CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
32 (2016), No. 1, 87 - 95 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

On strongly convex functions

JUDIT MAKÓ and ATTILA HÁZY

ABSTRACT. The main results of this paper give a connection between strong Jensen convexity and strong convexity type inequalities. We are also looking for the optimal Takagi type function of strong convexity. Finally a connection will be proved between the Jensen error term and an useful error function.

1. INTRODUCTION

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively.

Let X be a normed space and $D \subseteq X$ be a nonempty convex subset of X. Denote by D^* the set $\{\|x-y\|, x, y \in D\}$. It can be seen that D^* is an interval. Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative error function. We say that a function $f : D \to \mathbb{R}$ is *strongly α-Jensen convex*, if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} - \alpha(\|x-y\|). \tag{1.1}$$

Observe that if $\alpha \equiv 0$, we can get the classical definition of Jensen-convexity. When $\alpha(u) = cu^2$, we can get a kind of notion of strong convexity introduced by Polyak in [16] and examined by Azócar, Giménez, Nikodem and Sánchez (in [1]), Merentes and Nikodem, (in [12]) and Nikodem and Páles [14]. If $\alpha(u) = \varepsilon u^p$, then f is called *strongly (ε, p)-Jensen convex function*. In Section 2, we are looking connection between strong α-convexity and strong convexity type inequalities. Then, we are looking for the optimal error function. In Section 3, we will establish the connections between strong α-convexity and strong α-Jensen convexity, moreover the connections between strong convexity and Hermite–Hadamard type inequalities will be shown. These results will be the generalization of previous results of [1] and [12]. We say that $f : D \to \mathbb{R}$ is *locally upper bounded*, if for all $x, y \in D$, there exists a $K_{x,y}$ such that $f \leq K_{x,y}$ on $[x, y]$, where $[x, y] = \{tx + (1-t)y \mid t \in [0, 1]\}$.

In the sequel, we need the famous Bernstein–Doetsch theorem.

Theorem 1.1. *Let $f : D \to \mathbb{R}$ be locally upper bounded and Jensen-convex, then f is convex and continuous.*

Recently, some results concerning approximate convexity were proved. It is a natural questions, what happens when we consider a nonpositive error function, namely a strong convexity inequality.

In what follows we recall some Bernstein-Doetsch type theorem for approximately convex functions. A function $f : D \to \mathbb{R}$ is said to be *approximately α-Jensen convex* on D, if,

Received: 30.05.2014. In revised form: 17.10.2014. Accepted: 24.10.2014
2010 *Mathematics Subject Classification.* 39B12, 39B22.
Key words and phrases. *Strong convexity, Takagi type functions, Hermite–Hadamard type inequalities.*
Corresponding author: Judit Makó; matjudit@uni-miskolc.hu

for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \alpha(\|x-y\|). \tag{1.2}$$

Let introduce the Takagi type functions $\mathcal{T}_\alpha : \mathbb{R} \times D^* \to \mathbb{R}_+$ and $\mathcal{S}_\alpha : \mathbb{R} \times D^* \to \mathbb{R}_+$ by

$$\mathcal{T}_\alpha(t,u) := \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha\big(d_{\mathbb{Z}}(2^n t)u\big) \qquad \big((t,u) \in \mathbb{R} \times D^*\big) \tag{1.3}$$

and

$$\mathcal{S}_\alpha(t,u) := \sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) d_{\mathbb{Z}}(2^n t) \qquad \big((t,u) \in \mathbb{R} \times D^*\big), \tag{1.4}$$

where $d_{\mathbb{Z}}(t) := 2\,\mathrm{dist}(t, \mathbb{Z})$. Note that the first series converges uniformly if α is bounded, on the other hand, for the uniform convergence of the second series, it is sufficient if $\sum_{n=n_0}^{\infty} \alpha\big(2^{-n}\big) < \infty$ for some $n_0 \in \mathbb{N}$. The importance of the function $\mathcal{T}_\alpha$ introduced above is enlightened by the following result ([9], [18]) which can be considered as a generalization of the celebrated Bernstein-Doetsch theorem [2].

