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The order of convexity for an integral operator

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ABSTRACT. In this paper we consider an integral operator for analytic functions in the open unit disk and we derive the order of convexity for this integral operator, on certain classes of univalent functions.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, $f(0) = f'(0)-1 = 0$ and $\mathcal{S}$ be the subclass of univalent functions in the class $\mathcal{A}$. We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions by the order α, $0 \le \alpha < 1$. If $f \in \mathcal{S}^*(\alpha)$, then f verify the inequality

$$Re\frac{zf'(z)}{f(z)} > \alpha, \quad (z \in \mathcal{U}). \tag{1.1}$$

We denote with $\mathcal{K}(\alpha)$ the class of convex functions by the order α, $0 \le \alpha < 1$. The function $f \in \mathcal{K}(\alpha)$ verify the inequality

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in \mathcal{U}). \tag{1.2}$$

A function $f \in \mathcal{K}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$.

Petru T. Mocanu [4] defines the class of α-convex functions, which is denoted M_α, α be a real number. If the function $f \in M_\alpha$, then $f(0) = f'(0) - 1 = 0$ and f verifies the inequality

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \tag{1.3}$$

for all $z \in \mathcal{U}$.

J. Stankiewicz and A. Wisniowska [8] had introduced the class of univalent functions $\mathcal{SH}(\beta)$, for some $\beta > 0$. If $f \in \mathcal{SH}(\beta)$, then f verifies the next inequality:

$$Re\left(\sqrt{2}\frac{zf'(z)}{f(z)}\right) + 2\beta(\sqrt{2}-1) > \left|\frac{zf'(z)}{f(z)} - 2\beta(\sqrt{2}-1)\right|, \tag{1.4}$$

for some $\beta > 0$, $f \in \mathcal{S}$ and for all $z \in \mathcal{U}$.

F. Ronning [7] had defined the class of univalent functions denoted by $\mathcal{SP}$. The function $f \in \mathcal{S}$ is in $\mathcal{SP}$ if and only if

$$Re\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.5}$$

for all $z \in \mathcal{U}$.

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In the paper [7], F. Ronning introduced the class of univalent functions $\mathcal{SP}(\alpha,\beta)$, $\alpha>0$, $\beta\in[0,1]$, the class of all functions $f\in\mathcal{S}$ which have the property

$$Re\frac{zf'(z)}{f(z)}+\alpha-\beta\geq\left|\frac{zf'(z)}{f(z)}-(\alpha+\beta)\right|, \tag{1.6}$$

for all $z\in\mathcal{U}$.

Y. J. Kim and E. P. Merkes [3] defined the integral operator:

$$F_\alpha(z)=\int_0^z\left(\frac{f(u)}{u}\right)^\alpha du, \tag{1.7}$$

for α be a complex number and $f\in\mathcal{S}$.

J. Pfaltzgraff [6] introduced the integral operator:

$$G_\beta(z)=\int_0^z\left(f'(u)\right)^\beta du, \tag{1.8}$$

for β be a complex number and $f\in\mathcal{S}$.

The functions $F_\alpha(z)$ and $G_\beta(z)$ are regular functions in $\mathcal{U}$ and satisfies the normalization conditions $F_\alpha(0)=F'_\alpha(0)-1=0$ and $G_\beta(0)=G'_\beta(0)-1=0$.

Properties of certain integral operators were study by different authors in the following papers [1, 2, 5, 9, 10].

In this paper we consider the integral operator $I_{\alpha,\beta}$, which is defined by

$$I_{\alpha,\beta}(z)=\int_0^z\left(\frac{f(u)}{u}\right)^\alpha\cdot\left(f'(u)\right)^\beta du, \tag{1.9}$$

for α,β be complex numbers and $f\in\mathcal{A}$.

The function $I_{\alpha,\beta}(z)$ is regular in $\mathcal{U}$ normalized with the conditions
$I_{\alpha,\beta}(0)=I'_{\alpha,\beta}(0)-1=0$.

We investigate the conditions for convexity of the integral operator $I_{\alpha,\beta}$ and we determine the order of convexity of this integral operator, for the functions from the classes considered.

