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# A characterization of cone-convex vector-valued functions

DAISHI KUROIWA, NICOLAE POPOVICI and MATTEO ROCCA

ABSTRACT. An interesting result in convex analysis, established by J.-P. Crouzeix in 1977, states that a real-valued function defined on a linear space is convex if and only if each function obtained from it by adding a linear functional is quasiconvex. The aim of this paper is to extend this result for vector-valued functions taking values in a partially ordered linear space.

## 1. INTRODUCTION

The role of both convexity and quasiconvexity in optimization theory is nowadays well-recognized. Various generalizations of the classical notions of convexity and quasiconvexity of real-valued functions have been proposed in the literature for vector-valued functions taking values in a real linear space, partially ordered by a convex cone.

In this paper we will investigate only two of them, known as the cone-convexity and the cone-quasiconvexity. They preserve the characteristic properties of real-valued convex and quasiconvex functions concerning the convexity of the epigraph and the convexity of the lower level sets, respectively (cf. Luc [7]).

An interesting topic in vector optimization is to characterize the cone-(quasi)convexity of the vector-valued functions in terms of usual (quasi)convexity of certain real-valued functions, by means of appropriate scalarization functions, as for instance the Gerstewitz-type nonlinear scalarization functions or the extreme directions of the ordering cone's polar (see, e.g., La Torre, Popovici and Rocca [6]).

The principal aim of this paper is to present a new characterization of cone-convex vector-valued functions in terms of cone-quasiconvexity. Our main result (Theorem 3.1) represents an extension of a classical result by Crouzeix [4], stating that a real-valued convex function defined on a linear space is convex if and only if each function obtained from it by adding a linear functional is quasiconvex.

We also prove that, under certain additional assumptions, the cone-convex vector-valued functions can be characterized in terms of scalar quasiconvexity of certain real-valued functions defined by means of the extreme directions of the ordering cone's polar (Corollary 3.3).

In Section 2 we introduce some preliminary notations and we present the definitions of the main generalized convexity concepts used in the sequel. Section 3 is devoted to the characterization of cone-convexity in terms of cone-quasiconvexity and contains our main results and their corollaries.

## 2. PRELIMINARIES: NOTATIONS AND BASIC DEFINITIONS

For any real linear space $V$ (i.e., a linear space over the field $\mathbb{R}$ of reals) we adopt the following notations. The origin (i.e., the zero vector) of $V$ will be denoted by $0_V$. For

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Corresponding author: Nicolae Popovici; popovici@math.ubbcluj.ro

any nonempty sets $A, B \subseteq V$ and $\Lambda \subseteq \mathbb{R}$, and for any $t \in \mathbb{R}$ and $v \in V$, we denote $A + B := \{a + b \mid (a, b) \in A \times B\}$, $v + B := \{v\} + B$, $\Lambda \cdot A := \{\lambda a \mid (\lambda, a) \in \Lambda \times A\}$, $tA := \{t\} \cdot A$, and $\Lambda \cdot v := \Lambda \cdot \{v\}$.

Throughout this paper $X$ and $Y$ will be real linear spaces. As usual, the notation $L(X, Y)$ stands for the space of linear operators between $X$ and $Y$. Following Zălinescu [11], we will denote by $X' := L(X, \mathbb{R})$ and $Y' := L(Y, \mathbb{R})$ the algebraic duals of $X$ and $Y$, respectively. Whenever $Y$ will be a real topological linear space, we will also consider its topological dual $Y^* := \{y^* \in Y' \mid y^* \text{ is continuous}\}$.

In the sequel $D$ will denote a nonempty convex subset of the real linear space $X$, i.e., $tD + (1-t)D \subseteq D$ for all $t \in [0, 1]$. Recall that a function $f : D \to \mathbb{R}$ is called convex if

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) \tag{2.1}$$

for all $x_1, x_2 \in D$ and $t \in [0, 1]$, while $f$ is called quasiconvex if

$$f(tx_1 + (1-t)x_2) \le \max\{f(x_1), f(x_2)\} \tag{2.2}$$

for all $x_1, x_2 \in D$ and $t \in [0, 1]$. Clearly, any convex function is quasiconvex.

