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On the Stancu type bivariate approximation formula

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ABSTRACT. In the present paper we establish the form of remainder term associated to the bivariate approximation formula for Stancu type operators, using bivariate divided differences. We also shall establish an upper bound estimation for the remainder term, in the case when approximated function fulfills some given properties.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The operators $B_n : C[0,1] \to C[0,1]$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\tfrac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x)$ are the fundamental Bernstein's polynomials defined by

$$p_{n,k}(x) = \tbinom{n}{k} x^k (1-x)^{n-k}, \tag{1.2}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, are called Bernstein operators. These operators were introduced by S. N. Bernstein [8]. Let α be a non-negative parameter, which may depend only on the natural number n. The operators $P_n^{\langle\alpha\rangle} : C[0,1] \to C[0,1]$ given by

$$P_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x) f\left(\tfrac{k}{n}\right), \tag{1.3}$$

where $p_{n,k}^{\langle\alpha\rangle}(x)$ is a polynomial, which can be expressed by means of the factorial power $t^{[n,h]} = t(t-h)\cdot \ldots \cdot (t-(n-1)h)$, $t^{[0,h]} = 1$, (the nth factorial power of t with increment h), defined by

$$p_{n,k}^{\langle\alpha\rangle}(x) = \tbinom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}, \tag{1.4}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, are called Stancu operators. These operators were introduced by D. D. Stancu [19]. He investigated this linear polynomial operator of Bernstein type, in order to use it in the theory of uniform approximation of functions. In the case when $\alpha = 0$, the operators (1.3) reduce, obviously, to the classical Bernstein operators. For $\alpha = \frac{1}{n}$ one obtains a special case of the operators (1.3), introduced by L. Lupaş and A. Lupaş [14]. This is given by

$$P_n^{\left\langle\frac{1}{n}\right\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\left\langle\frac{1}{n}\right\rangle}(x) f\left(\tfrac{k}{n}\right) = \sum_{k=0}^{n} \tbinom{n}{k} \frac{x^{\left[k,-\frac{1}{n}\right]}(1-x)^{\left[n-k,-\frac{1}{n}\right]}}{1^{\left[n,-\frac{1}{n}\right]}} f\left(\tfrac{k}{n}\right). \tag{1.5}$$

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In [15], we established that an expression of the remainder term of Stancu operators, proved by D. D. Stancu [20], using only divided differences of second order is just an intermediate form of the relation

$$R_n^{\langle\alpha\rangle}(f;x) = -\tfrac{x(1-x)(1+\alpha n)}{n(1+\alpha)} \sum_{k=0}^{n-1} s_{n-1,k}^{\langle\alpha\rangle}(x+\alpha)\left[x, \tfrac{k}{n}, \tfrac{k+1}{n}; f\right], \tag{1.6}$$

for $x \in [0,1] \setminus \left\{ \frac{k}{n} \mid k = \overline{0,n} \right\}$, where $s_{n,k}^{\langle\alpha\rangle}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x+2\alpha)^{[n-k,-\alpha]}}{(1+2\alpha)^{[n,-\alpha]}}$, $x \geq 0$, $\alpha \geq 0$. Also in [15], the study on the remainder term associated to the particular case (1.5) of Stancu operators is made in analogous manner. In a recent paper [1] is given a new representation of the remainder in the Bernstein approximation based on divided differences, for arbitrary functions.

The aim of this paper is to revise, respectively establish the form of remainder term associated to the bivariate approximation formula of Stancu type operators, using bivariate divided differences. The revision is motivated by two ideas. One of them is contained in [15], where the revisited form of the reminder term associated to the univariate approximation formula of Stancu operators is established. The another one, is based on the fact that Stancu operators are not projectors and the decomposition formula of the identity operator for determining the form of the bivariate remainder term can not be applied. Concerning the second idea, the reader is invited to see the paper [5], where a detailed and complete exposure for the case of Bernstein operators was given. As a new direction of research, we note that, this revised form of remainder term associated to the bivariate approximation formula of Stancu type operators can be used in construction of quadrature and cubature formulas. An example in this sense could be the recent paper [7], where the Bernstein quadrature formula was revised.