Theorem 1.2. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^* \to \mathbb{R}_+$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\alpha(t, \|x-y\|) \tag{1.5}$$

for all $x, y \in D$ and $t \in [0,1]$.

The other Takagi type function $\mathcal{S}_\alpha$ was introduced by Jacek Tabor and Józef Tabor ([18]). Its role and importance in the theory of approximate convexity is shown by the next theorem.

Theorem 1.3. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^* \to \mathbb{R}_+$ be nondecreasing such that $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{S}_\alpha(t, \|x-y\|) \tag{1.6}$$

for all $x, y \in D$ and $t \in [0,1]$.

Let $\varepsilon, q \ge 0$ be arbitrary constants. When $\alpha(u) := \varepsilon u^q, (u \in D^*)$, the two corollaries below (see [6] and [18]) are immediately consequences of the previous theorems. For $q \ge 0$, define the Takagi type functions S_q and T_q by

$$T_q(t) := \sum_{n=0}^{\infty} \frac{\big(d_{\mathbb{Z}}(2^n t)\big)^q}{2^n}, \qquad S_q(t) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \qquad (t \in \mathbb{R}). \tag{1.7}$$

They generalize the classical Takagi function

$$T(t) := \sum_{n=0}^{\infty} \frac{\mathrm{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R})$$

in two ways, because $T_1 = S_1 = 2T$ holds obviously. This function was introduced by Takagi in [19] and it is a well-known example of a continuous but nowhere differentiable real function. It is less trivial, but it can be proved that $T_2(t) = S_2(t) = 4t(1-t)$ for $t \in [0,1]$.

Corollary 1.1. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and $\varepsilon, q \geq 0$. Then f is (ε, q)-Jensen convex on D, if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q \tag{1.8}$$

for all $x, y \in D$ and $t \in [0,1]$.

Corollary 1.2. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and $\varepsilon, q \geq 0$. Then f is (ε, q)-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q \tag{1.9}$$

for all $x, y \in D$ and $t \in [0,1]$.

In [3], Boros proved that if $q = 1$ and $t \in [0,1]$ is fixed, then $S_1(t) = T_1(t) = 2T(t)$ is the smallest possible. In [17] Tabor and Tabor showed that if $1 \leq q \leq 2$ and $t \in [0,1]$ is fixed, then $S_q(t)$ is the smallest possible value so that (1.9) be valid for all (ε, q)-Jensen convex functions f on D. Later in [8] and [11], the authors examined whether the error terms $\mathcal{T}_\alpha(t, \|x-y\|)$, $\mathcal{S}_\alpha(t, \|x-y\|)$ in (1.5) in (1.6) and $T_q(t)$ in (1.8) are the smallest possible ones. In other words, for all fixed $x, y \in D$, the exact upper bound of the convexity-difference of α-Jensen convex functions defined by

$$C_\alpha(x, y, t) := \sup_{f \in \mathcal{JC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{1.10}$$

where

$$\mathcal{JC}_\alpha(D) := \{f : D \to \mathbb{R} \,|\, f \text{ is } \alpha\text{-Jensen convex on } D\}$$

was examined. The statement of Theorem **1.2**, Theorem **1.3**, Corollary 1.1, and Corollary 1.2 can be stated as

$$C_\alpha(x, y, t) \leq \tau(t, \|x-y\|), \tag{1.11}$$

where $\tau : \mathbb{R} \times D^* \to \mathbb{R}_+$ is given by

$$\tau := \mathcal{T}_\alpha, \quad \tau := \mathcal{S}_\alpha, \quad \tau(t,u) := \varepsilon T_q(t) u^q, \quad \text{and} \quad \tau(t,u) := \varepsilon S_q(t) u^q,$$

respectively. To obtain also a lower bound for $C_\alpha(x, y, t)$, (and thus to prove the sharpness of the inequality (1.11)), the following important observation was done by Páles in [15].