2. Main results

Theorem 2.1. *Let $\alpha,\ \beta$ be real numbers with the properties $\alpha\geq0,\ \beta\geq0$ and*

$$0<\alpha+\beta<1. \tag{2.10}$$

If the functions $f\in\mathcal{S}^*(\alpha)$ and $z\cdot f'\in\mathcal{S}^*(\beta)$, then the integral operator $I_{\alpha,\beta}$ defined in (1.9) is convex by the order $\alpha^2+\beta^2-\alpha-\beta+1$.

Proof. We have

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}=\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+\beta\frac{zf''(z)}{f'(z)}, \tag{2.11}$$

for all $z\in\mathcal{U}$. From (2.11) we obtain

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1=\alpha\left(\frac{zf'(z)}{f(z)}-1\right)+\beta\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1, \tag{2.12}$$

$z\in\mathcal{U}$, and hence, we get

$$\begin{aligned}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right) &= \alpha Re\left(\frac{zf'(z)}{f(z)}\right)-\alpha+\\ &+ \beta Re\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1.\end{aligned} \tag{2.13}$$

Since $f \in \mathcal{S}^*(\alpha)$ and $z \cdot f' \in \mathcal{S}^*(\beta)$, we obtain

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right) > \alpha^2+\beta^2-\alpha-\beta+1 \tag{2.14}$$

and by hypothesis (2.10), it results that $I_{\alpha,\beta}$ is convex function by the order $\alpha^2+\beta^2-\alpha-\beta+1$. □

Corollary 2.1. *Let α be a real number, $0<\alpha<1$. If the function $f \in \mathcal{S}^*(\alpha)$, then the integral operator $F_\alpha(z)$ defined in (1.7) is convex by the order $1-\alpha+\alpha^2$.*

Proof. For $\beta = 0$ in Theorem 2.1, we obtain the Corollary 2.1. □

Corollary 2.2. *Let β be a real number, $0<\beta<1$. If the function $zf' \in \mathcal{S}^*(\beta)$, then the function $G_\beta(z)$ defined in (1.8) is convex function by the order $1-\beta+\beta^2$.*

Proof. We take $\alpha = 0$ in Theorem 2.1. □

Theorem 2.2. *Let $\alpha,\ \beta$ be real numbers with the properties $\alpha \geq 0,\ \beta \geq 0$ and*

$$0 < \alpha+\beta < 1. \tag{2.15}$$

We suppose that the functions $f \in \mathcal{SP}(\alpha)$ and $z \cdot f' \in \mathcal{SP}$, then the function $I_{\alpha,\beta}$ defined in (1.9) is convex function by the order $1-\alpha-\beta$.

Proof. We have

$$\begin{aligned} Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right) &= \alpha Re\left(\frac{zf'(z)}{f(z)}\right)-\alpha+ \\ &+ \beta Re\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1. \end{aligned} \tag{2.16}$$

Because $f \in \mathcal{SP}$ and $z \cdot f' \in \mathcal{SP}$, we apply in the relation (2.16), we obtain

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right) > \alpha\left|\frac{zf'(z)}{f(z)}-1\right|-\alpha+\beta\left|\frac{zf''(z)}{f'(z)}\right|-\beta+1$$

Because $\alpha\left|\frac{zf'(z)}{f(z)}-1\right| > 0$ and $\beta\left|\frac{zf''(z)}{f'(z)}\right| > 0$, for all $z \in \mathcal{U}$, we obtain that

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right) > 1-\alpha-\beta. \tag{2.17}$$

Using the hypothesis $\alpha+\beta<1$ in (2.17) we obtain that $I_{\alpha,\beta}$ is convex function by the order $1-\alpha-\beta$. □

Corollary 2.3. *Let α be a real number, $0<\alpha<1$. If $f \in \mathcal{SP}$, then the function $F_\alpha(z)$ defined in (1.7) is convex function by the order $1-\alpha$.*

Proof. For $\beta = 0$ in Theorem 2.2, we obtain Corollary 2.3. □

Corollary 2.4. *Let β be a real number, $0<\beta<1$. If $zf' \in \mathcal{SP}$, then the function $G_\beta(z)$ defined in (1.8) is convex function by the order $1-\beta$.*

Proof. We take $\alpha = 0$ in Theorem 2.2. □

Theorem 2.3. *Let $\alpha,\ \beta$ be real numbers with the properties $\alpha \geq 0,\ \beta \geq 0$, $\alpha = 1-\beta$, $f \in M_\alpha$, then the integral operator $I_{\alpha,\beta}$ is convex.*