The sum of two functions, $f_1 : D_1 \to \mathbb{R}$ and $f_2 : D_2 \to \mathbb{R}$, defined on any convex subsets $D_1$ and $D_2$ of $X$ having at least one point in common, is understood to be the function $f_1 + f_2 : D_1 \cap D_2 \to \mathbb{R}$, defined pointwise as

$$(f_1 + f_2)(x) := f_1(x) + f_2(x), \ \forall\, x \in D_1 \cap D_2.$$

In the next section we will often operate with functions defined on $D_1 = D$ and $D_2 = X$. Notice that the sum of any two convex functions is convex, but the sum of two quasiconvex functions is not necessarily quasiconvex.

**Example 2.1.** Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be defined for all $x \in \mathbb{R}$ by

$$f_1(x) = x^3 \quad \text{and} \quad f_2(x) = -x.$$

Obviously, both functions are quasiconvex, but their sum is not quasiconvex.

The notions of convexity and quasiconvexity of real-valued functions can be naturally extended to vector-valued functions taking values in a partially ordered linear space. Let $C \subseteq Y$ be a convex cone of the real linear space $Y$, i.e.,

$$0_Y \in C = \mathbb{R}_+ \cdot C = C + C.$$

It is known that $C$ induces a linear partial order on $Y$, defined for any $y_1, y_2 \in Y$ by

$$y_1 \le_C y_2 \quad :\Leftrightarrow \quad y_2 \in y_1 + C.$$

A vector-valued function $f : D \to Y$ is said to be $C$-convex if it satisfies the following vectorial version of (2.1) for all $x_1, x_2 \in D$ and $t \in [0, 1]$:

$$f(tx_1 + (1-t)x_2) \le_C tf(x_1) + (1-t)f(x_2).$$

Obviously every linear operator $A \in L(X, Y)$ is $C$-convex. Notice also that the sum of any two $C$-convex functions is $C$-convex.

In contrast to (2.1), in order to adapt (2.2) to vector functions, we cannot simply replace "$\le$" by "$\le_C$" since $\max\{f(x_1), f(x_2)\}$ does not make sense in general when $(Y, \le_C)$ is not a lattice. Following Luc [7], we say that $f : D \to Y$ is $C$-quasiconvex if for all $x_1, x_2 \in D$ and $t \in [0, 1]$ we have

$$f(tx_1 + (1-t)x_2) \le_C y$$

for every upper bound $y$ of $\{f(x_1), f(x_2)\}$. In other words, $f$ is $C$-quasiconvex if and only if for all $x_1, x_2 \in D$ and $t \in [0, 1]$ we have

$$(f(x_1) + C) \cap (f(x_2) + C) \subseteq f(tx_1 + (1-t)x_2) + C.$$

It is easily seen that every $C$-convex function is $C$-quasiconvex.

**Remark 2.1.** For any $f : D \to Y$ the following characterizations hold true:

$a)$ $f$ is $C$-convex if and only if the epigraph of $f$, i.e., set

$$\{(x, y) \in X \times Y \mid f(x) \leq_C y\},$$

is convex. In particular, when $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$, then $f = (f_1, \dots, f_n) : D \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-convex if and only if its scalar components, $f_1 : D \to \mathbb{R}, \dots, f_n : D \to \mathbb{R}$ are convex.

$b)$ $f$ is $C$-quasiconvex if and only if the level set

$$\{x \in D \mid f(x) \leq_C y\}$$

is convex for every $y \in Y$. In particular, when $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$, then a vector-valued function $f = (f_1, \dots, f_n) : D \to \mathbb{R}^n$ is $\mathbb{R}^n_+$-quasiconvex if and only if its scalar components, $f_1 : D \to \mathbb{R}, \dots, f_n : D \to \mathbb{R}$ are quasiconvex.

## 3. MAIN RESULTS: CHARACTERIZATIONS OF $C$-CONVEX FUNCTIONS

The next theorem represents a characterization of $C$-convex vector-valued functions in terms of $C$-quasiconvexity.

**Theorem 3.1.** *For any vector-valued function, $f : D \to Y$, the following assertions are equivalent:*

$1^\circ$ *$f$ is $C$-convex.*

$2^\circ$ *$f + A$ is $C$-quasiconvex for every $A \in L(X, Y)$.*

*Proof.* The implication $1^\circ \Rightarrow 2^\circ$ is obvious, since any operator $A \in L(X, Y)$ is $C$-convex and therefore, under the hypothesis $1^\circ$, function $f + A$ is $C$-convex hence $C$-quasiconvex.

