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Some properties for a general integral operator

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ABSTRACT. For certain classes of analytic functions in the open unit disk U, we study some convexity properties for a new general integral operator. Several corollaries of the main results are also considered.

1. INTRODUCTION

Let $U = \{z : |z| < 1\}$ be the unit disk and A be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U \tag{1.1}$$

which are analytic in U and satisfy the conditions

$$f(0) = f'(0) - 1 = 0.$$

We denote by S the subclass of A consisting of univalent functions on U.
A function $f \in A$ is a starlike function of order β, $0 \leq \beta < 1$ and we denote this class by $S^*(\beta)$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, z \in U$$

We denote by $K(\beta)$ the class of convex functions of order β, $0 \leq \beta < 1$ that satisfies the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \beta, z \in U$$

A function $f \in A$ is a convex function of complex order b, ($b \in \mathbb{C} \setminus \{0\}$) and type λ, ($0 \leq \lambda < 1$), if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > \lambda,\ z \in U.$$

We denote by $C^*_\lambda(b)$ the class of these functions.

A function $f \in A$ belongs to class $R(\beta)$, $0 \leq \beta < 1$, if

$$\mathrm{Re}(f'(z)) > \beta, z \in U$$

A function $f \in A$, is a starlike function of the complex order b, ($b \in \mathbb{C} \setminus \{0\}$) and type λ, ($0 \leq \lambda < 1$), if and only if

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$$\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>\lambda,\ z\in U.$$

We donote by $S^*_\lambda(b)$ the class of these functions.

F. Ronning introduce in [8] the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha>0,\beta\in[0,1)$ so:

We denote by $\mathcal{SP}(\alpha,\beta)$ the class of all functions $f\in S$ which satisfies the inequality

$$\left|\frac{zf'(z)}{f(z)}-(\alpha+\beta)\right|\le Re\frac{zf'(z)}{f(z)}+\alpha-\beta,\ z\in U. \tag{1.2}$$

Geometric interpretation is that $f\in\mathcal{SP}(\alpha,\beta)$ if and only if $zf'(z)/f(z)$, $z\in U$, takes all values in the parabolic region

$$\begin{aligned}\Omega_{\alpha,\beta}&=\{\omega:|\omega-(\alpha+\beta)|\le Re\omega+\alpha-\beta\}\\&=\{\omega=u+iv;v^2\le 4\alpha(u-\beta)\}\end{aligned}$$

Silverman defined in [9] the class G_b, so:

A function $f\in A$ is in the class G_b, $0<b\le 1$ if and only if

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<b\left|\frac{zf'(z)}{f(z)}\right|,\ z\in U \tag{1.3}$$

Uralegaddi (see [12]), Owa and Srivastava (see [4]) define the class $N(\beta)$, so:

A function $f\in A$ is in the class $N(\beta)$ if it verifies the inequality

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right)<\beta,\ z\in U;\beta>1 \tag{1.4}$$

2. Main results

In this paper, we introduce a new integral operator, defined by:

$$G_n(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(t)}{t}g_i'(t)\right)^{\alpha_i}dt \tag{2.5}$$

The integral operator defined above, generalizes integral operators introduced and studied by several authors:

Remark 2.1. If $f_i(z)=z$, for $i\in\{1,2...n\}$ from(2.5), we obtain the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z)=\int_0^z(g_1'(t))^{\alpha_1}\ldots(g_n'(t))^{\alpha_n}dt,$$

introduced and studied by D. Breaz et all in [1].

Remark 2.2. For $n=1$, $f(z)=z$, $g_1=g$, $\alpha_1=\gamma$ from (2.5), we obtain the integral operator

$$G(z)=\int_0^z(g'(t))^\gamma dt$$

studied in [5] and [7].

Remark 2.3. For $n = 1$, $f_1 = f$, $g_1 = g$, $g(z) = z$, $\alpha_1 = \alpha$ from (2.5), we obtain the integral operator

$$F_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$$

studied in [3] and [6].

