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A note on some positive linear operators associated with the Hermite polynomials

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ABSTRACT. In this paper we give direct approximation theorems and the Voronovskaya type asymptotic formula for certain linear operators associated with the Hermite polynomials. These operators extend the well-known Szász-Mirakjan operators.

1. INTRODUCTION

We introduce the class of operators G_n^α, $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $\alpha \geq 0$, given by the formula

$$G_n^\alpha(f;x) = e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+ := [0,\infty), \tag{1.1}$$

where H_k is the two variable Hermite polynomial (see [2]) defined by

$$H_k(n,\alpha) = k! \sum_{s=0}^{\left[\frac{k}{2}\right]} \frac{n^{k-2s}\alpha^s}{(k-2s)!s!}.$$

The Hermite polynomials and their properties were investigated in many papers, for example in [3, 15, 16]. Integrals of these polynomials are ubiquitous in problems concerning classical and quantum optics and in quantum mechanics (see [1, 22, 23]).

The operators (1.1) are linear and positive. Basic facts on positive linear operators, their generalizations and applications, can be found in [17, 18].

In this paper we shall study approximation properties of G_n^α for functions $f \in C_B\left(\mathbb{R}_0^+\right)$, where $C_B(\mathbb{R}_0^+)$ is the space of all real-valued functions f continuous and bounded on $\mathbb{R}_0^+$. The norm on $C_B(\mathbb{R}_0^+)$ is defined by

$$||f|| = \sup_{x\in\mathbb{R}_0^+} |f(x)|.$$

For $\alpha = 0$ we have

$$G_n^0(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{x^k}{k!} n^k f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0^+,$$

so G_n^0, $n \in \mathbb{N}$ are the classical Szász-Mirakjan operators. Approximation properties of Szász-Mirakjan operators in many different spaces were studied, for example, in [12, 13, 33, 35]. The above operators were modified by several authors (e.g. [4, 6, 7, 8, 9, 10, 11, 14, 19, 20, 24, 25, 27, 28, 29, 30, 31, 36]) which showed that new operators have similar or better approximation properties than G_n^0.

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Observe that $H_k(2n,-1) = \widetilde{H}_k(n)$, where $\widetilde{H}_k$ is the kth classical Hermite polynomial defined by

$$\widetilde{H}_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}, \quad k \in \mathbb{N}_0. \tag{1.2}$$

So, we can also consider operators of the form

$$\widetilde{G}_n^{-1}(f;x) = e^{-(2nx-x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \widetilde{H}_k(n) f\left(\frac{k}{2n}\right), \quad x \in \mathbb{R}_0^+.$$

The operators $\widetilde{G}_n^{-1}$, $n \in \mathbb{N}$ are linear, but not positive. In this case ($\alpha = -1$), approximation properties of $\widetilde{G}_n^{-1}$ should be considered which, however, will be done in a further note. It is worth mentioning that some approximation theorems for Poisson integrals associated with the classical Hermite polynomials (1.2) were presented in [34].

2. Auxiliary results

In this section we shall give some properties of the operators G_n^α, $\alpha \geq 0$, which we shall apply to the proofs of the main theorems.

In the sequel the following functions will be meaningful:

$$e_p(t) = t^p, \quad \phi_{x,p}(t) = (t-x)^p, \quad p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad x, t \in \mathbb{R}_0^+.$$

Using the generating function of two variable Hermite polynomials (see [2])

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(n,\alpha) = e^{nt+\alpha t^2},$$

we have

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_{k+r}(n,\alpha) = \frac{d^r}{dt^r} e^{nt+\alpha t^2}, \qquad r \in \mathbb{N},$$

and by simple computation we obtain the following lemma.

Lemma 2.1. *For any $p \in \mathbb{N}$ we have*

$$G_n^\alpha(e_p;x) = x^p + O\left(n^{-1}\right)$$

for each $x \in \mathbb{R}_0^+$. In particular

$$G_n^\alpha(1;x) = 1, \quad G_n^\alpha(e_1;x) = x + \frac{2\alpha x^2}{n},$$

$$G_n^\alpha(e_2;x) = x^2 + \frac{4\alpha x^3 + x}{n} + \frac{4\alpha^2 x^4 + 4\alpha x^2}{n^2},$$

$$G_n^\alpha(\phi_{x,1};x) = \frac{2\alpha x^2}{n}, \quad G_n^\alpha(\phi_{x,2};x) = \frac{x}{n} + \frac{4\alpha x^2(\alpha x^2+1)}{n^2},$$

$$G_n^\alpha(\phi_{x,4};x) = O\left(n^{-2}\right)$$

for each $x \in \mathbb{R}_0^+$.

