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# A new look on the truncated pentagonal number theorem

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ABSTRACT. Two new infinite families of inequalities are given in this paper for the partition function $p(n)$, using the truncated pentagonal number theorem.

## 1. INTRODUCTION.

Euler [3] began the mathematical theory of partitions in 1748 by discovering the generating function of the partition function $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the $q$-shifted factorial with $(a;q)_0 = 1$.

The pentagonal number theorem relates the product and the series representations of the Euler function,

$$(q;q)_\infty = \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} q^{G_k}, \tag{1.1}$$

where the exponents $G_k$ are called generalized pentagonal numbers, i.e.,

$$G_k = \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil.$$

This theorem leads to an efficient method of computing the partition function $p(n)$, i.e.,

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} p(n-G_k) = \delta_{0,n},$$

where $p(n) = 0$ for any negative integer $n$ and $\delta_{i,j}$ is the Kronecker delta. This formula is well-known as Euler's pentagonal number recurrence for computing $p(n)$. More details about these classical results in the partition theory can be found in Andrews's book [1].

Recently, Andrews and Merca [2] proved that, for $k > 0$,

$$(-1)^{k-1} \sum_{j=0}^{2k-1} (-1)^{\lceil j/2 \rceil} p(n-G_j) \geqslant 0 \tag{1.2}$$

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with strict inequality if $n \geqslant G_{2k}$. The proof of this inequality in [2] is based on the truncated formula of (1.1):

$$\frac{(-1)^{k-1}}{(q;q)_\infty}\sum_{j=0}^{2k-1}(-1)^{\lceil j/2\rceil}q^{G_j} = (-1)^{k-1} + \sum_{n=k}^{\infty}\frac{q^{(k+1)n+\binom{k}{2}}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}, \tag{1.3}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

are $q$-binomial coefficients. Whenever the base of a $q$-binomial coefficient is just $q$ it will be omitted.

In [2], the left side of the inequality (1.2) is denoted by $M_k(n)$ and counts the partitions of $n$ in which $k$ is the least integer that is not a part and there are more parts $> k$ than there are $< k$.

Motivated by these results, in this paper we shall prove:

**Theorem 1.1.** *For $n > 0$, $k \geqslant 0$,*

$$(-1)^{\lfloor k/2\rfloor}\sum_{j=0}^{k}(-1)^{\lceil j/2\rceil}p(n-G_j) \geqslant 0 \tag{1.4}$$

*with strict inequality if $n \geqslant G_{k+1}$.*

We remark that, the inequality (1.2) is the case $k$ odd in Theorem 1.1. The case $k = 2$ of this theorem is given by the well-known inequality

$$p(n) - p(n-1) - p(n-2) \leqslant 0.$$

It is an easy exercise to show that for $n \neq 1$ one has

$$p(n) - p(n-1) - p(n-2) \leqslant p(n) - 2p(n-1) + p(n-3).$$

In addition, the inequality

$$p(n) - 2p(n-1) + p(n-3) \leqslant 0, \quad n > 0 \tag{1.5}$$

is true, though

$$p(n-2) \geqslant p(n-1) - p(n-3), \quad n \neq 1.$$

To prove (1.5), we need to show that, the coefficient of $q^n$ in

$$\frac{1-2q+q^3}{(q;q)_\infty}$$

is nonpositive for $n > 0$. We have

$$\begin{aligned}
\frac{1-2q+q^3}{(q;q)_\infty} &= \frac{(1-q)(1-q-q^2)}{(1-q)(q^2;q)_\infty}\\
&= \frac{1}{(q^3;q)_\infty} - \frac{q}{(q^2;q)_\infty}\\
&= \sum_{n\geq 0}\frac{q^{3n}}{(q;q)_n} - \sum_{n\geq 0}\frac{q^{2n+1}}{(q;q)_n}\\
&= 1 - q + \sum_{n\geq 2}\frac{q^{2n+1}}{(q;q)_n}\left(q^{n-1}-1\right)\\
&= 1 - q - \sum_{n\geq 2}\frac{q^{2n+1}}{(q;q)_{n-2}(1-q^n)}
\end{aligned}$$

and we see that for $n > 0$ the coefficient of $q^n$ is nonpositive.

There is a more general result where the inequality (1.5) is a very special case, namely

**Theorem 1.2.** *For $n > 1$, $k \geqslant 0$,*

$$(-1)^{\lfloor k/2 \rfloor} \sum_{j=0}^{k} (-1)^{\lceil j/2 \rceil} \left( p(n - G_j) - p(n - 1 - G_j) \right) \geqslant 0. \tag{1.6}$$

In this theorem, for $n > 1$ we have an infinite family of inequalities for the partition function $p(n)$, where the first four entries are:

$$\begin{aligned} &p(n) - p(n-1) \geqslant 0, \\ &p(n) - 2p(n-1) + p(n-2) \geqslant 0, \\ &p(n) - 2p(n-1) + p(n-3) \leqslant 0, \text{ and} \\ &p(n) - 2p(n-1) + p(n-3) + p(n-5) - p(n-6) \leqslant 0. \end{aligned}$$

It is known that the left side of the first inequality counts the partitions of $n-1$ in which the least part occurs exactly once [5, A002865]. The left side of the second inequality is the number of partitions of $n-2$ with all parts $> 1$ and with the largest part occurring more than once [5, A053445]. In the third inequality, the absolute value of the left side counts the partitions of $n-3$ such that the least part occurs exactly twice [5, A096373]. In the last inequality, the absolute value of the left side is equal to the number of partitions of $n-4$ such that if the smallest part is $k$, then both $k$ and $k+1$ occur exactly once [5, A118267].