In order to prove the implication $2^\circ \Rightarrow 1^\circ$, assume that $2^\circ$ holds and consider $x_1, x_2 \in D$ and $t \in [0, 1]$. We will prove that

$$tf(x_1) + (1 - t)f(x_2) \in f(tx_1 + (1 - t)x_2) + C. \tag{3.3}$$

Let $x_0 = x_1 - x_2$. Since (3.3) obviously holds when $x_1 = x_2$, we can assume in what follows that $x_0 \neq 0_X$. Then, $\{x_0\}$ can be extended to a basis of $X$ by Zorn's lemma. Consequently, there is a linear functional $x^* \in X'$ (determined by its action on the basis) such that $x^*(x_0) = 1$. Define $A : X \to Y$ for all $x \in X$ by

$$A(x) = x^*(x)(f(x_2) - f(x_1)).$$

It is easily seen that $A \in L(X, Y)$ and $A(x_0) = A(x_1 - x_2) = f(x_2) - f(x_1)$, hence

$$(f + A)(x_1) = (f + A)(x_2). \tag{3.4}$$

By assumption $2^\circ$, the function $f + A$ is $C$-quasiconvex, hence

$$\big((f + A)(x_1) + C\big) \cap \big((f + A)(x_2) + C\big) \subseteq (f + A)(tx_1 + (1 - t)x_2) + C,$$

which, in view of (3.4), actually means that

$$(f + A)(x_1) + C - A(tx_1 + (1 - t)x_2) \subseteq f(tx_1 + (1 - t)x_2) + C.$$

Thus, in order to prove (3.3) it suffices to show that

$$tf(x_1) + (1 - t)f(x_2) \in (f + A)(x_1) + C - A(tx_1 + (1 - t)x_2).$$

Indeed, by linearity of $A$ this relation can be rewritten as

$$(1 - t)(f + A)(x_2) \in (1 - t)(f + A)(x_1) + C,$$

which is true due to (3.4) and the fact that $0_Y \in C$. $\square$

As a straightforward consequence of Theorem 3.1 we recover the following result, which is a counterpart of Proposition 9 in Crouzeix [4, Ch. 1].

**Corollary 3.1.** *For any $f : D \to \mathbb{R}$ the following assertions are equivalent:*
$1^\circ$ *The function $f$ is convex.*
$2^\circ$ *For each $x^* \in X'$, the function $f + x^*$ is quasiconvex.*

**Example 3.2.** Consider the functions $f_1$ and $f_2$ of Example 2.1. In this case, Corollary 3.1 neatly shows that function $f = f_1$ is not convex, since $x^* = f_2$ is a linear functional but $f + x^* = f_1 + f_2$ is not quasiconvex.

An interesting question in vector optimization, motivated by the practical importance of scalarization techniques, is to characterize the $C$-(quasi)convex vector-valued functions in terms of usual (quasi)convexity of certain real-valued functions (see, e.g., La Torre, Popovici and Rocca [6]). A possible approach to this aim is to use the functions belonging to the polar of the ordering cone $C$, defined as

$$C^+ := \{y^* \in Y^* \mid y^*(y) \geq 0,\ \forall\, y \in C\}.$$

Recall that a function $f : D \to Y$ is called $*$-quasiconvex w.r.t. $C$ in the sense of Jeyakumar, Oettli and Natividad [5] or scalarly-quasiconvex in the sense of Sach [9] if $y^* \circ f$ is quasiconvex for all $y^* \in C^+$.

**Lemma 3.1** (Luc [7], Jeyakumar, Oettli and Natividad [5])**.** *Assume that $Y$ is a real locally convex space and $C \subseteq Y$ is a closed convex cone. For any function $f : D \to Y$ the following hold true:*
*a) $f$ is $C$-convex if and only if $y^* \circ f$ is convex, for each $y^* \in C^+$.*
*b) $f$ is $C$-quasiconvex whenever it is $*$-quasiconvex w.r.t. $C$.*

**Corollary 3.2.** *Under the hypotheses of Lemma 3.1, the assertions below are equivalent for any function $f : D \to Y$.*
$1^\circ$ *$f$ is $C$-convex.*
$2^\circ$ *$f + A$ is $*$-quasiconvex w.r.t. $C$ for all $A \in L(X, Y)$.*

*Proof.* Assume that $1^\circ$ is true and consider an operator $A \in L(X, Y)$. Then function $f + A$ is $C$-convex as a sum of two $C$-convex functions. By Lemma 3.1.$a$) it follows that, for every $y^* \in C^+$ the function $y^* \circ (f + A)$ is convex, hence quasiconvex, which shows that $f + A$ is $*$-quasiconvex w.r.t. $C$. Thus $2^\circ$ holds true.