Using the definition (1.1) we can state the next result.

Theorem 2.1. *The operator G_n^α maps $C_B(\mathbb{R}_0^+)$ into $C_B(\mathbb{R}_0^+)$ and*

$$||G_n^\alpha(f)|| \leq ||f||$$

for $f \in C_B(\mathbb{R}_0^+)$.

3. Rate of convergence

In this part we shall state some estimates of the rate of convergence of the operators G_n^α, $\alpha \geq 0$ for functions $f \in C_B\left(\mathbb{R}_0^+\right)$. We shall use the modulus of continuity

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in\mathbb{R}_0^+ \\ |y-x|\leq\delta}} |f(y)-f(x)|, \qquad \delta>0$$

and the modulus of smoothness

$$\omega_2(f,\delta) = \sup_{\substack{x\in\mathbb{R}_0^+ \\ 0<h\leq\delta}} |f(x+2h)-2f(x+h)+f(x)|, \qquad \delta>0.$$

As is known (see, for example, [18, 21]), the modulus of smoothness $\omega_2(f,\delta)$ of $f \in C_B(\mathbb{R}_0^+)$ is equivalent to the Peetre $\mathcal{K}$-functional defined by

$$\mathcal{K}_2(f,\delta) = \inf_{g\in C_B^2(\mathbb{R}_0^+)} \{\|f-g\| + \delta\|g''\|\}, \quad \delta>0, \tag{3.3}$$

where $C_B^2(\mathbb{R}_0^+) = \{g \in C_B(\mathbb{R}_0^+) : g' \in AC_{loc}(\mathbb{R}_0^+),\, g'' \in C_B(\mathbb{R}_0^+)\}$. This means that there exist positive constants M and δ_0, independent of f, such that

$$M^{-1}\omega_2(f,\delta) \leq \mathcal{K}_2(f,\delta^2) \leq M\omega_2(f,\delta), \quad 0<\delta\leq\delta_0. \tag{3.4}$$

Of course, in this paper we only need the second inequality.

First, we present the quantitative estimate for G_n^α in terms of the classical first order modulus using a result of Shisha and Mond ([32], see also [5]).

Theorem 3.2. *For every $f \in C_B(\mathbb{R}_0^+)$, $x \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} |G_n^\alpha(f;x) - f(x)| &\leq 2\omega_1\left(f, \sqrt{G_n^\alpha(\phi_{x,2};x)}\right) \\ &= 2\omega_1\left(f, \sqrt{\frac{x}{n} + \frac{4\alpha x^2(\alpha x^2+1)}{n^2}}\right). \end{aligned}$$

Now, we state the estimate in terms of the first and second order moduli via $\mathcal{K}$- functionals.

Theorem 3.3. *If $f \in C_B(\mathbb{R}_0^+)$, then for every $x \in \mathbb{R}_0^+$ we have*

$$\begin{aligned} &|G_n^\alpha(f;x) - f(x)| \\ &\leq M\omega_2\left(f, \frac{1}{2}\sqrt{\frac{x}{n} + \frac{4\alpha x^2(\alpha x^2+1)}{n^2} + \left(\frac{2\alpha x^2}{n}\right)^2}\right) + \omega_1\left(f, \frac{2\alpha x^2}{n}\right), \end{aligned}$$

where M is some positive constant.