2. Auxiliary results

W. J. Gordon [11] has introduced the basic notions of the algebraic theory of multivariate functions approximation, a theory which was studied and developed by F. J. Delvos and W. Schempp [10]. The method of parametric extension is a procedure for constructing linear operators on the spaces of multivariate functions, starting from linear operators defined on spaces of univariate functions, (see [10]).

Let $S = [0,1] \times [0,1]$ be a polygonal domain and suppose that $f \in C(S)$ is given, $(x,y) \in S$ and $m, n \in \mathbb{N}$. Then, for $\alpha, \beta \geq 0$ the parametric extensions of (1.3) are defined by

$$ {}_xP_m^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x) p_{n,j}^{\langle\beta\rangle}(y) f\left(\tfrac{i}{m}, y\right), \text{ respectively} \tag{2.7}$$

$$ {}_yP_n^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x) p_{n,j}^{\langle\beta\rangle}(y) f\left(x, \tfrac{j}{n}\right). \tag{2.8}$$

Considering the operators (2.7) and (2.8) the bivariate Stancu operators [20], [9] $P_{m,n}^{\langle\alpha,\beta\rangle}$: $C(S) \to C(S)$ can be got by the tensorial product of parametric extensions and are given by

$$ {}_xP_m^{\langle\alpha,\beta\rangle}\left({}_yP_n^{\langle\alpha,\beta\rangle}; x, y\right) = P_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}^{\langle\alpha\rangle}(x) p_{n,j}^{\langle\beta\rangle}(y) f\left(\tfrac{i}{m}, \tfrac{j}{n}\right). \tag{2.9}$$

Let us recall some results concerning divided differences, which we will use afterwards in the paper. Suppose that $f : I \to \mathbb{R}$ is a real-valued function and $x_0, x_1 \in I$, such that

$x_0 \neq x_1$, I being a certain interval of the real axis. The first order divided difference of f with respect the distinct knots x_0, x_1 is defined by

$$[x_0, x_1; f] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{2.10}$$

If $x_0, x_1, \ldots, x_n \in I$ are distinct knots and $f : I \to \mathbb{R}$ is given, then the nth order divided difference of f with respect the mentioned knots is defined by the recurrence relation

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{2.11}$$

We note that, in the case of coalescing points, if f is a suitably differentiable function, the divided difference can be defined by using a limiting process. The divided differences were intensively studied by T. Popoviciu [18].
Now, let I, J be certain real intervals, $f : I \times J \to \mathbb{R}$ be a real-valued function and $(x_0, y_0), (x_1, y_1) \in I \times J$, such that $x_0 \neq x_1$ and $y_0 \neq y_1$. The bivariate divided differences of f with respect the knots (x_0, y_0), (x_1, y_1) are defined using the method of parametric extensions in [2], by

$$\begin{bmatrix} x_0, x_1 \\ y_0, y_1 \end{bmatrix}; f \Bigg] = \frac{f(x_1, y_1) - f(x_0, y_1) - f(x_1, y_0) + f(x_0, y_0)}{(x_1 - x_0)(y_1 - y_0)}. \tag{2.12}$$