Theorem 1.4. *Let $\alpha : D^* \to \mathbb{R}$ be continuous. Let $\tau : \mathbb{R} \times D^* \to \mathbb{R}$ be continuous in its first variable, with $\tau(0, u) = \tau(1, u) = 0$ for all $u \in D^*$, which is Jensen convex in the following sense, for all $u \in D^*$ and $s, t \in [0,1]$,*

$$\tau\Big(\frac{t+s}{2}, u\Big) \leq \frac{\tau(t,u) + \tau(s,u)}{2} + \alpha(|t-s|u).$$

Then,

$$C_\alpha(x, y, t) \geq \tau(t, \|x-y\|).$$

2. From strong α-Jensen convexity to strong convexity

With the help of the following theorem, we can "strengthen" our error function α. (See in [7].)

Theorem 2.5. *Let $f : D \to \mathbb{R}$ be a strongly α-Jensen convex function. Then f is strongly $\widetilde{\alpha}$-Jensen convex on D, where*

$$\widetilde{\alpha}(u) := \sup\left\{n^2 \alpha\left(\tfrac{u}{n}\right) \mid n \in \mathbb{N}\right\} \qquad (u \in D^*). \tag{2.12}$$

This means that, we can assume that $\alpha(2u) \geq 4\alpha(u)$ for all $u \in D^*$. In the case of strong (ε, q)-convexity, this means that $q \geq 2$. Similarly as in Theorem **1.2** and Theorem **1.3**, it can be proved two Bernstein–Doetsch type results for locally upper bounded strongly Jensen convex functions. Thus, these theorems give us connections between strong α-Jensen convexity and convexity type inequalities. See also in [4].

Theorem 2.6. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^* \to \mathbb{R}_+$. Then f is strongly α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mathcal{T}_\alpha(t, \|x - y\|) \tag{2.13}$$

for all $x, y \in D$ and $t \in [0, 1]$.

Theorem 2.7. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^* \to \mathbb{R}_+$ be $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mathcal{S}_\alpha(t, \|x - y\|) \tag{2.14}$$

for all $x, y \in D$ and $t \in [0, 1]$.

We can also look for the optimal Takagi type function. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of strongly α-Jensen convex functions defined by

$$SC_\alpha(x, y, t) := \sup_{f \in \mathcal{SJC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{2.15}$$

where

$\mathcal{SJC}_\alpha(D) := \{f : D \to \mathbb{R} \mid f$ is locally upper bounded and strongly α-Jensen convex on $D\}$.

By Theorem **1.4**, it is enough to prove the Jensen-convexity of $S_\alpha(\cdot, u)$ or $\mathcal{T}_\alpha(\cdot, u)$. We shall prove that the Takagi type function $S_\alpha(\cdot, u)$ will be the optimal choice. To show this suspicion let introduce the following Takagi type function $S_\varphi : [0, 1] \to \mathbb{R}$ defined by

$$S_\varphi(x) = \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) d_{\mathbb{Z}}(2^n x), \tag{2.16}$$

where $P := \{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots, \}$ and $\varphi : P \to \mathbb{R}_+$ is a nonnegative function. In fact, the proof of these results are very similar as in [11], so we ignore it. The main results of this section state that, under certain assumptions on the function $\varphi : P \to \mathbb{R}$, $(-S_\varphi)$ is well-defined and strongly Jensen convex in the following sense: For all $x, y \in [0, 1]$,

$$-S_\varphi\Big(\frac{x+y}{2}\Big) \leq \frac{-S_\varphi(x) - S_\varphi(y)}{2} - \varphi \circ d_{\mathbb{Z}}\Big(\frac{x-y}{2}\Big). \tag{2.17}$$

First we describe the situation when the definition of S_φ is correct.

Lemma 2.3. *Let $\varphi : P \to \mathbb{R}_+$ be a nonnegative function. Then S_φ is well-defined, i.e., the series on the right hand side of* (2.16) *is convergent everywhere if and only if*

$$\sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty. \tag{2.18}$$

In the sequel, the class of nonnegative functions $\varphi : P \to \mathbb{R}_+$ satisfying the condition (2.18) will be denoted by $\mathcal{H}$:

$$\mathcal{H} := \Big\{\varphi : P \to \mathbb{R}_+ \mid \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty\Big\}.$$

The next theorem, which was discovered by Jacek Tabor and Józef Tabor, has an important role in the proof of the main theorem of this section.

