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Second order differential equations with an irregular singularity at the origin and a large parameter: convergent and asymptotic expansions

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ABSTRACT. We consider the second order linear differential equation

$$y'' = \left[\frac{\Lambda^2}{t^\alpha} + g(t)\right] y,$$

where Λ is a large complex parameter and g is a continuous function. In previous works we have considered the case $\alpha \in (-\infty, 2]$ and designed a convergent and asymptotic method for the solution of the corresponding initial value problem with data at $t = 0$. In this paper we complete the research initiated in those works and analyze the remaining case $\alpha \in (2, \infty)$. We use here the same fixed point technique; the main difference is that for $\alpha \in (2, \infty)$ the convergence of the method requires that the initial datum is given at a point different from the origin; for convenience we choose the point at the infinity. We obtain a sequence of functions that converges to the unique solution of the problem. This sequence has also the property of being an asymptotic expansion for large Λ (not of Poincaré-type) of the solution of the problem. The generalization to non-linear problems is straightforward. An application to a quantum mechanical problem is given as an illustration.

1. INTRODUCTION

The most famous asymptotic method for second order linear differential equations containing a large parameter is, without any doubt, Olver's method [6, Chaps. 10, 11, 12]. Olver's theory considers the equation

$$\ddot{y} - \frac{\tilde{\Lambda}^2}{z^\alpha} y = f(z) y, \qquad \tilde{\Lambda} \to \infty, \qquad z \in \mathbb{C}, \qquad \alpha \neq 2, \tag{1.1}$$

with special attention to the cases $\alpha = 0, -1, 1$ ($x = 0$ is a regular point, a transition point or a regular singular point respectively). In [6, Chaps. 10, 11 and 12], Olver gives, for $\alpha = 0, -1, 1$ respectively, the Poincaré-type asymptotic expansion of two independent solutions of (1.1) for large $\tilde{\Lambda}$, including error bounds, sectors of validity, etc. In [6, Chap. 12, Sec. 14] we can also find indications about the generalization of the study to any integer value of α, except $\alpha = 2$.

In summary, we have that for any $\alpha \in \mathbb{Z} \setminus \{2\}$, two independent solutions of (1.1) have the form

$$y(z) = P_\alpha(z) \sum_{k=0}^{n-1} \frac{A_k(z)}{\tilde{\Lambda}^{2k}} + \frac{1}{\tilde{\Lambda}^2} P'_\alpha(z) \sum_{k=0}^{n-1} \frac{B_k(z)}{\tilde{\Lambda}^{2k}} + R_{\alpha,n}(z), \tag{1.2}$$

where $R_{\alpha,n}(z) = \mathcal{O}(\tilde{\Lambda}^{-2n})$ uniformly for z in a certain region in the complex plane. In this formula, $P_\alpha(z)$ is one of the two following basic solutions of (1.1), that is, independent

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solutions of (1.1) for $\tilde{g}=0$

$$P_\alpha(z):=\begin{cases}\sqrt{z}I_{\hat{\alpha}}(2\hat{\alpha}\tilde{\Lambda}z^{1/(2\hat{\alpha})}),\\ \sqrt{z}K_{\hat{\alpha}}(2\hat{\alpha}\tilde{\Lambda}z^{1/(2\hat{\alpha})}),\end{cases}\qquad \hat{\alpha}:=\frac{1}{2-\alpha}, \tag{1.3}$$

where $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions. For example, for $\alpha=0,1,3,4,5,\ldots$, the coefficients A_k and B_k are given by the following system of recurrence relations: $A_0(z)=1$ and

$$\begin{cases}B_n(z)= & \dfrac{z^{\alpha/2}}{2}\displaystyle\int^z z^{\alpha/2}[\tilde{g}(z)A_n(z)-A_n''(z)]dz,\\ A_{n+1}(z)= & -\dfrac{1}{2}B_n'(z)+\dfrac{1}{2}\displaystyle\int^z \tilde{g}(z)B_n(z)dz.\end{cases}\qquad n=0,1,2,\ldots \tag{1.4}$$

Both families of coefficients A_n and B_n are analytic at $z=0$ when $\tilde{g}(z)$ is also analytic there. Olver's important contribution is the proof of the asymptotic character of the expansions (1.2) and the derivation of error bounds for the remainder $R_{\alpha,n}(z)$.