Conversely, assume that assertion $2^\circ$ is true. Then, according to Lemma 3.1.$b$), $f + A$ is $C$-quasiconvex for all $A \in L(X, Y)$. By Theorem 3.1 we infer that function $f$ is $C$-convex, i.e., $1^\circ$ holds true. □

**Remark 3.2.** Assume that the hypotheses on $Y$ and $C$ in Lemma 3.1 are fulfilled and let $f : D \to Y$. According to Theorem 3.1 and Corollary 3.2, the following conditions are equivalent:
$(C1)$ $f + A$ is $C$-quasiconvex for every $A \in L(X, Y)$.
$(C2)$ $f + A$ is $*$-quasiconvex w.r.t. $C$ for every $A \in L(X, Y)$.

However, if $f + A$ is $C$-quasiconvex for some $A \in L(X, Y)$, we cannot guarantee the $*$-quasiconvexity of $f + A$. More precisely, although $(C1) \Leftrightarrow (C2)$ holds, the following assertions are not equivalent in general for an a priori given $A \in L(X, Y)$:
$(C1')$ $f + A$ is $C$-quasiconvex.
$(C2')$ $f + A$ is $*$-quasiconvex w.r.t. $C$.

**Example 3.3.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$, and let $A = 0 \in L(\mathbb{R}, \mathbb{R}^2)$ be the null operator. Let $f = (f_1, f_2) : D = \mathbb{R} \to \mathbb{R}^2$ be the function defined by

$$f(x) = (x^3, -x),\ \forall\, x \in \mathbb{R}.$$

By using the definition of $C$-quasiconvexity (or the componentwise characterization of $C$-quasiconvexity, mentioned in Remark 2.1) it is easy to check that function $f$ (i.e., $f + A$) is $C$-quasiconvex. However, by choosing $y^* \in C^+$ defined as $y^*(y) = y_1 + y_2$ for all $y = (y_1, y_2) \in \mathbb{R}^2$, we get $y^* \circ (f + A) = f_1 + f_2$, which is not quasiconvex, as we have already seen in Example 2.1. Thus the function $f + A$ is not $*$-quasiconvex.

In what follows we denote by $\operatorname{extd} C^+$ the set of all extreme directions of $C^+$. Recall that $y^* \in \operatorname{extd} C^+$ if and only if $y^* \in C^+ \setminus \{0_{Y^*}\}$ and for all $y_1^*, y_2^* \in C^+$ such that $y^* = y_1^* + y_2^*$ we actually have $y_1^*, y_2^* \in \mathbb{R}_+ \cdot y^*$.

The following result gives a characterization of $C$-convex vector-valued functions, which is similar to Lemma 3.1.$a$), but involves only the extreme directions of $C^+$.

**Lemma 3.2** (Popovici [8, Th. 2.1]). *Assume that $Y$ is a real locally convex space and $C$ is a closed convex cone of $Y$ satisfying the property that $C^+$ is the weak*-closed convex hull of $\operatorname{extd} C^+$. For any function $f : D \to Y$ the following assertions are equivalent*:

$1^\circ$ *$f$ is $C$-convex.*

$2^\circ$ *$y^* \circ f$ is convex, for each $y^* \in \operatorname{extd} C^+$.*

**Remark 3.3.** Every closed convex cone $C$ with nonempty interior in a Banach space $Y$ satisfies the property that $C^+$ is the weak$^*$-closed convex hull of $\operatorname{extd} C^+$. However, there are cones with empty interior which satisfy this condition as well (see Example 3.5 below).