Theorem 2.8. *For every $x, y \in \mathbb{R}$*

$$S_2\Big(\frac{x+y}{2}\Big) \le \frac{S_2(x)+S_2(y)}{2} + d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

The following simple observation is a direct consequence of the previous theorem.

Corollary 2.4. *For every $x, y \in [0,1]$*

$$-S_2\Big(\frac{x+y}{2}\Big) = \frac{-S_2(x)-S_2(y)}{2} - d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

In the next result we give a representation of $S_\varphi(x)$ as an infinite linear combination of the values $S_2(2^n x)$, $n = 1, 2, \ldots$.

Theorem 2.9. *Let $\varphi \in \mathcal{H}$. Then, for every $x \in \mathbb{R}$,*

$$S_\varphi(x) = \varphi(1)S_2(x) + \sum_{n=1}^{\infty}\Big(\varphi\big(\tfrac{1}{2^n}\big) - \tfrac{1}{4}\varphi\big(\tfrac{1}{2^{n-1}}\big)\Big)S_2(2^n x). \tag{2.19}$$

An immediate consequence of the previous two theorems is the next result which states the strong convexity of $(-S_\varphi)$.

Theorem 2.10. *Let $\varphi \in \mathcal{H}$ such that, for all $u \in \frac{1}{2}P$, $\varphi(2u) \ge 4\varphi(u)$. Then, for all $x, y \in [0,1]$,*

$$-S_\varphi\Big(\frac{x+y}{2}\Big) \le \frac{-S_\varphi(x)-S_\varphi(y)}{2} - \Phi_2\Big(\frac{x-y}{2}\Big), \tag{2.20}$$

where $\Phi_2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi_2(u) := \sum_{n=0}^{\infty}\varphi\big(\tfrac{1}{2^n}\big)\Big(d_{\mathbb{Z}}^2(2^n u) - \frac{1}{4}d_{\mathbb{Z}}^2(2^{n+1}u)\Big). \tag{2.21}$$

In the next proposition we describe a decomposition property of the function Φ_2.

Proposition 2.5. *For $\varphi \in \mathcal{H}$, for all $u \in]0, \frac{1}{2}]$,*

$$\Phi_2(u) = \Phi_2\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]}} - u\Big) + \varphi\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]-1}}\Big)\Big(1 - 2\cdot 2^{[\log_2 \frac{1}{u}]}u\Big). \tag{2.22}$$

The next theorem has an important role in the proof of our subsequent main results.

Theorem 2.11. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$, then, for all $u \in [0,1]$,*

$$-\Phi_2(u) \le -\varphi \circ d_{\mathbb{Z}}(u). \tag{2.23}$$

The main result of this section is stated in the following theorem. The proof of is this theorem is based on the previous propositions and lemmas.

Theorem 2.12. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$. Then $(-S_\varphi)$ is strongly Jensen convex in the sense of* (2.17).

We shall prove that the error terms $-\mathcal{S}_\alpha(t, \|x-y\|)$ in (1.6) under certain assumptions on the error function α is the smallest possible one. In other words, the next theorem will provide exact upper bound for the convexity-difference of strongly α-Jensen convex functions defined by (2.15).

Theorem 2.13. *Let $\alpha : D^* \to \mathbb{R}$ be an error function such that $\alpha(0) = 0$ and the map $u \mapsto \frac{\alpha(u)}{u}$ is convex on $D^* \setminus \{0\}$. Then, for all $x, y \in D$ and $t \in [0,1]$,*

$$SC_\alpha(x, y, t) = -\mathcal{S}_\alpha(t, \|x-y\|). \tag{2.24}$$

Taking an error function α which is a combination of power functions of exponents from $[2, \infty[$, we obtain the following result.