Proof. We define T_n^α as follows

$$T_n^\alpha(f;x) = G_n^\alpha(f;x) - f\left(x + \frac{2\alpha x^2}{n}\right) + f(x).$$

Let $x \in \mathbb{R}_0^+$ and $g \in C_B^2(\mathbb{R}_0^+)$. We can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du, \quad t \in \mathbb{R}_0^+.$$

From this and by $T_n^{\alpha}(\phi_{x,1};x)=0$ we obtain

$$\begin{aligned}
\left|T_n^{\alpha}(g-g(x);x)\right| &= \left|T_n^{\alpha}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\
&= \left|G_n^{\alpha}\left(\int_x^t (t-u)g''(u)\,du;x\right)-\int_x^{x+\frac{2\alpha x^2}{n}}\left(x+\frac{2\alpha x^2}{n}-u\right)g''(u)\,du\right|.
\end{aligned}$$

Observe that

$$\begin{aligned}
&G_n^{\alpha}\left(\int_x^t (t-u)g''(u)\,du;x\right) \\
&\quad = e^{-(nx+\alpha x^2)}\sum_{k=0}^{\infty}\frac{x^k}{k!}H_k(n,\alpha)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)g''(u)\,du \\
&\quad \leq \frac{\|g''\|}{2}G_n^{\alpha}(\phi_{x,2};x)
\end{aligned}$$

and

$$\left|\int_x^{x+\frac{2\alpha x^2}{n}}\left(x+\frac{2\alpha x^2}{n}-u\right)g''(u)\,du\right| \leq \frac{\|g''\|}{2}\left(\frac{2\alpha x^2}{n}\right)^2.$$

Hence

$$\left|T_n^{\alpha}(g-g(x);x)\right| \leq \frac{\|g''\|}{2}\left(\frac{x}{n}+\frac{4\alpha x^2(\alpha x^2+1)}{n^2}+\left(\frac{2\alpha x^2}{n}\right)^2\right)$$

for $g\in C_B^2(\mathbb{R}_0^+)$.

Let $f\in C_B(\mathbb{R}_0^+)$. From the above we have

$$\begin{aligned}
&\left|G_n^{\alpha}(f-f(x);x)\right| \\
&\quad \leq \left|T_n^{\alpha}(f-g;x)-(f-g)(x)\right|+\left|T_n^{\alpha}(g;x)-g(x)\right|+\left|f\left(x+\frac{2\alpha x^2}{n}\right)-f(x)\right| \\
&\quad \leq 2\|f-g\|+\frac{\|g''\|}{2}\left(\frac{x}{n}+\frac{4\alpha x^2(\alpha x^2+1)}{n^2}+\left(\frac{2\alpha x^2}{n}\right)^2\right)+\omega_1\left(f,\frac{2\alpha x^2}{n}\right) \\
&\quad \leq 2\left\{\|f-g\|+\frac{1}{4}\left(\frac{x}{n}+\frac{4\alpha x^2(\alpha x^2+1)}{n^2}+\left(\frac{2\alpha x^2}{n}\right)^2\right)\|g''\|\right\}+\omega_1\left(f,\frac{2\alpha x^2}{n}\right).
\end{aligned}$$

Using (3.3) and (3.4) we obtain

$$\begin{aligned}
&\left|G_n^{\alpha}(f-f(x);x)\right| \\
&\quad \leq 2\mathcal{K}\left(f,\frac{1}{4}\left(\frac{x}{n}+\frac{4\alpha x^2(\alpha x^2+1)}{n^2}+\left(\frac{2\alpha x^2}{n}\right)^2\right)\right)+\omega_1\left(f,\frac{2\alpha x^2}{n}\right) \\
&\quad \leq M\omega_2\left(f,\frac{1}{2}\sqrt{\frac{x}{n}+\frac{4\alpha x^2(\alpha x^2+1)}{n^2}+\left(\frac{2\alpha x^2}{n}\right)^2}\right)+\omega_1\left(f,\frac{2\alpha x^2}{n}\right)
\end{aligned}$$

for some constant $M>0$. $\square$

Remark 3.1. *Observe that from Păltănea's theorem in* [26] *we can state the following result:*

$$\begin{aligned} & |G_n^\alpha(f;x)-f(x)| \\ \leq & \left(1+\frac{1}{2h^2}G_n^\alpha(\phi_{x,2};x)\right)\omega_2(f,h)+\frac{1}{h}\left|G_n^\alpha(\phi_{x,1};x)\right|\omega_1(f,h), \end{aligned}$$

where $h>0$, $f\in C_B(\mathbb{R}_0^+)$, $x\in\mathbb{R}_0^+$. Setting $h=\sqrt{\frac{2\alpha x^2}{n}}$ we get

$$\begin{aligned} & |G_n^\alpha(f;x)-f(x)| \\ \leq & \left(1+\frac{1}{4\alpha x}+\frac{\alpha x^2+1}{n}\right)\omega_2\left(f,\sqrt{\frac{2\alpha x^2}{n}}\right)+\sqrt{\frac{2\alpha x^2}{n}}\omega_1\left(f,\sqrt{\frac{2\alpha x^2}{n}}\right) \end{aligned}$$

for $\alpha,x>0$. The above estimate gives a similar result like in Theorem 3.3, but it leads to the estimate with precise constants.