Proof. From (2.13), for $\alpha = 1 - \beta$, we have

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) = (1-\beta)Re\left(\frac{zf'(z)}{f(z)}\right) + \beta Re\left(\frac{zf''(z)}{f'(z)} + 1\right),$$

and since $f \in M_\beta$, we obtain

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) > 0, \quad (z \in \mathcal{U}),$$

hence, it results that the integral operator $I_{\alpha,\beta}$ is convex. □

Theorem 2.4. *Let $\alpha,\ \beta,\ \gamma,\ \delta$ be real numbers $\alpha \geq 0,\ \beta \geq 0$, $\gamma \in (0,1)$, $\delta > 0$, $f \in \mathcal{K}(\gamma)$ and $f \in \mathcal{SH}(\delta)$.*

If

$$0 < \sqrt{2}\alpha\delta + \beta\gamma + 1 - 2\alpha\delta - \alpha - \beta < 1 \tag{2.18}$$

then the integral operator $I_{\alpha,\beta}$ is convex by the order $\sqrt{2}\alpha\delta + \beta\gamma + 1 - 2\alpha\delta - \alpha - \beta$.

Proof. From (2.12) we have

$$\sqrt{2}\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) = \sqrt{2}\alpha\frac{zf'(z)}{f(z)} - \sqrt{2}\alpha + \sqrt{2}\beta\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}\beta + \sqrt{2},$$

hence, we obtain

$$\sqrt{2}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) = \sqrt{2}\alpha Re\frac{zf'(z)}{f(z)} - \sqrt{2}\alpha +$$

$$+\sqrt{2}\beta Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}\beta + \sqrt{2}, \tag{2.19}$$

From 2.19 we get

$$\sqrt{2}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) = \alpha\left\{Re\left[\sqrt{2}\frac{zf'(z)}{f(z)}\right] + 2\delta(\sqrt{2}-1)\right\} - 2\alpha\delta(\sqrt{2}-1) -$$

$$-\sqrt{2}\alpha + \beta\sqrt{2}Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}\beta + \sqrt{2}. \tag{2.20}$$

Since, $f \in \mathcal{K}(\gamma)$ and $f \in \mathcal{SH}(\delta)$, by (2.20) we have

$$\sqrt{2}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) > \alpha\left|\frac{zf'(z)}{f(z)} - 2\delta(\sqrt{2}-1)\right| - 2\alpha\delta(\sqrt{2}-1) -$$

$$-\sqrt{2}\alpha + \beta\gamma\sqrt{2} - \sqrt{2}\beta + \sqrt{2}.$$

Because $\alpha\left|\frac{zf'(z)}{f(z)} - 2\delta(\sqrt{2}-1)\right| > 0$ we obtain that

$$\sqrt{2}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) > \sqrt{2}(\beta\gamma - 2\alpha\delta - \alpha - \beta + 1) + 2\alpha\delta$$

and we get

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) > \beta\gamma - 2\alpha\delta - \alpha - \beta + 1 + \sqrt{2}\alpha\delta \tag{2.21}$$

Using the hypothesis (2.18) and (2.21), it results that the integral operator $I_{\alpha,\beta}$ is convex by the order $\sqrt{2}\alpha\delta + \beta\gamma + 1 - 2\alpha\delta - \alpha - \beta$. □

Corollary 2.5. *Let α be a real number, $0 < \alpha < 1$, $\beta \in (0,1)$, $f \in \mathcal{K}(\beta)$ and $f \in \mathcal{SH}(\beta)$.*
If

$$0 < \alpha < \frac{\beta^2 - \beta + 1}{\beta(2 - \sqrt{2}) + 1}, \tag{2.22}$$

then the function $I_{\alpha,\beta}(z)$ is convex function by the order

$$\beta^2 - \beta + 1 - \alpha(2\beta + 1 - \sqrt{2}\beta).$$

Proof. For $\gamma = \beta$, $\delta = \beta$, $\beta \in (0,1)$, from Theorem 2.4, we obtain Corollary 2.5. □