Theorem 2.1. *Let $f_i \in S^*(\beta_i)$, $0 \le \beta_i < 1$ and $g_i \in K(\lambda_i)$, $0 \le \lambda_i < 1$ for $i \in \{1, 2, ...n\}$. If α_i are real numbers with $\alpha_i > 0$, for $i \in \{1, 2, ...n\}$ so that*

$$\sum_{i=1}^n \alpha_i(2 - \beta_i - \lambda_i) < 1.$$

In these conditions the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} g_i'(t)\right)^{\alpha_i} dt \tag{2.6}$$

is convex of order

$$1 + \sum_{i=1}^n \alpha_i(\beta_i - 2 + \lambda_i), \quad \textit{for} \quad i \in \{1, 2, ...n\}$$

Proof. We calculate the first and second order derivatives for G_n and we obtain:

$$G_n'(z) = \prod_{i=1}^n \left(\frac{f_i(z)}{z} g_i'(z)\right)^{\alpha_i}$$

and

$$G_n''(z) = \sum_{i=1}^n \alpha_i \left(\frac{f_i(z)}{z} g_i'(z)\right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z^2} g_i'(z) + \frac{f_i(z)}{z} g_i''(z)\right) \prod_{\substack{k=1 \\ k \neq i}}^n \left(\frac{f_k(z)}{z} g_k'(z)\right)^{\alpha_k} \tag{2.7}$$

We have:

$$\frac{z G_n''(z)}{G_n'(z)} = \sum_{i=1}^n \alpha_i \left[\left(\frac{z f_i'(z)}{f_i(z)} - 1\right) + \frac{z g_i''(z)}{g_i'(z)}\right] =$$

$$\sum_{i=1}^n \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \frac{z g_i''(z)}{g_i'(z)}. \tag{2.8}$$

Relation (2.8) is equivalent to:

$$\frac{z G_n''(z)}{G_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \frac{z g_i''(z)}{g_i'(z)} + 1$$

We calculate the real part of both terms in the above expression and we obtain:

$$\mathrm{Re}\left(\frac{z G_n''(z)}{G_n'(z)} + 1\right) = \sum_{i=1}^n \alpha_i \mathrm{Re}\left(\frac{z f_i'(z)}{f_i(z)}\right) - \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \mathrm{Re}\frac{z g_i''(z)}{g_i'(z)} + 1 =$$

$$= \sum_{i=1}^n \alpha_i \mathrm{Re}\frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \mathrm{Re}\left(\frac{z g_i''(z)}{g_i'(z)} + 1\right) - \sum_{i=1}^n \alpha_i + 1 \tag{2.9}$$

Since $f_i \in S^*(\beta_i)$, for $i \in \{1, 2, ...n\}$, so that $\text{Re}\frac{zf_i'(z)}{f_i(z)} > \beta_i$, $(i \in \{1, 2, ...n\})$ and functions $g_i \in K(\lambda_i)$, for $i \in \{1, 2, ...n\}$, so that $\text{Re}\left(\frac{zg_i''(z)}{g_i'(z)} + 1\right) > \lambda_i$, $(i \in \{1, 2, ..., n\})$, we have:

$$Re\left(\frac{zG_n''(z)}{G_n'(z)} + 1\right) > \sum_{i=1}^{n} \alpha_i\beta_i - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1 \tag{2.10}$$

$$= \sum_{i=1}^{n} \alpha_i\beta_i - 2\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i\lambda_i + 1$$

$$= \sum_{i=1}^{n} \alpha_i(\beta_i - 2 + \lambda_i) + 1.$$

So, the integral operator G_n is convex of order $1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 2 + \lambda_i)$, for $i \in \{1, 2, ...n\}$. □

If we consider $n = 1$ in Theorem 2.1, we get the following corollary:

Corollary 2.1. *Let $f \in S^*(\beta)$, with $0 \le \beta < 1$ and $g \in K(\lambda)$, $0 \le \lambda < 1$. Let α be a real number so that $\alpha > 0$ and $\alpha(2 - \lambda - \beta) < 1$. Then the integral operator*

$$G(z) = \int_0^z \left(\frac{f(t)}{t} g'(t)\right)^{\alpha} dt \tag{2.11}$$

is convex of order $1 + \alpha(\beta - 2 + \lambda)$.