Other equivalent definitions for univariate, respectively bivariate divided differences can be found in the excellent monographs [12] and [13]. In definition of divided differences the number of abscissas in general is not equal with the number of coordinates. It follows

$$\left[\begin{matrix} x_0, x_1 \\ y_0 \end{matrix} ; f \right] = \frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}, \quad \left[\begin{matrix} x_0, x_1, x_2 \\ y_0 \end{matrix} ; f \right] = \frac{\left[\begin{matrix} x_1, x_2 \\ y_0 \end{matrix} ; f \right] - \left[\begin{matrix} x_0, x_1 \\ y_0 \end{matrix} ; f \right]}{x_2 - x_1},$$

where x_0, x_1, x_2 are distinct knots. If $x_0, x_1, \ldots, x_p \in I$ and $y_0, y_1, \ldots, y_q \in J$ are distinct knots and $f : I \times J \to \mathbb{R}$ is given, the following recurrence formula

$$\begin{aligned} \left[\begin{matrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{matrix} ; f \right] &= \frac{1}{(x_p - x_0)(y_q - y_0)} \left(\left[\begin{matrix} x_1, \ldots, x_p \\ y_1, \ldots, y_q \end{matrix} ; f \right] \right. \\ &\left. - \left[\begin{matrix} x_0, \ldots, x_{p-1} \\ y_1, \ldots, y_q \end{matrix} ; f \right] - \left[\begin{matrix} x_1, \ldots, x_p \\ y_0, \ldots, y_{q-1} \end{matrix} ; f \right] + \left[\begin{matrix} x_0, \ldots, x_{p-1} \\ y_0, \ldots, y_{q-1} \end{matrix} ; f \right] \right) \end{aligned} \tag{2.13}$$

holds (see [2]), for $p, q \in \mathbb{N}$, $p, q \geq 2$ and

$$\left[\begin{matrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{matrix} ; f \right] = \left[\begin{matrix} x_{i_0}, x_{i_1}, \ldots, x_{i_p} \\ y_{j_0}, y_{j_1}, \ldots, y_{j_q} \end{matrix} ; f \right], \tag{2.14}$$

where $(i_0, i_1, \ldots, i_p)$, $(j_0, j_1, \ldots, j_q)$ are permutations of $(0, 1, \ldots, p)$, respectively $(0, 1, \ldots, q)$. Another interesting results concerning univariate, respectively bivariate divided differences with multiple knots can be found in a recent paper of O. T. Pop and D. Bărbosu [17].

3. Main results

In the following, let $f : S \to \mathbb{R}$ be given. The parametric extensions of the particular case (1.5) of Stancu operators are defined in [16] and are given by

$$
{}_xP_m^{\left\langle \frac{1}{m},\frac{1}{n}\right\rangle}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\left\langle\frac{1}{m}\right\rangle}(x)p_{n,j}^{\left\langle\frac{1}{n}\right\rangle}(y)f\left(\tfrac{i}{m},y\right) \tag{3.15}
$$
$$
=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{x^{\left[i,-\frac{1}{m}\right]}(1-x)^{\left[m-i,-\frac{1}{m}\right]}y^{\left[j,-\frac{1}{n}\right]}(1-y)^{\left[n-j,-\frac{1}{n}\right]}}{1^{\left[m,-\frac{1}{m}\right]}1^{\left[n,-\frac{1}{n}\right]}}f\left(\tfrac{i}{m},y\right),\ \text{respectively}
$$

$$
{}_yP_n^{\left\langle \frac{1}{m},\frac{1}{n}\right\rangle}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\left\langle\frac{1}{m}\right\rangle}(x)p_{n,j}^{\left\langle\frac{1}{n}\right\rangle}(y)f\left(x,\tfrac{j}{n}\right) \tag{3.16}
$$
$$
=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{x^{\left[i,-\frac{1}{m}\right]}(1-x)^{\left[m-i,-\frac{1}{m}\right]}y^{\left[j,-\frac{1}{n}\right]}(1-y)^{\left[n-j,-\frac{1}{n}\right]}}{1^{\left[m,-\frac{1}{m}\right]}1^{\left[n,-\frac{1}{n}\right]}}f\left(x,\tfrac{j}{n}\right).
$$