Theorem 2.14. *Let ν be a nonnegative bounded Borel measure on $[2, \infty[$. Define the error function $\alpha_\nu : D^* \to \mathbb{R}_+$ by*

$$\alpha_\nu(u) := \int_{[2,\infty[} u^q d\nu(q) \qquad (u \in D^*).$$

Then, for all $x, y \in D$ and $t \in [0,1]$,

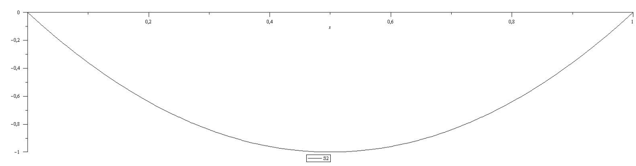
$$SC_{\alpha_\nu}(x, y, t) = -\int_{[2,\infty[} S_q(t)\|x - y\|^q d\nu(q),$$

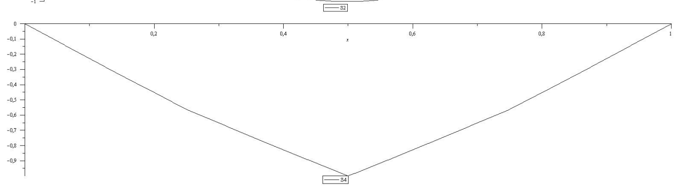
where $S_q : \mathbb{R} \to \mathbb{R}$ is given by (1.7).

Corollary 2.6. *Let $q \in [2, \infty[$ and $\varepsilon \geq 0$. Define the error function $\alpha : D^* \to \mathbb{R}_+$ by $\alpha(u) := \varepsilon u^q$. Then, for all $x, y \in D$ and $t \in [0, 1]$,*

$$SC_\alpha(x, y, t) = -\varepsilon S_q(t)\|x - y\|^q.$$

The next figures demonstrate the strong convexity of $(-S_q)$, when $q = 2$ and $q = 4$.





3. On a strong convexity type inequality

Given a nonnegative function $\alpha : D^* \to \mathbb{R}_+$, we say that a map $f : D \to \mathbb{R}$ is *strongly α-convex*, if for all $x, y \in D$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t\alpha\big((1-t)\|x - y\|\big) - (1-t)\alpha\big(t\|y - x\|\big) \tag{3.25}$$

holds. In [9], a similar approximate convexity type inequality was examined. If (3.25) holds with a $t \in]0,1[$, we say that f is *strongly (t, α)-convex* on D. If (3.25) holds with $t = \frac{1}{2}$, we can get the strong $\alpha(\frac{\cdot}{2})$-Jensen convexity of the function f. By the nonnegativity of α, we have that strongly α-Jensen convex and strongly α-convex functions are always convex in the same sense, respectively.

In [7], strong α-Jensen convexity was examined and the following result was established:

Theorem 3.15. *For any function $f : D \to \mathbb{R}$, the following conditions are equivalent:*

(i) f is strongly α-convex.
(ii) f is directionally differentiable at every point of D, and for all $x_0 \in D$, the map $h \mapsto f'(x_0, h)$ is sublinear on X, furthermore for all $x_0, x \in D$,

$$f(x) \geq f(x_0) + f'(x_0, x - x_0) + \alpha(\|x - x_0\|). \tag{3.26}$$

(iii) For all $x_0 \in D$, there exits an element $A \in X'$ such that

$$f(x) \geq f(x_0) + A(x - x_0) + \alpha(\|x - x_0\|) \quad \textit{for all} \quad x \in D. \tag{3.27}$$

Thus, it can be important to look for connections between the strong (λ, α)-convexity and strong α-convexity.

Theorem 3.16. *If $f : D \to \mathbb{R}$ is locally upper bounded and strongly (λ, α)-convex, with $\lambda \in]0, 1[$ then f is strongly $\frac{1}{\lambda}\alpha$-convex on D.*

Proof. Since f is strongly (λ, α)-convex and locally upper bounded, we immediately have that f is convex. Let $x, y \in D$ be arbitrary. First using that the directional derivative of $f'(y, \cdot)$ is positive homogeneuos, then appying Theorem **3.15**, with $\alpha = 0$, finally using the strong (λ, α)-convexity of f, we can get that:

$$\begin{aligned} f'(y, x - y) &= \frac{1}{\lambda} f'\big(y, \lambda(x - y)\big) \leq \frac{1}{\lambda}\big(f\big(y + \lambda(x - y)\big) - f(y)\big) \\ &\leq \frac{1}{\lambda}\big(\lambda f(x) + (1 - \lambda) f(y) - f(y) - \alpha(\|x - y\|)\big) = f(x) - f(y) - \frac{1}{\lambda}\alpha(\|x - y\|), \end{aligned}$$

which is (by Theorem **3.15**) equivalent to the strong $\frac{1}{\lambda}\alpha$-convexity of f. □

It is not difficult to prove Hermite–Hadamard type inequalities for strongly (λ, α)-convex function. If $f : D \to \mathbb{R}$ is strongly (λ, α)-convex, we can get that f is strongly $\frac{1}{\lambda}\alpha$-convex. Applying Theorem **2.5** from [10], we get the following theorem.