**Corollary 3.3.** *Let $f : D \to Y$. Under the hypotheses of Lemma* 3.2 *the following assertions are equivalent*:

$1^\circ$ *$f$ is $C$-convex.*

$2^\circ$ *$y^* \circ f + x^*$ is quasiconvex, for all $y^* \in C^+$ and $x^* \in X'$.*

$3^\circ$ *$y^* \circ f + x^*$ is quasiconvex, for all $y^* \in \operatorname{extd} C^+$ and $x^* \in X'$.*

*Proof.* The equivalence $1^\circ \Leftrightarrow 2^\circ$ follows by Lemma 3.1.$a$) and Corollary 3.1, while the equivalence $1^\circ \Leftrightarrow 3^\circ$ follows by Lemma 3.2 and Corollary 3.1 (applied for $y^* \circ f$). □

**Example 3.4.** Let $Y = \mathbb{R}^n$ be the real Euclidean space, partially ordered by a polyhedral cone $C$ with nonempty interior in $\mathbb{R}^n$. By identifying $Y^* = (\mathbb{R}^n)^*$ with $\mathbb{R}^n$, the polar cone $C^+ \subseteq \mathbb{R}^n$ is also polyhedral, being generated by a finite number of its extreme directions, $d_1, \ldots, d_m$. Corollary 3.3 actually shows that a vector function $f = (f_1, \ldots, f_n) : D \to \mathbb{R}^n$ is $C$-convex if and only if, for all $i \in \{1, \ldots, m\}$, the function $\langle d_i, f \rangle + x^*$ is quasiconvex for every $x^* \in X'$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

In particular, if $C = \mathbb{R}^n_+$ is the usual ordering cone, then $C^+ = \mathbb{R}^n_+$ and

$$\operatorname{extd} C^+ = ]0, \infty[ \cdot B,$$

where $B$ is the canonical basis of $\mathbb{R}^n$. Hence $f$ is $\mathbb{R}^n_+$-convex if and only if the scalar functions $\alpha_1 f_1 + x^*, \ldots, \alpha_n f_n + x^*$ are quasiconvex for all positive numbers $\alpha_1, \ldots, \alpha_n$ and every functional $x^* \in X'$, which indeed is equivalent to the fact that $f_1 + x^*, \ldots, f_n + x^*$ are quasiconvex for every $x^* \in X'$.

**Example 3.5.** Let $p \in ]1, \infty[$ be a real number and consider the sequence space $Y = l^p$, partially ordered by

$$C = l^p_+ = \{y = (y_i)_{i \in \mathbb{N}} \in l^p \mid \forall\, i \in \mathbb{N},\, y_i \geq 0\}.$$

Then, by identifying $Y^* = (l^p)^*$ with $l^q$ (where $1/p + 1/q = 1$), we have $C^+ = l^q_+$. Notice that $\operatorname{int} C = \emptyset$ while $C^+$ is the weak$^*$-closed convex hull of

$$\operatorname{extd} C^+ = \{y^* = (y_i^*)_{i \in \mathbb{N}} \in l^q \mid \exists\, i \in \mathbb{N} : y_i^* > 0, y_j^* = 0, \forall\, j \in \mathbb{N}, j \neq i\}.$$

In this case, Corollary 3.3 shows that a function $f = (f_i)_{i\in\mathbb{N}} : D \to l^p$ is $l^p_+$-convex if and only if, for all $i \in \mathbb{N}$, $y_i^* > 0$ and $x^* \in X'$, the function $y_i^* f_i + x^*$ is quasiconvex, which indeed is equivalent to the fact that $f_i + x^*$ is quasiconvex for all $i \in \mathbb{N}$ and $x^* \in X'$.

In contrast to $C$-convex functions, the $C$-quasiconvex ones cannot be easily handled by scalarization. In order to obtain a characterization of $C$-quasiconvexity similar to Lemma 3.2, some additional assumptions must be imposed on the ordered space $(Y, \leq_C)$.

**Lemma 3.3** (Benoist, Borwein and Popovici [3, Th. 3.1]). *Assume that $Y$ is a real Banach space and $C \subseteq Y$ is a closed convex cone such that $C - C = Y$ and $C^+$ is the weak*-closed convex hull of $\operatorname{extd} C^+$. For any function $f : D \to Y$ the following assertions are equivalent:*

$1^\circ$ *$f$ is $C$-quasiconvex.*

$2^\circ$ *$y^* \circ f$ is quasiconvex, for each $y^* \in \operatorname{extd} C^+$.*

As a consequence of Theorem 3.1 and Lemma 3.3 we can derive the following result.