Using the parametric extensions (3.15) and (3.16) we get the tensorial product, given by

$$
{}_xP_m^{\left\langle \frac{1}{m},\frac{1}{n}\right\rangle}\left({}_yP_n^{\left\langle \frac{1}{m},\frac{1}{n}\right\rangle};x,y\right)=P_{m,n}^{\left\langle \frac{1}{m},\frac{1}{n}\right\rangle}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\left\langle\frac{1}{m}\right\rangle}(x)p_{n,j}^{\left\langle\frac{1}{n}\right\rangle}(y)f\left(\tfrac{i}{m},\tfrac{j}{n}\right). \tag{3.17}
$$

We shall prove:

Theorem 3.1. *For $x \in [0,1]\setminus\left\{\frac{i}{m} \mid i=\overline{0,m}\right\}$ and $y \in [0,1]\setminus\left\{\frac{j}{n} \mid j=\overline{0,n}\right\}$, the remainder term associated to the Stancu bivariate approximation formula can be represented under the following form*

$$
\begin{aligned}
R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)=&-\tfrac{x(1-x)(1+m\alpha)}{m(1+\alpha)}\sum_{i=0}^{m-1}\sum_{j=0}^{n}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix} x,\frac{i}{m},\frac{i+1}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}\\
&-\tfrac{y(1-y)(1+n\beta)}{n(1+\beta)}\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\begin{bmatrix} \frac{i}{m}, & ;f\\ y,\frac{j}{n},\frac{j+1}{n} & \end{bmatrix}\\
&+\tfrac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\begin{bmatrix} x,\frac{i}{m},\frac{i+1}{m} & ;f\\ y,\frac{j}{n},\frac{j+1}{n} & \end{bmatrix},
\end{aligned} \tag{3.18}
$$

where $s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)=\binom{m-1}{i}\frac{(x+\alpha)^{[i,-\alpha]}(1-x+\alpha)^{[m-1-i,-\alpha]}}{(1+2\alpha)^{[m-1,-\alpha]}}$.

Proof. Starting with the approximation formula $f(x;y)=P_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)+R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)$, in order to evaluate the remainder term, using the Vandermonde convolution formula we notice that the bivariate Stancu operators (2.9) reproduce constants, such that

$$
R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\left(f(x,y)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right). \tag{3.19}
$$

Using the identity

$$
\begin{aligned}
f(x,y)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)=&\left(f\left(x,\tfrac{j}{n}\right)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)+\left(f\left(\tfrac{i}{m},y\right)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)\\
&+\left(f(x,y)-f\left(x,\tfrac{j}{n}\right)-f\left(\tfrac{i}{m},y\right)+f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)
\end{aligned}
$$

and taking into account (3.19), it follows

$$R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\left(f\left(x,\tfrac{j}{n}\right)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)$$
$$+\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\left(f\left(\tfrac{i}{m},y\right)-f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)$$
$$+\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\left(f(x,y)-f\left(x,\tfrac{j}{n}\right)-f\left(\tfrac{i}{m},y\right)+f\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right)$$
$$=S_1+S_2+S_3.$$

For S_1, it follows $S_1=\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\frac{mx-i}{m}\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$, next taking into account $mx-i=(m-i)(x+i\alpha)-i(1-x+(m-i)\alpha)$ and the following relation $t^{[i+j,h]}=t^{[i,h]}(t-ih)^{[j,h]}$, for any $i,j\in\mathbb{N}$, $h\neq 0$, we get