Corollary 2.6. *Let α, β be real numbers, $\alpha \in (0,1)$, $f \in \mathcal{K}(\alpha)$ and $f \in \mathcal{SH}(\alpha)$.*
If

$$0 < \beta < \frac{1 + \alpha^2(\sqrt{2} - 2) - \alpha}{1 - \alpha}, \tag{2.23}$$

then the integral operator $I_{\alpha,\beta}(z)$ is convex, by the order

$$\alpha^2(\sqrt{2} - 2) - \alpha + 1 + \beta(\alpha - 1).$$

Proof. We take $\gamma = \alpha$, $\delta = \alpha$, $\alpha \in (0,1)$ in Theorem 2.4. □

Corollary 2.7. *Let α, β be real numbers, $\alpha \in (\frac{1}{2\sqrt{2}-1}, 1)$, $f \in \mathcal{K}(\alpha)$ and $f \in \mathcal{SH}(\beta)$.*
If

$$0 < \beta < \frac{\alpha}{(2\sqrt{2} - 1)\alpha - 1}, \tag{2.24}$$

then the integral operator $I_{\alpha,\beta}(z)$ is convex, by the order

$$\beta[(2\sqrt{2} - 1)\alpha - 1] - \alpha + 1.$$

Proof. For $\gamma = \alpha$, $\delta = \beta$, $\alpha \in (0,1)$, $\beta > 0$ from Theorem 2.4 we obtain the Corollary 2.7. □

Corollary 2.8. *Let α, β be real numbers, $\alpha > 0$, $\beta \in (0,1)$, $f \in \mathcal{K}(\beta)$ and $f \in \mathcal{SH}(\alpha)$.*
If

$$\beta^2 - \beta + 1 - (2 - \sqrt{2})\alpha^2 - \alpha > 0, \tag{2.25}$$

then the integral operator $I_{\alpha,\beta}(z)$ is convex, by the order

$$\beta^2 - \beta + 1 - (2 - \sqrt{2})\alpha^2 - \alpha.$$

Proof. We take $\gamma = \beta$, $\delta = \alpha$, $\beta \in (0,1)$, $\alpha > 0$ in Theorem 2.4. □

Theorem 2.5. *Let $\alpha, \beta, \gamma, \delta, \eta$ be real numbers, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta \in (0,1)$, $\eta \in [0,1)$, $f \in \mathcal{K}(\eta)$ and $f \in \mathcal{SH}(\gamma, \delta)$.*
If

$$0 < 1 - \alpha(\gamma - \delta + 1) + \beta(\eta - 1) < 1, \tag{2.26}$$

then the integral operator $I_{\alpha,\beta}(z)$ is convex, by the order

$$1 - \alpha(\gamma - \delta + 1) + \beta(\eta - 1).$$

Proof. From (2.13) we get

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)=\alpha\left[Re\frac{zf'(z)}{f(z)}+\gamma-\delta\right]-\alpha(\gamma-\delta)-\alpha+$$

$$+\beta Re\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1 \tag{2.27}$$

Since $f\in\mathcal{K}(\eta)$ and $f\in\mathcal{SH}(\gamma,\delta)$, we have

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)\geq\alpha\left|\frac{zf'(z)}{f(z)}-(\gamma+\delta)\right|-\alpha(\gamma-\delta)-\alpha+$$

$$+\beta\eta-\beta+1 \tag{2.28}$$

Because $\alpha\left|\frac{zf'(z)}{f(z)}-(\gamma+\delta)\right|>0$, we obtain that

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)>1-\alpha(\gamma-\delta+1)+\beta(\eta-1)$$

and using the hypothesis (2.26), it results that the integral operator $I_{\alpha,\beta}$ is convex by the order $1-\alpha(\gamma-\delta+1)+\beta(\eta-1)$. ☐

Corollary 2.9. *Let α,β be real numbers, $\alpha>0$, $\beta\in(0,1)$, $f\in\mathcal{K}(\beta)$ and $f\in\mathcal{SH}(\alpha,\beta)$. If*

$$0<1-\alpha(\alpha-\beta+1)+\beta(\beta-1)<1, \tag{2.29}$$

then the integral operator $I_{\alpha,\beta}(z)$ is convex, by the order $1-\alpha(\alpha-\beta+1)+\beta(\beta-1)$.

Proof. From Theorem 2.5, for $\gamma=\alpha$, $\delta=\eta=\beta$, $\alpha>0$, $\beta\in(0,1)$ we have the Corollary 2.9. ☐

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