Theorem 3.17. *Let μ be a probability Borel measure on $[0, 1]$ and $\alpha : D^* \to \mathbb{R}$ be bounded and Borel measurable function. If $f : D \to \mathbb{R}$ is locally upper bounded and strongly (λ, α)-convex, then, for all $x, y \in D$, f satisfies the following lower Hermite–Hadamard type inequality*

$$\begin{aligned} f(\mu_1 x + (1 - \mu_1) y) \leq &\int_{[0,1]} f(tx + (1 - t)y) d\mu(t) \\ &- \frac{1}{\lambda} \int_{[0,1]} \Big(t\alpha((1 - t)\|x - y\|) + (1 - t)\alpha(t\|x - y\|)\Big) d\mu(t) \end{aligned} \tag{3.28}$$

with $\mu_1 = \int_{[0,1]} t d\mu(t)$.

Applying Theorem **3.14** from [10], we can get the following upper Hermite–Hadamard type inequality.

Theorem 3.18. *Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such that the support of μ is not a singleton. Denote*

$$\mu_1 := \int\limits_{[0,1]} t d\mu(t) \qquad \textit{and} \qquad S(\mu) := \mu\big([0, \mu_1]\big) \int\limits_{]\mu_1,1]} t d\mu(t) - \mu\big(]\mu_1, 1]\big) \int\limits_{[0,\mu_1]} t d\mu(t).$$

Assume that $f : D \to \mathbb{R}$ is μ-integrable and strongly (λ, α)-convex. Moreover, for all $(x, y) \in D^2$,

$$I(x, y) := \int\limits_{]\mu_1,1]} \int\limits_{[0,\mu_1]} (t'' - \mu_1)\alpha\big((\mu_1 - t')\|x - y\|\big) + (\mu_1 - t')\alpha\big((t'' - \mu_1)\|x - y\|\big) d\mu(t') d\mu(t'')$$

exists in $[0, \infty]$. Then, for all $(x, y) \in D^2$, the function f also satisfies the lower Hermite–Hadamard type inequality

$$f((1 - \mu_1)x + \mu_1 y) \leq \int\limits_{[0,1]} f\big((1 - t)x + ty\big) d\mu(t) - \frac{1}{\lambda S(\mu)} I(x, y).$$

In the following theorems, we have established relations between Hermite–Hadamard type inequalities and strong (Jensen) convexity.

Theorem 3.19. *Let* μ *be a Borel probability measure on* $[0,1]$ *and assume that* $\alpha : D^* \to \mathbb{R}_+$ *be a given error function. Denote* $\mu_1 := \int_{[0,1]} t d\mu(t)$. *If* $f : D \to \mathbb{R}$ *is continuous and satisfies the following upper Hermite–Hadamard type inequality*

$$\int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t)dt \le \mu_1 f(x) + (1-\mu_1)f(y) - \alpha(\|x-y\|), \quad (x, y \in D), \tag{3.29}$$

then f *is strongly* $\frac{1}{\mu_1}\alpha$*-convex on* D.

Proof. Let $x, y \in D$ arbitrary. By (3.29), we have that

$$\int\limits_{[0,1]} (f(y + t(x-y)) - f(y))d\mu(t) \le \mu_1(f(y) - f(x)) - \alpha(\|x-y\|), \qquad (x, y \in D).$$

Since α is nonnegative, f satisfies (3.29) with $\alpha = 0$ and hence f is convex, which implies

$$f'(y, t(x-y)) \le f(y + t(x-y)) - f(y), \qquad (t \in [0,1]).$$

Combining the above two inequalities, and using the positive homogeneity of the directional derivative the proof is complete. □

Theorem 3.20. *Let* μ *be a Borel probability measure on* $[0,1]$ *and assume that* $\alpha : D^* \to \mathbb{R}_+$ *be a given error function. Denote* $\mu_1 := \int_{[0,1]} t d\mu(t)$. *If* $f : D \to \mathbb{R}$ *is continuous and satisfies the following lower Hermite–Hadamard type inequality,*

$$f(\mu_1 x + +(1-\mu_1)y) \le \int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t) - \alpha(\|x-y\|) \tag{3.30}$$

then f *is* $\frac{1}{\mu_1}\alpha$*-convex on* D.