**Corollary 3.4.** *Let $f : D \to Y$. Under the hypotheses of Lemma 3.3 the following assertions are equivalent:*

$1^\circ$ *$f$ is $C$-convex.*

$2^\circ$ *$y^* \circ (f + A)$ is quasiconvex, for all $y^* \in \operatorname{extd} C^+$ and $A \in L(X, Y)$.*

**Remark 3.4.** Obviously $y^* \circ (f + A) = y^* \circ f + y^* \circ A$, for all $y^* \in Y'$ and $A \in L(X, Y)$. By comparing assertion $3^\circ$ of Corollary 3.3 with assertion $2^\circ$ in Corollary 3.4, a natural question arises on whether any $x^* \in X'$ could be expressed as a composite function $y^* \circ A$ for some $A \in L(X, Y)$, when $y^* \in \operatorname{extd} C^+$ is a priori given. We end our paper by giving a positive answer to this question. To this aim, we will use linear processes.

Recall that a set-valued function $\mathcal{F} : X \to 2^Y$ is said to be a linear process in the sense of Aubin and Frankowska [2] if its graph,

$$\{(x, y) \in X \times Y \mid x \in X,\ y \in \mathcal{F}(x)\},$$

is a linear subspace of $X \times Y$. Notice that the linear processes are also known in the literature under different names, as for instance in the early paper by Arens [1] or in the recent one by Száz [10], where they are called linear relations, a set-valued function being identified with its graph, which in its turn can be seen as a binary relation.

**Lemma 3.4** (Száz [10, C. 8.3]). *Every linear process, $\mathcal{A} : X \to 2^Y$, taking nonempty values, admits a linear single-valued selection function, i.e., a linear operator $A \in L(X, Y)$ such that $A(x) \in \mathcal{A}(x)$ for all $x \in X$.*

**Theorem 3.2.** *For any linear functionals, $x^* \in X'$ and $y^* \in Y' \setminus \{0_{Y'}\}$, there exists a linear operator $A \in L(X, Y)$ such that $x^* = y^* \circ A$.*

*Proof.* Let $x^* \in X'$ and let $y^* \in Y'$ be such that $y^*(\tilde{y}) \neq 0$ for some $\tilde{y} \in Y$. Consider the set-valued map $\mathcal{A} : X \to 2^Y$, defined for all $x \in X$ by

$$\mathcal{A}(x) := \{y \in Y \mid y^*(y) = x^*(x)\}. \tag{3.5}$$

It is a simple exercise to check that for all $x \in X$ we have

$$\mathcal{A}(x) = \frac{x^*(x)}{y^*(\tilde{y})}\tilde{y} + \ker y^*.$$

By linearity of $x^*$ it follows that $\mathcal{A}(x_1) + \mathcal{A}(x_2) \subseteq \mathcal{A}(x_1 + x_2)$ and $t\mathcal{A}(x) \subseteq \mathcal{A}(tx)$ for all $x_1, x_2, x \in X$ and $t \in \mathbb{R}$. Thus the graph of $\mathcal{A}$ is a linear subspace of $X \times Y$, i.e., $\mathcal{A}$ is a linear process. Due to Lemma 3.4, we can choose a selection $A \in L(X, Y)$ of $\mathcal{A}$. By definition (3.5) of $\mathcal{A}$ we conclude that $y^*(A(x)) = x^*(x)$ for all $x \in X$, i.e., $x^* = y^* \circ A$. □

The following result is a straightforward consequence of Theorem 3.2.

**Corollary 3.5.** *Let $f : D \to Y$ be a function and let $y^* \in Y' \setminus \{0_{Y'}\}$. The following assertions are equivalent*:

$1^\circ$ $y^* \circ f + x^*$ *is quasiconvex, for all* $x^* \in X'$.

$2^\circ$ $y^* \circ (f + A)$ *is quasiconvex, for all* $A \in L(X, Y)$.

**Remark 3.5.** Since $\operatorname{extd} C^+ \subseteq C^+ \setminus \{0_{Y^*}\} \subseteq Y' \setminus \{0_{Y'}\}$, Corollary 3.5 allows us to recover the conclusion of Corollary 3.4 directly from Corollary 3.3, under the mild hypotheses of Lemma 3.2, by avoiding so the use of Lemma 3.3.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
SHIMANE UNIVERSITY
690-8504 MATSUE, JAPAN
*E-mail address*: kuroiwa@math.shimane-u.ac.jp

BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
KOGĂLNICEANU 1, 400084 CLUJ-NAPOCA, ROMANIA
*E-mail address*: popovici@math.ubbcluj.ro

DEPARTMENT OF ECONOMICS
UNIVERSITY OF INSUBRIA
MONTE GENEROSO 71, 21100 VARESE, ITALY
*E-mail address*: matteo.rocca@uninsubria.it