$$S_1=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\frac{(m-i)(x+i\alpha)-i(1-x+(m-i)\alpha)}{m}\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$=\sum_{i=0}^{m-1}\sum_{j=0}^{n}\binom{m-1}{i}\frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}}{1^{[m,-\alpha]}}p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$-\sum_{i=1}^{m}\sum_{j=0}^{n}\binom{m-1}{i-1}\frac{x^{[i,-\alpha]}(1-x)^{[m+1-i,-\alpha]}}{1^{[m,-\alpha]}}p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$=\sum_{i=0}^{m-1}\sum_{j=0}^{n}\binom{m-1}{i}\frac{x^{[1,-\alpha]}(1-x)^{[1,-\alpha]}(1+\alpha m)(x+\alpha)^{[i,-\alpha]}(1-x+\alpha)^{[m-1-i,-\alpha]}}{1^{[2,-\alpha]}(1+2\alpha)^{[m-1,-\alpha]}}p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$-\sum_{i=0}^{m-1}\sum_{j=0}^{n}\binom{m-1}{i}\frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}}{1^{[m,-\alpha]}}p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i+1}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$=\frac{x(1-x)(1+\alpha m)}{1+\alpha}\sum_{i=0}^{m-1}\sum_{j=0}^{n}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$-\frac{x(1-x)(1+\alpha m)}{1+\alpha}\sum_{i=0}^{m-1}\sum_{j=0}^{n}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i+1}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}$$
$$=\frac{x(1-x)(1+\alpha m)}{1+\alpha}\sum_{i=0}^{m-1}\sum_{j=0}^{n}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\left(\begin{bmatrix}x,\frac{i}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}-\begin{bmatrix}x,\frac{i+1}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}\right)$$
$$=-\frac{x(1-x)(1+\alpha m)}{m(1+\alpha)}\sum_{i=0}^{m-1}\sum_{j=0}^{n}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\begin{bmatrix}x,\frac{i}{m},\frac{i+1}{m} & ;f\\ \frac{j}{n} & \end{bmatrix}.$$

Analogously for S_2, it follows $S_2=\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\frac{ny-j}{n}\begin{bmatrix}\frac{i}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix}$, next taking into account $ny-j=(n-j)(y+j\beta)-j(1-y+(n-j)\beta)$, we get

$$S_2=\sum_{i=0}^{m}\sum_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\beta\rangle}(y)\tfrac{(n-j)(y+j\beta)-j(1-y+(n-j)\beta)}{n}\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$=\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)\tbinom{n-1}{j}\tfrac{y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[n,-\beta]}}\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$-\sum_{i=0}^{m}\sum_{j=1}^{n}p_{m,i}^{\langle\alpha\rangle}(x)\tbinom{n-1}{j-1}\tfrac{y^{[j,-\beta]}(1-y)^{[n+1-j,-\beta]}}{1^{[n,-\beta]}}\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$=\tfrac{y(1-y)(1+\beta n)}{1+\beta}\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$-\tfrac{y(1-y)(1+\beta n)}{1+\beta}\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j+1}{n}\end{array};f\right]$$
$$=\tfrac{y(1-y)(1+\beta n)}{1+\beta}\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left(\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]-\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j+1}{n}\end{array};f\right]\right)$$
$$=-\tfrac{y(1-y)(1+\beta n)}{n(1+\beta)}\sum_{i=0}^{m}\sum_{j=0}^{n-1}p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left[\begin{array}{c}\frac{i}{m}\\ y,\frac{j}{n},\frac{j+1}{n}\end{array};f\right].$$

For S_3, it follows $S_3=\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}p_{m,i}^{\langle\alpha\rangle}(x)p_{n,j}^{\langle\alpha\rangle}(y)\left(\frac{mx-i}{m}\right)\left(\frac{ny-j}{n}\right)\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$, next taking into account the identities $mx-i=(m-i)(x+i\alpha)-i(1-x+(m-i)\alpha)$, $ny-j=(n-j)(y+j\beta)-j(1-y+(n-j)\beta)$, we get

$$S_3=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\tbinom{m-1}{i}\tbinom{n-1}{j}\tfrac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}}\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$-\sum_{i=0}^{m-1}\sum_{j=1}^{n}\tbinom{m-1}{i}\tbinom{n-1}{j-1}\tfrac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j,-\beta]}(1-y)^{[n+1-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}}\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$-\sum_{i=1}^{m}\sum_{j=0}^{n-1}\tbinom{m-1}{i-1}\tbinom{n-1}{j}\tfrac{x^{[i,-\alpha]}(1-x)^{[m+1-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}}\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\tbinom{m-1}{i-1}\tbinom{n-1}{j-1}\tfrac{x^{[i,-\alpha]}(1-x)^{[m+1-i,-\alpha]}y^{[j,-\beta]}(1-y)^{[n+1-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}}\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$=T_1-T_2-T_3+T_4,$$