Is it possible to have a general combinatorial interpretation for the left side of the inequality (1.6)?

## 2. Proof of Theorem 1.1

We need to prove only the case $k$ even. We define the integer $S_k(n)$ by

$$S_k(n) = (-1)^k \sum_{j=0}^{2k} (-1)^{\lceil j/2 \rceil} p(n - G_j). \tag{2.7}$$

We have

$$S_k(n) = M_{k+1}(n) + p(n - G_{2k+1}). \tag{2.8}$$

By (1.3), we obtain the generating function of $S_k(n)$

$$\begin{aligned} &(-1)^k + \frac{q^{G_{2k+1}}}{(q;q)_\infty} + \sum_{n=k+1}^{\infty} \frac{q^{(k+2)n + \binom{k+1}{2}}}{(q;q)_n} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\ &= (-1)^k + \frac{q^{G_{2k+1}}}{(q;q)_\infty} + \sum_{n=k}^{\infty} \frac{q^{(k+2)(n+1) + \binom{k+1}{2}}}{(q;q)_{n+1}} \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned} \tag{2.9}$$

Clearly the coefficients of $q^k$ for $1 \leqslant k \leqslant G_{2k}$ are all 0, and for $k > G_{2k}$ all the coefficients are positive. Thus the theorem is proved.

## 3. Proof of Theorem 1.2

Denoting the left side of (1.6) by $A_k(n)$, we see that

$$A_{2k-1}(n) = M_k(n) - M_k(n-1)$$

and

$$A_{2k}(n) = S_k(n) - S_k(n-1).$$

By (1.3), we notice that the generating function for $A_{2k-1}(n)$ is

$$(-1)^{k-1}(1-q)+\sum_{n=k}^{\infty}\frac{q^{(k+1)n+\binom{k}{2}}}{(q^2;q)_{n-1}}\begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

In this expression, the coefficients of $q^n$ for $n>1$ are all nonnegative. It is clear that $A_{2k-1}(n)\geqslant 0$ for $n>1$.

On the other hand, according to (2.9), the generating function of $A_{2k}(n)$ is given by

$$(-1)^{k}(1-q)+\frac{q^{G_{2k+1}}}{(q^2;q)_{\infty}}+\sum_{n=k}^{\infty}\frac{q^{(k+2)(n+1)+\binom{k+1}{2}}}{(q^2;q)_{n}}\begin{bmatrix}n\\k\end{bmatrix},$$

where the coefficients of $q^n$ for $n>1$ are all nonnegative. Clearly $A_{2k}(n)\geqslant 0$ for $n>1$.

Theorem 1.2 is proved.

## 4. Concluding remarks

The case $k$ even of Theorem 1.1 is given by

**Corollary 4.1.** *For $n>0$, $k\geqslant 0$,*

$$(-1)^k\sum_{j=0}^{2k}(-1)^{\lceil j/2\rceil}p(n-G_j)\geqslant 0$$

*with strict inequality if $n\geqslant G_{2k+1}$. For example,*

$$\begin{aligned}&p(n)\geqslant 0,\\&p(n)-p(n-1)-p(n-2)\leqslant 0,\\&p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)\geqslant 0.\end{aligned}$$

In addition, due to D. Shanks [2, eq. 4.1], we have the following identity

$$\sum_{j=0}^{2k}(-1)^{\lceil j/2\rceil}q^{G_j}=(q;q)_k\sum_{j=0}^{k}\frac{(-1)^j q^{jk+\binom{j+1}{2}}}{(q;q)_j}$$

and we can write

**Corollary 4.2.** *For $k\geqslant 0$,*

$$\frac{1}{(q^{k+1};q)_{\infty}}\sum_{j=0}^{k}\frac{(-1)^j q^{jk+\binom{j+1}{2}}}{(q;q)_j}$$

*has nonnegative coefficients if $k$ is even and nonpositive coefficients if $k$ is odd.*

Taking into account the definitions of the numbers $M_k(n)$ and $S_k(n)$, we obtain the following relation

$$S_k(n)=-M_k(n)+p(n-G_{2k}).$$

In this context, Corollary 4.1 can be written as

**Corollary 4.3.** *Let $n$ and $k$ be two positive integers. The number of partitions of $n$ in which $k$ is the least integer that is not a part and there are more parts $>k$ than there are $<k$ is less than or equal to the number of partitions of $n-G_{2k}$.*

On the other hand, the case $k$ even of Theorem 1.2 can be written as

**Corollary 4.4.** *Let $n$ and $k$ be two positive integers. The number of partitions of $n$ in which $k$ is the least integer that is not a part, there are more parts $> k$ than there are $< k$ and the largest part occurs at least twice is less than or equal to the number of partitions of $n - G_{2k}$ that do not contain $1$ as a part.*

It is still an open problem to give combinatorial interpretations for the numbers $S_k(n)$ or $S_k(n) - S_k(n-1)$.

There is a follow-up of Andrews and Merca's paper [2] by Guo and Zeng [4]. A similar extension of the results in the last paper should be possible.

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