Proof. Using again the convexity of f in (3.30), we can have that f is strongly (μ_1, α)-convex on D. Applying Theorem **3.16**, with $\lambda = \mu_1$, we have $\frac{1}{\mu_1}\alpha$-convexity of f. □

Remark 3.1. A lot of theorems and propositions are true in linear space, but we would like to work in normed space in the whole paper.

Acknowledgements. The first named author research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 'National Excellence Program'. The second named author research has been supported by the János Bolyai research scholarship of the Hungarian Academy of Sciences.

References

[1] Azócar, A., Giménez, J., Nikodem, K. and Sánchez, J. L., *On strongly midconvex functions*, Opuscula Math., **31** (2011), No. 1, 15–26

[2] Bernstein, F. and Doetsch. G., *Zur Theorie der konvexen Funktionen*, Math. Ann., **76** (1915), No. 4, 514–526

[3] Boros, Z., *An inequality for the Takagi function*, Math. Inequal. Appl., **11** (2008), No. 4, 757–765

[4] Conzález, C., Nikodem, K., Páles, Zs., and Roa, G., *Bernstein-Doetsch type theorems for set-valued maps of strongly and approximately convex and concave type*, Publ. Math. Debrecen, **84** (2014), 229–252

[5] Hadamard, J., *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215

[6] Házy, A. and Páles, Zs., *On approximately t-convex functions*, Publ. Math. Debrecen, **66** (2005), 489–501

[7] Makó, J. and Nikodem, K. and Páles, Zs., *On strong* $(\alpha, \mathbb{F})$*-convexity*, Math. Inequal. Appl., **15** (2012), 289–299

[8] Makó, J. and Páles, Zs., *Approximate convexity of Takagi type functions*, J. Math. Anal. Appl., **369** (2010), 545–554
[9] Makó, J. and Páles, Zs., *On φ-convexity*, Publ. Math. Debrecen, **80** (2012), 107–126
[10] Makó, J. and Páles, Zs., *Approximate Hermite–Hadamard type inequalities for approximately convex functions*, Math. Inequal. Appl., **16** (2013), 507–526
[11] Makó, J. and Páles, Zs., *On approximately convex Takagi type functions*, Proc. Amer. Math. Soc., **141** (2013), 2069–2080
[12] Merentes, N. and Nikodem, K., *Remarks on strongly convex functions*, Aequat. Math., **80** (2010), 193–199
[13] Ng, C. T. and Nikodem, K., *On approximately convex functions*, Proc. Amer. Math. Soc., **118** (1993), No. 1, 103–108
[14] Nikodem, K. and Páles, Zs., *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal., **5** (2011), No. 1, 83–87
[15] Páles, Zs., *The Forty-first International Symposium on Functional Equations*, June 8–15, 2003, Noszvaj, Hungary, Aequat. Math., **67** (2004), 285–320
[16] Polyak, B. T., *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl., **7** (1966), 72–75
[17] Tabor, Ja. and Tabor, Jó., *Generalized approximate midconvexity*, Control Cybernet., **38** (2009), No. 3, 655–669
[18] Tabor, Ja. and Tabor, Jó., *Takagi functions and approximate midconvexity*, J. Math. Anal. Appl., **356** (2009), No. 2, 729–737
[19] Takagi. T., *A simple example of the continuous function without derivative*, J. Phys. Math. Soc. Japan, **1** (1903), 176–177

UNIVERSITY OF MISKOLC
INSTITUTE OF MATHEMATICS
H-3515 MISKOLC-EGYETEMVÁROS, HUNGARY
E-mail address: {matjudit,matha}@uni-miskolc.hu