where

$$T_1=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\tbinom{m-1}{i}\tbinom{n-1}{j}\tfrac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}}\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right]$$
$$=\tfrac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{(1+\alpha)(1+\beta)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\left[\begin{array}{c}x,\frac{i}{m}\\ y,\frac{j}{n}\end{array};f\right].$$

$$T_2 = \sum_{i=0}^{m-1}\sum_{j=1}^{n} \binom{m-1}{i}\binom{n-1}{j-1} \frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j,-\beta]}(1-y)^{[n+1-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i}{m} & ; f \\ y, \frac{j}{n} & \end{bmatrix}$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \binom{m-1}{i}\binom{n-1}{j} \frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i}{m} & ; f \\ y, \frac{j+1}{n} & \end{bmatrix}$$

$$= \frac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s^{\langle\alpha\rangle}_{m-1,i}(x+\alpha)s^{\langle\beta\rangle}_{n-1,j}(y+\beta) \begin{bmatrix} x, \frac{i}{m} & ; f \\ y, \frac{j+1}{n} & \end{bmatrix}.$$

$$T_3 = \sum_{i=1}^{m}\sum_{j=0}^{n-1} \binom{m-1}{i-1}\binom{n-1}{j} \frac{x^{[i,-\alpha]}(1-x)^{[m+1-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i}{m} & ; f \\ y, \frac{j}{n} & \end{bmatrix}$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \binom{m-1}{i}\binom{n-1}{j} \frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i+1}{m} & ; f \\ y, \frac{j}{n} & \end{bmatrix}$$

$$= \frac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s^{\langle\alpha\rangle}_{m-1,i}(x+\alpha)s^{\langle\beta\rangle}_{n-1,j}(y+\beta) \begin{bmatrix} x, \frac{i+1}{m} & ; f \\ y, \frac{j}{n} & \end{bmatrix}.$$

$$T_4 = \sum_{i=1}^{m}\sum_{j=1}^{n} \binom{m-1}{i-1}\binom{n-1}{j-1} \frac{x^{[i,-\alpha]}(1-x)^{[m+1-i,-\alpha]}y^{[j,-\beta]}(1-y)^{[n+1-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i}{m} & ; f \\ y, \frac{j}{n} & \end{bmatrix}$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \binom{m-1}{i}\binom{n-1}{j} \frac{x^{[i+1,-\alpha]}(1-x)^{[m-i,-\alpha]}y^{[j+1,-\beta]}(1-y)^{[n-j,-\beta]}}{1^{[m,-\alpha]}1^{[n,-\beta]}} \begin{bmatrix} x, \frac{i+1}{m} & ; f \\ y, \frac{j+1}{n} & \end{bmatrix}$$

$$= \frac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s^{\langle\alpha\rangle}_{m-1,i}(x+\alpha)s^{\langle\beta\rangle}_{n-1,j}(y+\beta) \begin{bmatrix} x, \frac{i+1}{m} & ; f \\ y, \frac{j+1}{n} & \end{bmatrix}.$$

Using the last expressions of T_1, T_2, T_3, T_4 and the relation (2.13), it follows

$$S_3 = \frac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s^{\langle\alpha\rangle}_{m-1,i}(x+\alpha)s^{\langle\beta\rangle}_{n-1,j}(y+\beta) \begin{bmatrix} x, \frac{i}{m}, \frac{i+1}{m} & ; f \\ y, \frac{j}{n}, \frac{j+1}{n} & \end{bmatrix}.$$