The inequality obtained in Theorem 3.2 (also in Theorem 3.3 and Remark 3.1) implies the following corollary.

Corollary 3.1. *If f is an uniformly continuous bounded function on $\mathbb{R}_0^+$, then*

$$\lim_{n\to\infty}G_n^\alpha(f;x)=f(x)$$

uniformly on every interval $[a,b]\subset\mathbb{R}_0^+$, $a<b$.

4. The Voronovskaya type theorem

In this section we shall establish the Voronovskaya type asymptotic formula for the operators G_n^α.

We first need the following lemma, which immediately follows from Lemma 2.1.

Lemma 4.2. *For every fixed $x\in\mathbb{R}_0^+$, it holds*

$$\lim_{n\to\infty}nG_n^\alpha(\phi_{x,1};x)=2\alpha x^2,\qquad \lim_{n\to\infty}nG_n^\alpha(\phi_{x,2};x)=x,$$

$$\lim_{n\to\infty}n^2G_n^\alpha(\phi_{x,4};x)=M_{x,\alpha}, \tag{4.5}$$

where $M_{x,\alpha}$ is some positive constant.

Theorem 4.4. *Let $x\in\mathbb{R}_0^+$ be a fixed point and let f be an uniformly continuous bounded function on $\mathbb{R}_0^+$. If f is of the class $C^1(\mathbb{R}_0^+)$ in a certain neighbourhood of a point x and $f''(x)$ exists, then*

$$\lim_{n\to\infty}n\left[G_n^\alpha(f;x)-f(x)\right]=2\alpha x^2f'(x)+\frac{x}{2}f''(x).$$

Proof. Let $x\in\mathbb{R}_0^+$. Define

$$\psi_x(t)=\begin{cases}\dfrac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x)}{(t-x)^2}, & t\neq x,\\ 0, & t=x.\end{cases}$$

Then from Taylor's formula we have $\lim_{t\to x}\psi_x(t)=0$ and the function ψ_x is uniformly continuous and bounded on $\mathbb{R}_0^+$.

Remark that

$$\begin{aligned} G_n^\alpha(f;x)-f(x) &= f'(x)G_n^\alpha(\phi_{x,1};x) \\ &\quad +\frac{1}{2}f''(x)G_n^\alpha(\phi_{x,2};x)+G_n^\alpha(\psi_x\,\phi_{x,2};x). \end{aligned} \tag{4.6}$$

Using the Cauchy-Schwarz inequality we obtain

$$n\left|G_n^{\alpha}\left(\psi_x\,\phi_{x,2};x\right)\right| \leq \left|G_n^{\alpha}\left(\psi_x^2;x\right)\right|^{1/2}\left|n^2G_n^{\alpha}\left(\phi_{x,4};x\right)\right|^{1/2}.$$

Let $\eta_x(t)=\psi_x^2(t)$. We have $\eta_x(x)=0$ and η_x is bounded and uniformly continuous on $\mathbb{R}_0^+$. Then it follows from Corollary 3.1 that

$$\lim_{n\to\infty} G_n^{\alpha}\left(\psi_x^2;x\right)=\lim_{n\to\infty} G_n^{\alpha}\left(\eta_x;x\right)=\eta_x(x)=0.$$

Using (4.5) we obtain

$$\lim_{n\to\infty} nG_n^{\alpha}\left(\psi_x\,\phi_{x,2};x\right)=0. \tag{4.7}$$

From (4.6), (4.7) and Lemma 4.2 we get the assertion. □

Corollary 4.2. *Let $x\in\mathbb{R}_0^+$. If f satisfies the assumption of Theorem 4.4, then*

$$\left|G_n^{\alpha}(f;x)-f(x)\right|=O\left(n^{-1}\right).$$

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