In [5] and [3] we gave an alternative approximation to Olver's expansions (in the cases $\alpha=0,-1,1$) that are not only asymptotic, but also convergent. In [3] and [4] we generalized the method to $\alpha\in(-\infty,2]$, not necessary an integer. In this paper we complete the theory initiated in [5], [2], [3] and [4] by considering the remaining cases: $\alpha\in(2,\infty)$.

Following the methodology of [5], [2], [3] and [4], we formulate a convenient initial value problem associated to equation (1.1). In those papers we have considered an initial value problem with data given at $z=0$. When $\alpha\in(-\infty,2]$, the degree of the singularity at $z=0$ is controllable by the iterated technique and the method converges, as it is shown in [5], [2], [3] and [4]. But now, for $\alpha\in(2,\infty)$, the singularity at $z=0$ is too strong and the iterated integrals involved in the fixed point technique are divergent. To overcome this problem we must consider an initial value problem with datum given at a point z_0 different from $z=0$. The most interesting possibility from a practical point of view is $z_0=\infty$, that is a regular singular point for $\alpha=3,4,5,\ldots$.

In the remaining of this section we introduce some definitions and considerations necessary to develop our theory. The branch cut chosen for z^α in (1.1) is the negative real axis, that is, we choose the principal value for z^α. In this paper we are going to analyze equation (1.1) in rays emanating from the origin: $z=te^{i\theta}$ with fixed $\theta\in\mathbb{R}$ and positive and unbounded t: $t\in[t_0,\infty)$ with $t_0>0$ fixed. Therefore, we may absorb the argument of the independent variable z in a redefinition of $\tilde{\Lambda}$ and $f(z)$: $\tilde{\Lambda}\to\Lambda:=e^{i\theta(1-\alpha/2)}\tilde{\Lambda}$ and $f(z)\to\tilde{g}(t):=e^{2i\theta}f(te^{i\theta})$ and consider the differential equation in the real variable t

$$\ddot{y}-\frac{\Lambda^2}{t^\alpha}y-\tilde{g}(t)y=0,\qquad \Lambda\in\mathbb{C},\quad \alpha>2. \tag{1.5}$$

We require for the function $\tilde{g}:[t_0,\infty)\to\mathbb{C}$ to be continuous. For the later convenience, we define the function

$$\mathrm{H}_\alpha(z):=1+\left|\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2}{\alpha-2}z^{\alpha/2-1}\right)\right|,\qquad z\in\mathbb{C}, \tag{1.6}$$

where $\mathrm{I}_\nu(z)$ and $\mathrm{K}_\nu(z)$ denote the principal values of the modified Bessel functions.

In the following section, we use the Banach fixed point theorem and the Green function of an auxiliary initial value problem to obtain uniformly convergent expansions of a solution of (1.5) in terms of iterated integrals of Bessel functions. In Section 3 we show that this expansion is an asymptotic expansion, for large $\tilde{\Lambda}$, of the unique solution of the initial

In Olver's analysis, the function $\tilde{g}$ is required to be analytic in a certain region of the complex plane and α integer.

value problem. As an application, an example extracted from atomic physics is presented. Finally, in Section 4, we summarize and compare the problems so far analyzed in [5], [2], [3] and [4] and in this paper.

2. The iterative method

The point $t=\infty$ is a regular singular point of the differential equation (1.5) for $\alpha = 3,4,5,\dots$ and, for any $\alpha\in(2,\infty)$, the following initial value problem selects one of the solutions of the equation

$$\begin{cases} \ddot{y}-\dfrac{\Lambda^2}{t^\alpha}y-\tilde{g}(t)y=0 \quad \text{in } [t_0,\infty), \quad t_0>0, \\ \lim\limits_{t\to\infty} y(t)=y_0, \end{cases} \tag{2.7}$$

with $y_0, \Lambda\in\mathbb{C}$, $y_0=\mathcal{O}(1)$ as $\Lambda\to\infty$.