Taking into account the last expressions of S_1, S_2 and S_3, it follows the desired equality. □

Let $C^{2,2}(S)$ be the space of bidimensional functions with continuous partial derivatives in S of order less than or equal to two. The upper bound estimation for the remainder term is given in:

Corollary 3.1. *If the function f has the following properties*
i) $f \in C^{2,2}(S)$,
ii) there exists $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ on S,
iii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on S, then the inequalities hold

(3.20)
$$\left| R^{\langle\alpha,\beta\rangle}_{m,n}(f;x,y) \right|$$
$$\le \frac{x(1-x)(1+\alpha m)}{2m(1+\alpha)} M_{2,0}[f] + \frac{y(1-y)(1+\beta n)}{2n(1+\beta)} M_{0,2}[f] + \frac{xy(1-x)(1-y)(1+\alpha m)(1+\beta n)}{4mn(1+\alpha)(1+\beta)} M_{2,2}[f]$$
$$\le \frac{1+\alpha m}{8m(1+\alpha)} M_{2,0}[f] + \frac{1+\beta n}{8n(1+\beta)} M_{0,2}[f] + \frac{(1+\alpha m)(1+\beta n)}{64mn(1+\alpha)(1+\beta)} M_{2,2}[f],$$

for $x \in [0,1]\setminus\left\{\frac{i}{m} \mid i=\overline{0,m}\right\}$, $y \in [0,1]\setminus\left\{\frac{j}{n} \mid j=\overline{0,n}\right\}$, $\alpha=\alpha(m)\to 0$ *as* $m\to\infty$, $\beta=\beta(n)\to 0$ *as* $n\to\infty$ *and* $m,n\in\mathbb{N}$, *where* $M_{2,0}[f] := \sup\limits_{(x,y)\in S}\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|$, $M_{0,2}[f] := \sup\limits_{(x,y)\in S}\left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|$, $M_{2,2}[f] := \sup\limits_{(x,y)\in S}\left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|$.

Proof. Applying the mean value theorem for the bivariate divided differences, it follows that exist $(\xi_1(i,j),\eta_1(i,j)),(\xi_2(i,j),\eta_2(i,j)),(\xi_3(i,j),\eta_3(i,j))\in(0,1)\times(0,1)$, such that

$$R_{m,n}^{\langle\alpha,\beta\rangle}(f;x,y) = -\frac{x(1-x)(1+m\alpha)}{m(1+\alpha)}\sum_{i=0}^{m-1}\sum_{j=0}^{n} s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)p_{n,j}^{\langle\beta\rangle}(y)\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\xi_1(i,j),\eta_1(i,j))$$
$$-\frac{y(1-y)(1+n\beta)}{n(1+\beta)}\sum_{i=0}^{m}\sum_{j=0}^{n-1} p_{m,i}^{\langle\alpha\rangle}(x)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\xi_2(i,j),\eta_2(i,j))$$
$$+\frac{xy(1-x)(1-y)(1+m\alpha)(1+n\beta)}{mn(1+\alpha)(1+\beta)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s_{m-1,i}^{\langle\alpha\rangle}(x+\alpha)s_{n-1,j}^{\langle\beta\rangle}(y+\beta)\frac{1}{4}\frac{\partial^4 f}{\partial x^2\partial y^2}(\xi_3(i,j),\eta_3(i,j)).$$

Next by using modulus, the fact that partial derivatives of function f are bounded on S and the Vandermonde convolution formula, one arrives at (3.20). □

Remark 3.1. For $\alpha=0$, respectively $\beta=0$ one remarks that the results of Theorem 3.1 and Corollary 3.1 reduce to the classical case of bivariate Bernstein operators, (see [3], [4], [5], [6]).

