$ is the following set:

$$Cl_u(T_0) = \bigcup_{n \geq 0} M_n$$

$M_0 = T_0 = \{\rho_1, \rho_2, \rho_3\}$
$M_1 = M_0 \cup \{\rho_4\} = u(\rho_2, \rho_3)$
$M_2 = M_1 \cup \{\rho_5\} = u(\rho_1, \rho_4)$
$M_3 = M_2$

We obtain $Cl_u(T_0) = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$.

We have the stratified graph $\mathcal{G} = (G_0, L, T, u, f)$ over G_0 where $L = \{CAT\ Det, CAT\ Noun, CAT\ Adj, \sigma_L(CAT\ Noun, CAT\ Adj), \sigma_L(CAT\ Det, \sigma_L(CAT\ Noun, CAT\ Adj))\}$ and $T = Cl_u(T_0)$, where:

- $(f_0(CAT\ Noun), f_0(CAT\ Adj)) \in dom(u)$, therefore $\sigma_L(CAT\ Noun, CAT\ Adj) \in L$ and $f(\sigma_L(CAT\ Noun, CAT\ Adj)) = u(f(CAT\ Noun), f(CAT\ Adj)) = u(\rho_2, \rho_3) = \rho_4$.
- $(f(CAT\ Det), f(\sigma_L(CAT\ Noun, CAT\ Adj))) \in dom(u)$ therefore $\sigma_L(CAT\ Det, \sigma_L(CAT\ Noun, CAT\ Adj)) \in L$ and $f(\sigma_L(CAT\ Det, \sigma_L(CAT\ Noun, CAT\ Adj))) = u(f(CAT\ Det), f(\sigma_L(CAT\ Noun, CAT\ Adj))) = u(\rho_1, \rho_4) = \rho_5$.

The labeled stratified graph $\mathcal{G}$ is shown in Figure 6, where the accepted maximal structured path $([x_1, x_2, x_3, x_4], [CAT\ Det, [CAT\ Noun, CAT\ Adj]])$ is represented.

Let us suppose we have a set of algorithms for Natural Language Generation (shortly NLG). We are not interested here what is the method used for generation but we suppose that it contains at least two elements: non-terminals (denoting the syntactic classes of the text components) and terminals (i.e. word forms).

$Alg_{\ CAT\ syn\_cat}(x, y)$
take $word\_form \leftarrow$ pick a word from lexicon with the specified syn_cat
Output: $word\_form$
end algorithm

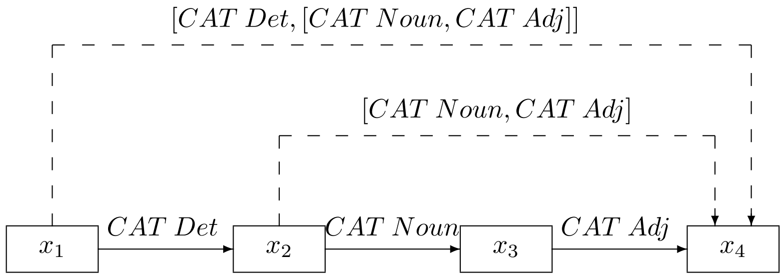

FIGURE 6. Labeled stratified graph for Natural Language Generation

$Alg_{\sigma(CAT\ syn\_cat1, CAT\ syn\_cat2)}(o_1, o_2)$
  load the agreement rules between the syntactic categories syn_cat1, syn_cat2
    generate inflected forms of o_1 and o_2
    Output: *inflected form*(o_1) + " " + *inflected form*(o_2)
end algorithm

$Alg_{\sigma(u,v)}(o_1, o_2)$
  take n the number of words of o_1 sequence, $n \geq 1$
  take m the number of words of o_2 sequence, $m \geq 1$
  IF $(n > 1) \vee (m > 1)$ THEN
    take $head(o_1) \leftarrow$ the head word of o_1 sequence
    take $head(o_2) \leftarrow$ the head word of o_2 sequence
    load the agreement rules between the syntactic categories of $head(o_1)$ and $head(o_2)$
      generate inflected forms of o_1 and o_2
      Output: *inflected form*(o_1) + " " + *inflected form*(o_2)
  ENDIF
end algorithm

$Alg_{\sigma(CAT\ syn\_cat, u)}(o_1, o_2)$
  take n the number of words of o_2 sequence, $n \geq 1$
  IF $(n > 1)$ THEN
    take $head(o_2) \leftarrow$ the head word of o_2 sequence
    load the agreement rules between syn_cat and the syntactic category of $head(o_2)$
      generate inflected forms of o_1 and o_2
      Output: *inflected form*(o_1) + " " + *inflected form*(o_2)
  ENDIF
end algorithm

In what follows all Romanian constructions are marked in quotes and the English equivalents follow the Romanian examples in brackets. Let us consider:

$Alg_{CAT\ Noun}(x_1, x_2)$ = "un" ("a"),
$Alg_{CAT\ Noun}\ (x_2, x_3)$ = "fată" ("the girl"),
$Alg_{CAT\ Adj}(x_2, x_3)$ = "frumos" ("beautiful").

With these interpretations, we can obtain:

$Alg_{\sigma(CAT\ Noun, CAT\ Adj)}(o_1, o_2)$ = "fată frumoasă" (in English"the beautiful girl") with

o_1 = $Alg_{CAT\ Noun}$ (x_2, x_3) = "fată" ("the girl"), o_2 = $Alg_{CAT\ Adj}$ (x_3, x_4) = "frumos" ("beautiful"). The sequence "fată frumoasă" ("the beautiful girl") obey the agreement in gender realization between o_1 and o_2 ("fata": number=sg., gender=fem., case = direct, "frumosă": number=sg., gender=fem).

$Alg_{\sigma(CAT\ Det, \sigma(CAT\ Noun, CAT\ Adj))}(o_1, o_2)$= "o fată frumoasă" ("the beautiful girl") with o_1 = 'un" ("a") , o_2 = "fată frumoasă" ("the beautiful girl"). In "o fată frumoasă" ("the beautiful girl") the sequence of o_1 takes the gender of the head word of o_2, that is "fată" ("fată": number=sg., gender=fem., case=direct).

The inference process for the considered study case is exemplified in what follows:
Input:
$SKP = (\mathcal{G}, (Y, \odot), ob, \{Alg_u\}_{u \in L})$; $(x_1, x_4) \in S \times S$
Method:
Compute $C_{(x_1,x_4)} = \{([x_1, x_2, x_3, x_4], [CAT\ Det, [CAT\ Noun, CAT\ Adj]])\}$;
Output:
$IP_{SKP}(C_{(x_1,x_4)}) = \{$"o fată frumoasă" $\}$

7. Conclusions

In this paper we treat from the mathematical point of view the concept of inference based on stratified graphs. We define the concept of knowledge processing system with stratified graphs and the concept of inference of such systems. We exemplify the inference process in such system by means of a new mechanism for interpreting the relations encoded in stratified graphs that was defined in [3]. This interpretation can be used to generate natural language constructions.

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