Equivalently, after the change of variable $t\to 1/x$, we may write (2.7) in the form

$$\begin{cases} x^{4-\alpha}y''+2x^{3-\alpha}y'-\Lambda^2y-g(x)y=0 \quad \text{in } [0,X], \\ y(0)=y_0, \end{cases} \tag{2.8}$$

where $g(x):=\tilde{g}(1/x)$, $X:=1/t_0$. The function g is continuous in $[0,X]$, but when $\alpha\in(2,3)$, as it will be clear later, we require an extra condition for g

$$\left\| x^{\alpha-3}g(x)\right\|_\infty \le L, \tag{2.9}$$

where L is a positive constant independent of x. Here, $\|\cdot\|_\infty$ is the supremum norm in $[0,X]$

$$\|u\|_\infty = \sup_{x\in[0,X]} |u(x)|. \tag{2.10}$$

We have the following theorem:

Theorem 2.1. *Let $g:[0,X]\to\mathbb{C}$ be continuous and satisfy* (2.9) *when $\alpha\in(2,3)$. Then, problem* (2.8) *has a unique solution $y(x)$. Moreover,*

(i) *For $n=0,1,2,\dots$, the sequence*

$$\begin{aligned} y_{n+1}(x) = \quad & \phi(x)+\frac{2}{\alpha-2}\int_0^x \frac{t^{\alpha-2}}{\sqrt{xt}}\left[\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\right. \\ & \left. -\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\right] g(t)y_n(t)dt, \end{aligned} \tag{2.11}$$

with

$$y_0(x)=\phi(x):=\left(\frac{1}{\alpha-2}\right)^{\frac{3-\alpha}{2-\alpha}}\Gamma\left(\frac{1}{\alpha-2}\right)\Lambda^{1/(2-\alpha)}\frac{y_0}{\sqrt{x}}\,\mathrm{I}_{\frac{1}{\alpha-2}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right), \tag{2.12}$$

converges to $y(x)$ uniformly for $x\in[0,X]$.

(ii) *The remainder*

$$R_n(x):=H_\alpha^{-1}\left(\Lambda^{\frac{2}{\alpha-2}}x\right)[y(x)-y_n(x)], \tag{2.13}$$

is bounded by

$$|R_n(x)|\le\frac{A^n}{n!}\left|\frac{X}{\Lambda}\right|^n\left\|H_\alpha^{-1}\left(\Lambda^{\frac{2}{\alpha-2}}\cdot\right)(y-\phi)\right\|_\infty, \tag{2.14}$$

where A is a constant independent of x and Λ and H_α is defined in (1.6).

Proof. The function $\phi(x)$ defined in (2.11) is the unique solution of the auxiliary initial value problem

$$\begin{cases} x^{4-\alpha}\phi'' + 2x^{3-\alpha}\phi' - \Lambda^2\phi = 0 \quad \text{in} \quad [0,X], \\ \phi(0) = y_0. \end{cases} \tag{2.15}$$

Then, after the change of unknown $y(x) \to u(x) := y(x) - \phi(x)$, problem (2.8) reads

$$\begin{cases} x^{4-\alpha}u'' + 2x^{3-\alpha}u' - \Lambda^2 u = F(x,u) := (u+\phi)g \quad \text{in} \quad [0,X], \\ u(0) = 0. \end{cases} \tag{2.16}$$

We seek solutions of the equation $\mathbf{L}[u] := x^{4-\alpha}u'' + 2x^{3-\alpha}u' - \Lambda^2 u - F(x,u) = 0$ in the Banach space $\mathcal{B} = \{u : [0,X] \to \mathbb{C}, u \in \mathcal{C}[0,X]\}$ equipped with the norm (2.10). Then, we solve the equation $\mathbf{L}[u] = 0$ for u by using the Green function of the operator $\mathbf{M}[u] := x^{4-\alpha}u'' + 2x^{3-\alpha}u' - \Lambda^2 u$, accompanied by a homogeneous initial condition [8]. That is, $G(x,t)$ is the unique solution of the initial value problem

$$\begin{cases} x^{4-\alpha}G_{xx}(x,t) + 2x^{3-\alpha}G_x(x,t) - \Lambda^2 G(x,t) = \delta(x-t) \quad \text{in} \quad (x,t) \in [0,X]^2, \\ G(0,t) = 0. \end{cases} \tag{2.17}$$