Theorem 2.2. *Let $f_i, g_i \in A$, where $g_i \in G_{b_i}$, $0 < b_i \le 1$, for $i \in \{1, 2, ...n\}$. For any $M_i \ge 1$, which verify*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \le M_i\ , \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1, \text{ for all } z \in U. \tag{2.12}$$

there are α_i real numbers , with $\alpha_i > 0$ so that

$$\lambda = 1 - \sum_{i=1}^{n} \alpha_i(2 + M_i + 2b_i) > 0.$$

In these conditions, the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} g_i'(t)\right)^{\alpha_i} dt \tag{2.13}$$

is in the class $K(\lambda)$.

Proof. After the same steps as in the proof of Theorem 2.1, we get:

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 + \frac{zg_i''(z)}{g_i'(z)}\right) =$$

$$\sum_{i=1}^{n} \left[\alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \alpha_i \left(\frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1\right) + \alpha_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right)\right]$$

So, we have :

$$\left|\frac{zG_n''(z)}{G_n'(z)}\right| \le \sum_{i=1}^{n} \left[\alpha_i \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + \alpha_i \left|\frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1\right| + \alpha_i \left|\frac{zg_i'(z)}{g_i(z)} - 1\right|\right]$$

Since functions $g_i \in G_{b_i}$, $0 < b_i \leq 1$, for $i \in \{1, 2, ...n\}$, using inequalities (2.12) and (1.3), we get:

$$\left|\frac{zG''_n(z)}{G'_n(z)}\right| \leq \sum_{i=1}^{n}\left[\alpha_i(M_i+1)+\alpha_i b_i\left|\frac{zg'_i(z)}{g_i(z)}\right|+\alpha_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right] \leq \tag{2.14}$$

$$\leq \sum_{i=1}^{n}\left[\alpha_i(M_i+1)+\alpha_i b_i\left(\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+1\right)+\alpha_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right] \leq$$

$$\leq \sum_{i=1}^{n}\left[\alpha_i(M_i+1)+\alpha_i b_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+\alpha_i b_i+\alpha_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right]$$

$$\leq \sum_{i=1}^{n}\left[\alpha_i(M_i+1)+(\alpha_i b_i+\alpha_i)\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+\alpha_i b_i\right] \leq$$

$$\leq \sum_{i=1}^{n}\left[\alpha_i(M_i+1)+2\alpha_i b_i+\alpha_i\right] \leq \sum_{i=1}^{n}\alpha_i(2+M_i+2b_i)=1-\lambda.$$

So, the integral operator G_n is in the class $K(\lambda)$. □

If we consider $n = 1$ in Theorem 2.2, we get the following corollary:

Corollary 2.2. *Let $f, g \in A$, where $g \in G_b$, $0 < b \leq 1$. For any $M \geq 1$, verify the next conditions*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M\ ,\left|\frac{zg'(z)}{g(z)}-1\right| < 1, \text{ for all } z \in U, \tag{2.15}$$

$\alpha > 0$, real number, with

$$\lambda = 1 - \alpha(2 + M + 2b) > 0.$$

In these conditions, the integral operator

$$G(z) = \int_0^z \left(\frac{f(t)}{t}g'(t)\right)^{\alpha} dt$$

is in the class $K(\lambda)$.

Theorem 2.3. *Let functions $f_i \in \mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$ and $g_i \in K(\lambda_i)$, $0 \leq \lambda_i < 1$ for $i \in \{1, 2, ..., n\}$. If $\alpha_i \in \mathbb{R}$, with $\alpha_i > 0$ for $i \in \{1, 2, ..., n\}$ so that*

$$\rho = 1 + \sum_{i=1}^{n}\alpha_i(\beta - \alpha - 2 + \lambda_i) = 1 + (\beta - \alpha - 2)\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\alpha_i\lambda_i > 0.$$

In these conditions, the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}g'_i(t)\right)^{\alpha_i} dt \tag{2.16}$$

is in the class $K(\rho)$.