The study on the remainder term associated to the particular case of Stancu operators, can be done in analogous manner, i.e., starting with the approximation formula $f(x,y) = P_{m,n}^{\langle\frac{1}{m},\frac{1}{n}\rangle}(f;x,y) + R_{m,n}^{\langle\frac{1}{m},\frac{1}{n}\rangle}(f;x,y)$, it follows:

Theorem 3.2. *For* $x\in[0,1]\setminus\left\{\frac{i}{m}\mid i=\overline{0,m}\right\}$ *and* $y\in[0,1]\setminus\left\{\frac{j}{n}\mid j=\overline{0,n}\right\}$, *the representation of the remainder term for the particular case of Stancu operators, is given by*

$$R_{m,n}^{\langle\frac{1}{m},\frac{1}{n}\rangle}(f;x,y) = -\frac{2x(1-x)}{m+1}\sum_{i=0}^{m-1}\sum_{j=0}^{n} s_{m-1,i}^{\langle\frac{1}{m}\rangle}\left(x+\frac{1}{m}\right)p_{n,j}^{\langle\frac{1}{n}\rangle}(y)\begin{bmatrix} x,\frac{i}{m},\frac{i+1}{m} & ;f \\ \frac{j}{n} & \end{bmatrix} \tag{3.21}$$
$$-\frac{2y(1-y)}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n-1} p_{m,i}^{\langle\frac{1}{m}\rangle}(x)s_{n-1,j}^{\langle\frac{1}{n}\rangle}\left(y+\frac{1}{n}\right)\begin{bmatrix} \frac{i}{m} & ;f \\ y,\frac{j}{n},\frac{j+1}{n} & \end{bmatrix}$$
$$+\frac{4xy(1-x)(1-y)}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} s_{m-1,i}^{\langle\frac{1}{m}\rangle}\left(x+\frac{1}{m}\right)s_{n-1,j}^{\langle\frac{1}{n}\rangle}\left(y+\frac{1}{n}\right)\begin{bmatrix} x,\frac{i}{m},\frac{i+1}{m} & ;f \\ y,\frac{j}{n},\frac{j+1}{n} & \end{bmatrix},$$

where $s_{m-1,i}^{\langle\frac{1}{m}\rangle}\left(x+\frac{1}{m}\right) = \binom{m-1}{i}\frac{\left(x+\frac{1}{m}\right)^{\left[i,-\frac{1}{m}\right]}\left(1-x+\frac{1}{m}\right)^{\left[m-1-i,-\frac{1}{m}\right]}}{\left(1+\frac{2}{m}\right)^{\left[m-1,-\frac{1}{m}\right]}}$.

Proof. Using the same idea as in Theorem 3.1, we get the relation (3.21). □

The above result follows also from (3.18), if we take $\alpha=\frac{1}{m}$, respectively $\beta=\frac{1}{n}$.

Corollary 3.2. *If the function* f *has the following properties*
i) $f\in C^{2,2}(S)$,
ii) there exists $\frac{\partial^4 f}{\partial x^2\partial y^2}$ *on* S,

iii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* S*, then the inequalities hold*

$$\left| R_{m,n}^{\left\langle \frac{1}{m}, \frac{1}{n} \right\rangle}(f;x,y) \right| \leq \frac{x(1-x)}{m+1} M_{2,0}[f] + \frac{y(1-y)}{n+1} M_{0,2}[f] + \frac{xy(1-x)(1-y)}{(m+1)(n+1)} M_{2,2}[f] \tag{3.22}$$

$$\leq \frac{1}{4(m+1)} M_{2,0}[f] + \frac{1}{4(n+1)} M_{0,2}[f] + \frac{1}{16(m+1)(n+1)} M_{2,2}[f],$$

for $x \in [0,1] \setminus \left\{ \frac{i}{m} \mid i = \overline{0,m} \right\}$, $y \in [0,1] \setminus \left\{ \frac{j}{n} \mid j = \overline{0,n} \right\}$ *and* $m, n \in \mathbb{N}$*, where* $M_{2,0}[f]$, $M_{0,2}[f]$, $M_{2,2}[f]$ *were already presented.*

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