After a straightforward computation we obtain

$$\begin{aligned} G(x,t) = \frac{2t^{\alpha-2}}{\alpha-2}\frac{1}{\sqrt{xt}} \quad & \left[\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\right. \\ & \left. -\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\right]\chi_{[0,x]}(t), \end{aligned} \tag{2.18}$$

where $\chi_{[0,x]}(t)$ is the characteristic function of the interval $[0,x]$. Then, any solution $u(x)$ of (2.16) is a solution of the Volterra integral equation of the second kind

$$\begin{aligned} u(x) = \quad & \frac{2}{\alpha-2}\int_0^x \frac{t^{\alpha-2}}{\sqrt{xt}}\left[\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\right. \\ & \left. -\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\right] g(t)[u(t)+\phi(t)]dt. \end{aligned}$$

Or equivalently, defining

$$\tilde{u}(x) := \mathrm{H}_\alpha^{-1}\left(\Lambda^{\frac{2}{\alpha-2}}x\right)u(x) \quad \text{and} \quad \tilde{\phi}(x) := \mathrm{H}_\alpha^{-1}\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\phi(x), \tag{2.19}$$

we have that for any solution $u(x) = \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\tilde{u}(x)$ of (2.16), $\tilde{u}(x)$ is a solution of the Volterra integral equation of the second kind

$$\tilde{u}(x) = [\mathbf{T}\tilde{u}](x), \tag{2.20}$$

where $\mathbf{T}$ is the integral operator

$$[\mathbf{T}\tilde{u}](x) := \frac{1}{\Lambda}\int_0^x K_\Lambda(x,t)g(t)[\tilde{u}(t)+\tilde{\phi}(t)]dt, \tag{2.21}$$

with kernel

(2.22)
$$K_\Lambda(x,t) := \frac{2\Lambda t^{\alpha-2}}{(\alpha-2)\sqrt{xt}} \frac{\mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}t\right)}{\mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)} \left[\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\right.$$
$$\left. -\mathrm{I}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}t^{\alpha/2-1}\right)\mathrm{K}_{\frac{1}{2-\alpha}}\left(\frac{2\Lambda}{\alpha-2}x^{\alpha/2-1}\right)\right].$$

Now we study separately the cases $\alpha \in (2,3)$ and $\alpha \geq 3$.

When $\alpha \geq 3$, the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are continuous functions of z in $\mathbb{C} \setminus (-\infty, 0]$ and $\mathrm{H}_\alpha(z) \neq 0$ for all $z \in \mathbb{C}$. Using in addition the asymptotic behavior of these functions at $z = 0$ [7, Eqs. 10.25.2, 10.27.4 and 10.27.5] and at $z = \infty$ [7, Sec. 10.40(i)], we find that the kernel $K_\Lambda(x,t)$ of the operator $\mathbf{T}$ is uniformly bounded by a constant N independent of x and Λ

$$|K_\Lambda(x,t)| \leq N \quad \text{for} \quad 0 \leq t \leq x \leq X \quad \text{and} \quad \Lambda \in \mathbb{C}. \tag{2.23}$$

Using this bound, from [8, Chap 4. Eq. (4.13)] with $M = N\|g\|_\infty/|\Lambda|$ and $a = 0$ we find that, for any couple $z, w \in \mathcal{B}$ and $x \in [0, X]$,

$$|[\mathbf{T}^n z](x) - [\mathbf{T}^n w](x)| \leq \frac{\|g\|_\infty^n}{n!}\left|\frac{Nx}{\Lambda}\right|^n \|z - w\|_\infty \tag{2.24}$$

and then

$$\|\mathbf{T}^n z - \mathbf{T}^n w\|_\infty \leq \frac{\|g\|_\infty^n}{n!}\left|\frac{NX}{\Lambda}\right|^n \|z - w\|_\infty. \tag{2.25}$$

This means that the operator $\mathbf{T}^n$ is contractive in $\mathcal{B}$ for large enough n and the successive approximations $\tilde{u}_{n+1} = \mathbf{T}(\tilde{u}_n)$, $n = 0, 1, 2, \ldots$, $\tilde{u}_0(x) = 0$, converge uniformly in $x \in [0, X]$ to $\tilde{u}(x)$ [8, Chap 4. Sec. 4]. In other words, $u(x) = \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\tilde{u}(x)$ is the unique solution of (2.16). Or equivalently, the sequence $y_n(x) = \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)[\tilde{u}_n(x) + \tilde{\phi}(x)]$ given in (2.11) converges uniformly in $x \in [0, X]$ to the unique solution of (2.8), and we have thesis (i) with $A = N\|g\|_\infty$.