Proof. We calculate the first and second order derivatives for $G_n(z)$ and we obtain:

$$\frac{zG_n''(z)}{G_n'(z)}+1=\sum_{i=1}^{n}\alpha_i\left[\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\frac{zg_i''(z)}{g_i'(z)}\right]+1=$$
$$\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i\frac{zg_i''(z)}{g_i'(z)}+1=$$
$$\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}+\alpha-\beta\right)+$$
$$+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i\left(\frac{zg_i''(z)}{g_i'(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1.$$

We calculate the real part of both terms in the above expression and we obtain:

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\left(\mathrm{Re}\frac{zf_i'(z)}{f_i(z)}+\alpha-\beta\right)+$$

$$+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg_i''(z)}{g_i'(z)}+1\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{2.17}$$

Since $f_i\in\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1)$ and $g_i\in K(\lambda_i)$, $0\le\lambda_i<1$, for $i\in\{1,2,...,n\}$, using the relation (1.2) in (2.17) , we get:
(2.18)

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right)\ge\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-(\alpha+\beta)\right|+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i\lambda_i-\sum_{i=1}^{n}\alpha_i+1.$$

Since $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-(\alpha+\beta)\right|>0$ for $i\in\{1,2,...n\}$, using inequality (2.18), we have:

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)}+1\right)\ge(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i(\lambda_i-1)+1\ge$$
$$\sum_{i=1}^{n}\alpha_i(\beta-\alpha-2+\lambda_i)+1>0$$

So, the integral operator G_n is in the class $K(\rho)$. □

If we consider $n=1$ in Theorem 2.3 we obtain the following corollary:

Corollary 2.3. *Let functions $f\in\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1)$ and $g\in K(\lambda)$, $0\le\lambda<1$. If there is the number $\alpha\in\mathbb{R}$, with $\alpha>0$ so that*

$$\rho=1+(\beta-\alpha-2+\lambda)\alpha>0.$$

In these conditions, the integral operator

$$G(z)=\int_0^z\left(\frac{f(t)}{t}g'(t)\right)^{\alpha}dt \tag{2.19}$$

is in the class $K(\rho)$.

Theorem 2.4. *Let $f_i \in S^*_{\lambda_i}(b)$, $g_i \in C_{\lambda_i}(b)$, with $0 \le \lambda_i < 1$ for $i \in \{1,2,...n\}$ and $b \in \mathbb{C}-\{0\}$. Also, let $\alpha_i (i \in \{1,2,....n\})$ be real numbers, with $\alpha_i > 0$ for $i \in \{1,2,....n\}$. If*

$$0 \le 1 + \sum_{i=1}^{n} 2\alpha_i(\lambda_i - 1) < 1, \tag{2.20}$$

then the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} g_i'(t)\right)^{\alpha_i} dt \tag{2.21}$$

is in class the $C_\mu(b)$, with $\mu = 1 + \sum_{i=1}^{n} 2\alpha_i(\lambda_i - 1)$, for $i \in \{1,2,...n\}$.

Proof. After the same steps with previous theorems, we obtain:

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \alpha_i \left[\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \frac{zg_i''(z)}{g_i'(z)}\right] \tag{2.22}$$

Multiplying relation (2.22) with $\frac{1}{b}$ we get:

$$\frac{1}{b}\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \alpha_i \left[\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \frac{1}{b}\frac{zg_i''(z)}{g_i'(z)}\right] = \tag{2.23}$$

$$= \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right) - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{b}\frac{zg_i''(z)}{g_i'(z)}\right) - \sum_{i=1}^{n} \alpha_i.$$

Relation (2.23) is equivalent to:

$$1 + \frac{1}{b}\frac{zG_n''(z)}{G_n'(z)} = \tag{2.24}$$

$$1 + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right) - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \left(1 + \frac{1}{b}\frac{zg_i''(z)}{g_i'(z)}\right) - \sum_{i=1}^{n} \alpha_i.$$

Since $f_i \in S^*_{\lambda_i}(b)$ and $g_i \in C_{\lambda_i}(b)$, for $i \in \{1,2,...,n\}$, we have:

$$\operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right) > \lambda_i \ (z \in U), \ \operatorname{Re}\left(1 + \frac{1}{b}\left(\frac{zg_i''(z)}{g_i'(z)}\right)\right) > \lambda_i \ (z \in U) \tag{2.25}$$