When $\alpha \in (2,3)$, the kernel (2.22) is not bounded at $t = 0$ and the previous argument is not valid. In this case we rewrite the operator $\mathbf{T}$ in the form:

$$[\mathbf{T}\tilde{u}](x) := \frac{1}{\Lambda}\int_0^x \widetilde{K}(x,t)t^{\alpha-3}g(t)[\tilde{u}(t) + \tilde{\phi}(t)]dt, \tag{2.26}$$

with

$$\widetilde{K}_\Lambda(x,t) = t^{3-\alpha}K_\Lambda(x,t). \tag{2.27}$$

The factor $t^{3-\alpha}$ makes the new kernel $\widetilde{K}_\Lambda(x,t)$ bounded at $t = 0$ and we have the same uniform bound for $\widetilde{K}_\Lambda(x,t)$ when $\alpha \in (2,3)$ as we had for $K_\Lambda(x,t)$ when $\alpha > 3$:

$$|\widetilde{K}_\Lambda(x,t)| \leq N \quad \text{for} \quad 0 \leq t \leq x \leq X \quad \text{and} \quad \Lambda \in \mathbb{C}, \tag{2.28}$$

where N is a constant independent of x and Λ.

Using this bound, from [8, Chap 4. Eq. (4.13)] with $M = NL/|\Lambda|$ and $a = 0$ we find that, for any couple $z, w \in \mathcal{B}$ and $x \in [0, X]$,

$$|[\mathbf{T}^n z](x) - [\mathbf{T}^n w](x)| \leq \frac{L^n}{n!}\left|\frac{Nx}{\Lambda}\right|^n \|z - w\|_\infty, \tag{2.29}$$

and then

$$\|\mathbf{T}^n z - \mathbf{T}^n w\|_\infty \leq \frac{L^n}{n!}\left|\frac{NX}{\Lambda}\right|^n \|z-w\|_\infty . \tag{2.30}$$

From here, the proof is identical to the proof for $\alpha \geq 3$, and we have thesis (i) with $A = NL$.

To prove thesis (ii) we set $z = \tilde{u}$ and $w = \tilde{u}_0 = 0$ in (2.24) and (2.29) for $\alpha \geq 3$ and $\alpha \in (2,3)$ respectively. Using that $\mathbf{T}^n\tilde{u} = \tilde{u}$ and $\mathbf{T}^n\tilde{u}_0 = \tilde{u}_n$ we find

$$|\tilde{u}(x) - \tilde{u}_n(x)| \leq \frac{A^n}{n!}\left|\frac{X}{\Lambda}\right|^n \|\tilde{u}\|_\infty , \tag{2.31}$$

with $A = N\|g\|_\infty$ when $\alpha \geq 3$ and $A = NL$ when $\alpha \in (2,3)$. Using $y(x) = \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)$ $\tilde{u}(x) + \phi(x)$ and $y_n(x) = \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\tilde{u}_n(x) + \phi(x)$ in (2.31) we find (2.14). □

3. Asymptotic property of the expansion

We have seen in the Theorem 2.1 that the unique solution $y(x)$ of problem (2.8) may be obtained from the limit $y(x) = \lim_{n\to\infty} y_n(x)$ uniformly in $[0,X]$, where $y_n(x)$ is the recurrence relation defined in (2.11). In other words, $y(x)$ admits the series expansion

$$y(x) = \phi(x) + \sum_{k=0}^{\infty}[y_{k+1}(x) - y_k(x)] = \phi(x) + \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\sum_{k=0}^{\infty}[\tilde{u}_{k+1}(x) - \tilde{u}_k(x)], \tag{3.32}$$

with

$$\tilde{u}_n(x) := \mathrm{H}_\alpha^{-1}\left(\Lambda^{\frac{2}{\alpha-2}}x\right)[y_n(x) - \phi(x)], \qquad n = 0,1,2,\ldots. \tag{3.33}$$