So, we get:

$$\operatorname{Re}\left(1 + \frac{1}{b}\frac{zG_n''(z)}{G_n'(z)}\right) =$$

$$1 + \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right\} - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left\{1 + \frac{1}{b}\frac{zg_i''(z)}{g_i'(z)}\right\} - \sum_{i=1}^{n} \alpha_i >$$

$$> 1 + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i \ge$$

$$\ge 1 + 2\sum_{i=1}^{n} \alpha_i\lambda_i - 2\sum_{i=1}^{n} \alpha_i \ge 1 + \sum_{i=1}^{n} 2\alpha_i(\lambda_i - 1).$$

Since $0 \le 1 + \sum_{i=1}^{n} 2\alpha_i(\lambda_i - 1) < 1$, we get that the integral operator G_n defined by (2.25) is in the class $C_\mu(b)$, where $\mu = 1 + \sum_{i=1}^{n} 2\alpha_i(\lambda_i - 1)$. □

If we consider $n = 1$ in Theorem 2.4, we get the following corollary:

Corollary 2.4. *Let $f \in S^*_\lambda(b)$, $g \in C_\lambda(b)$, with $0 \leq \lambda < 1$ and $b \in \mathbb{C} - \{0\}$. Also, let α be a real number, with $\alpha > 0$. If*

$$0 \leq 1 + 2\alpha(\lambda - 1),$$

then the integral operator

$$G(z) = \int_0^z \left(\frac{f(t)}{t} g'(t) \right)^\alpha dt$$

is in the class $C_\mu(b)$, with

$$\mu = 1 + 2\alpha(\lambda - 1).$$

Theorem 2.5. *Let $f_i, g_i \in A$, where $g_i \in N(\lambda_i)$, with $\lambda_i > 1$ for $i \in \{1, 2, ...n\}$. For any $\lambda_i > 1$, f_i verifying conditions*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq 1, z \in U \tag{2.26}$$

there are numbers $\alpha_i \in \mathbb{R}$, with $\alpha_i > 0$ so that $\mu = \sum_{i=1}^n \alpha_i \lambda_i + 1$ for $i \in \{1, 2, ...n\}$. In these conditions, the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} g_i'(t) \right)^{\alpha_i} dt \tag{2.27}$$

is in the class $N(\mu)$.

Proof. From the previous theorems, we obtain:

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^n \alpha_i \left[\left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \frac{z g_i''(z)}{g_i'(z)} \right] \tag{2.28}$$

From (2.28), we get:

$$\frac{zG_n''(z)}{G_n'(z)} + 1 = \sum_{i=1}^n \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \sum_{i=1}^n \alpha_i \left(\frac{z g_i''(z)}{g_i'(z)} + 1 \right) - \sum_{i=1}^n \alpha_i + 1 \tag{2.29}$$

We calculate the real part for both terms in the above expression and we get:

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1 \right) = \sum_{i=1}^n \alpha_i \mathrm{Re} \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \sum_{i=1}^n \alpha_i \mathrm{Re} \left(\frac{z g_i''(z)}{g_i'(z)} + 1 \right) - \sum_{i=1}^n \alpha_i + 1. \tag{2.30}$$

Since $g_i \in N(\lambda_i)$, for $i \in \{1, 2, ...n\}$, we have:

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1 \right) \leq \sum_{i=1}^n \alpha_i \mathrm{Re} \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) + \sum_{i=1}^n \alpha_i \lambda_i - \sum_{i=1}^n \alpha_i + 1 \tag{2.31}$$

Since $\mathrm{Re}\omega \leq |\omega|$ and applying the condition (2.26) from the hypothesis of the theorem, we get:

$$\mathrm{Re}\left(\frac{zG_n''(z)}{G_n'(z)} + 1 \right) \leq \sum_{i=1}^n \alpha_i \left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^n \alpha_i \lambda_i - \sum_{i=1}^n \alpha_i + 1 \leq$$
$$\leq \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \lambda_i - \sum_{i=1}^n \alpha_i + 1 \leq \sum_{i=1}^n \alpha_i \lambda_i + 1.$$

So, $G_n \in N(\mu)$, where $\mu = \sum_{i=1}^n \alpha_i \lambda_i + 1$. □

Remark 2.4. If in the integral operator

$$F_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} (g_i'(t))^{\gamma_i}\, dt,$$

introduced and studied by D. Breaz and L. Stanciu in [2] and studied by L. Stanciu in [10], [11] $\alpha_i = \gamma_i$, then we obtain the integral operator G_n defined by (2.5).

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