and $\mathrm{H}_\alpha(x)$ defined in (1.6). Then, from (2.13), we may write (3.32) in the form

$$\begin{aligned} y(x) &= \phi(x) + \sum_{k=0}^{n-1}[y_{k+1}(x) - y_k(x)] + \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)R_n(x) \\ &= \phi(x) + \mathrm{H}_\alpha\left(\Lambda^{\frac{2}{\alpha-2}}x\right)\left[\sum_{k=0}^{n-1}[\tilde{u}_{k+1}(x) - \tilde{u}_k(x)] + R_n(x)\right], \end{aligned} \tag{3.34}$$

where $R_n(x)$ is defined in (2.13).

Theorem 3.2. *Under the conditions of Theorem 2.1, the expansion* (3.34) *is an asymptotic expansion for large* Λ *of the unique solution of* (2.8), *uniformly for* $x \in [0,X]$. *More precisely, for* $n = 1,2,3,\ldots$,

$$\tilde{u}_n(x) - \tilde{u}_{n-1}(x) = \mathcal{O}(\tilde{\phi}\Lambda^{-n}) \quad \textit{and} \quad R_n(x) = \mathcal{O}(\tilde{\phi}\Lambda^{-n-1}) \tag{3.35}$$

as $\Lambda \to \infty$ *uniformly for* $x \in [0,X]$.

Proof. We prove only the case $\alpha \geq 3$, the proof for $\alpha \in (2,3)$ is similar. From definition (2.21) we have

$$\tilde{u}_n(x) = [\mathbf{T}\tilde{u}_{n-1}](x) = \frac{1}{\Lambda}\int_0^x K_\Lambda(x,t)g(t)[\tilde{u}_{n-1}(t) + \tilde{\phi}(t)]dt \tag{3.36}$$

and

$$\tilde{u}_{n+1}(x) = [\mathbf{T}\tilde{u}_n](x) = \frac{1}{\Lambda}\int_0^x K_\Lambda(x,t)g(t)[\tilde{u}_n(t) + \tilde{\phi}(t)]dt, \tag{3.37}$$

with $K_\Lambda(x,t)$ defined in (2.22). Subtracting (3.36) and (3.37) and using the bound (2.23) we find that

$$\|\tilde{u}_{n+1} - \tilde{u}_n\|_\infty \le \frac{NX}{|\Lambda|} \|g\|_\infty \|\tilde{u}_n - \tilde{u}_{n-1}\|_\infty . \tag{3.38}$$

We have $\tilde{u}_0(x) = 0$ and $\tilde{u}_1(x) = [\mathbf{T}\tilde{u}_0](x) = \mathcal{O}(\tilde{\phi}\Lambda^{-1})$ uniformly for $x \in [0, X]$. Using this and (3.38), the first thesis in (3.35) follows by induction over n.

Observe that $\tilde{u} = \lim_{n\to\infty} \tilde{u}_n = \sum_{k=0}^{\infty}[\tilde{u}_{k+1} - \tilde{u}_k] = \sum_{k=0}^{\infty} \mathcal{O}(\tilde{\phi}\Lambda^{-k-1}) = \mathcal{O}(\tilde{\phi}\Lambda^{-1})$. This and inequality (2.31) prove the second thesis in (3.35). □

Observe that the expansion (3.34) is not of Poincaré-type (in terms of pure negative powers of Λ), however, this expansion is convergent.

Remark 3.1. We have considered in (2.8) a linear differential equation. It is straightforward to generalize the method of Section 2 to non-linear problems of the form

$$\begin{cases} x^{4-\alpha}y'' + 2x^{3-\alpha}y' - \Lambda^2 y = f(x,y) \quad \text{in } [0,X], \\ y(0) = y_0, \qquad y_0, \Lambda \in \mathbb{C}, \quad y_0 = \mathcal{O}(1) \text{ as } \Lambda \to \infty, \end{cases} \tag{3.39}$$

where the function $f : [0, X] \times \mathbb{C} \to \mathbb{C}$ is continuous in its two variables and satisfies a Lipschitz condition in its second variable:

$$|f(x,y) - f(x,z)| \le M|y - z| \qquad \alpha \ge 3, \tag{3.40}$$

$$|x^{\alpha-3}[f(x,y) - f(x,z)]| \le M|y - z| \qquad \alpha \in (2,3), \tag{3.41}$$

$\forall y, z \in \mathbb{C}$ and $x \in [0, X]$, with M a positive constant independent of x, y, z. From here, it is straightforward to derive, for problem (3.39), the same conclusions of Theorems 1 and 2, but replacing $g(t)y_n(t)$ by $f(t, y_n(t))$ in the right hand side of (2.11); and replacing A, by M in (2.14).

Example 3.1. The radial part of the Schrödinger equation with supersingular plus Coulomb potential can be reduced to the following double confluent Heun equation [1]

$$t^2\ddot{y}(t) + \left(-\frac{C}{t^2} - l(l+1) + Zt + Et^2\right) y(t) = 0, \tag{3.42}$$

where l is the angular momentum, E is related to the energy of the particle, and C and Z represent the intensities of the supersingular and the Coulomb parts respectively. Without loss of generality, let us consider the $l = 0$ ground state. When we specify the behavior of the wave function $y(t)$ at the infinity and perform the change of variable $t = 1/x$, we obtain the initial value problem

$$\begin{cases} y'' + \dfrac{2}{x}y' + (E + Zx - C)\,y = 0 \quad \text{in} \quad [0, X], \\ y(0) = 1. \end{cases} \tag{3.43}$$

This is problem (2.8) with $\alpha = 4$, $\Lambda^2 = C$ and $g(x) = -E - Zx$. Table 1 contains several numerical experiments of the approximation supplied by (2.11) for several degrees of approximation n, and different values of the parameters.

$x = 1.5, Z = 1, E = -0.09$				
C	$R_0(x)$	$R_1(x)$	$R_2(x)$	$R_3(x)$
5	0.2	0.01	0.0003	7.e-6
10	0.1	0.006	1.6e-4	6.e-6
50	0.07	0.002	2.3e-5	8.e-6

$x = 2, Z = 1, E = -0.09$				
C	$R_0(x)$	$R_1(x)$	$R_2(x)$	$R_3(x)$
5	0.4	0.05	0.003	1.3e-4
10	0.3	0.03	0.002	6.e-5
50	0.1	0.007	2.3e-4	1.3e-5

$x = 1.5, Z = 2, E = -0.09$				
C	$R_0(x)$	$R_1(x)$	$R_2(x)$	$R_3(x)$
5	0.4	0.05	0.003	1.2e-4
10	0.3	0.03	0.002	6.7e-5
50	0.1	0.01	3.e-4	1.5e-5

$x = 2, Z = 1, E = -0.14$				
C	$R_0(x)$	$R_1(x)$	$R_2(x)$	$R_3(x)$
5	0.36	0.04	0.003	1.e-4
10	0.26	0.025	0.001	5.e-5
50	0.1	0.006	2.e-4	1.3e-5

TABLE 1. Numerical experiments about the relative errors $R_n(x)$ in the approximation of the solution of problem (3.43) with $l = 0$, different values of Z and E, and several values of C given by (2.11) for several degrees of approximation n.

4. Final remarks

In this paper we have completed the research line initiated in [5] and continued in [2], [3] and [4], analyzing the differential equation in (2.8) for any $\alpha \in \mathbb{R}$. For different regions of α, we have constructed a sequence of functions $y_n(x)$ that converges to the unique solution of an appropriate initial value problem. As it is shown in [5], [2], [3] and [4], for any real α, that sequence consists of iterated integrals of Bessel functions similar to (2.11); that reduce to exponential functions in the case $\alpha = 0$ [5] or Airy functions in the case $\alpha = -1$ [3]. Depending on the value of α, a different number of conditions must be specified in order to have a well-posed initial value problem, as it is summarized in Table 2.

	$\alpha < 1$	$1 \leq \alpha < 2$	$\alpha = 2$	$\alpha > 2$
Initial data	$y(0), y'(0)$	$y'(0)$	$y(0)$	$y(\infty)$

TABLE 2. The initial conditions that must be added to the differential equation in (2.8) to set a well-posed problem depend on the value of α.

For any real α, the sequence $y_n(x)$ is not only convergent, but it is also an asymptotic expansion for large Λ of the unique solution of the initial value problem, different from the one supplied by Olver's method.

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