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Five short lemmas in Banach spaces

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ABSTRACT. Consider a commutative diagram of bounded linear operators between Banach spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{J} & Y & \xrightarrow{Q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle A}\big\downarrow & & {\scriptstyle B}\big\downarrow & & {\scriptstyle C}\big\downarrow & & \\ 0 & \longrightarrow & X & \underset{J}{\longrightarrow} & Y & \underset{Q}{\longrightarrow} & Z & \longrightarrow & 0 \end{array}$$

with exact rows. In what ways are the spectral and local spectral properties of B related to those of the pairs of operators A and C? In this paper, we give our answers to this general question using tools from local spectral theory.

1. INTRODUCTION

In homological algebra, the short five lemma states that in abelian category, or in the category of groups, for the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{J_1} & Y_1 & \xrightarrow{Q_1} & Z_1 & \longrightarrow & 0 \\ & & {\scriptstyle A}\big\downarrow & & {\scriptstyle B}\big\downarrow & & {\scriptstyle C}\big\downarrow & & \\ 0 & \longrightarrow & X_2 & \underset{J_2}{\longrightarrow} & Y_2 & \underset{Q_2}{\longrightarrow} & Z_2 & \longrightarrow & 0 \end{array}$$

with exact rows, if A and C are isomorphisms, then B is an isomorphism as well.

The present work is concerned with the analogues in Banach category. More precisely, for the following commutative diagram of bounded linear operators between Banach spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{J} & Y & \xrightarrow{Q} & Z & \longrightarrow & 0 \\ & & {\scriptstyle A}\big\downarrow & & {\scriptstyle B}\big\downarrow & & {\scriptstyle C}\big\downarrow & & \\ 0 & \longrightarrow & X & \underset{J}{\longrightarrow} & Y & \underset{Q}{\longrightarrow} & Z & \longrightarrow & 0 \end{array} \tag{1.1}$$

with exact rows, what are the relationships between the (local) spectral properties of B and those of the pairs of operators A and C? In this paper, we give our answers to this general question using local spectral theory. For some pioneering work in this direction, we refer the reader to the seminal monograph by Laursen and Neumann [5, pp. 112-120 and p. 145].

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2. NOTATIONS

We first fix some notations in spectral theory. Throughout this paper, $\mathcal{B}(X)$ will denote the set of all bounded linear operators on a complex Banach space X. For an operator $T \in \mathcal{B}(X)$, let $\mathcal{N}(T)$ denote its kernel, $\alpha(T)$ its nullity, $\mathcal{R}(T)$ its range, $\beta(T)$ its defect, $\sigma(T)$ its spectrum, $\sigma_{ap}(T)$ its approximate point spectrum, $\sigma_{su}(T)$ its surjective spectrum and $\rho(T)$ its resolvent set. If the range $\mathcal{R}(T)$ of $T \in \mathcal{B}(X)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp. *lower semi-Fredholm*). If $T \in \mathcal{B}(X)$ is both upper and lower semi-Fredholm, then T is said to be *Fredholm*. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is said to be *semi-Fredholm*, and its index is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. The *upper semi-Weyl operators* (resp. *lower semi-Weyl operators*) are defined as the class of upper semi-Fredholm operators with index less than or equal to zero (resp. lower semi-Fredholm operators with index greater than or equal to zero), while *Weyl operators* are defined as the class of Fredholm operators of index zero. Recall that the *descent* and the *ascent* of $T \in \mathcal{B}(X)$ are $dsc(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and $asc(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, respectively. It is known that if $asc(T)$ and $dsc(T)$ are both finite, then they are equal ([1, Theorem 3.3]). We call an operator $T \in \mathcal{B}(X)$ *Drazin invertible* if $asc(T) = dsc(T) < \infty$. An operator $T \in \mathcal{B}(X)$ is called *upper semi-Browder* (resp. *lower semi-Browder*) if it is upper semi-Fredholm of finite ascent (resp. lower semi-Fredholm of finite desent), while T is called *Browder* if it is Fredholm of finite ascent and finite descent.

For $T \in \mathcal{B}(X)$, let us define the *upper semi-Fredholm spectrum*, *lower semi-Fredholm spectrum*, *essential spectrum*, *upper semi-Weyl spectrum*, *lower semi-Weyl spectrum*, *Weyl spectrum*, *upper semi-Browder spectrum*, *lower semi-Browder spectrum*, *Browder spectrum* and *Drazin spectrum* of T as follows respectively:

$$\sigma_{usf}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm}\},$$
$$\sigma_{lsf}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Fredholm}\},$$
$$\sigma_{e}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},$$
$$\sigma_{usw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Weyl}\},$$
$$\sigma_{lsw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Weyl}\},$$
$$\sigma_{w}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$
$$\sigma_{usb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Browder}\},$$
$$\sigma_{lsb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Browder}\},$$
$$\sigma_{b}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\},$$
$$\sigma_{d}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin}\}.$$

We next fix some notations in local spectral theory. Let $D(\lambda, r)$ be the open disc centred at $\lambda \in \mathbb{C}$ with radius $r > 0$. We say that $T \in \mathcal{B}(X)$ has the *single valued extension property* at $\lambda \in \mathbb{C}$ (SVEP at λ for brevity), if there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$, the only analytic function $f : U \to X$ which satisfies $(T - \mu)f(\mu) = 0$ for all $\mu \in U$ is the function $f \equiv 0$. Let $\mathcal{S}(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. An operator $T \in \mathcal{B}(X)$ is said to have SVEP if $\mathcal{S}(T) = \varnothing$.

Let $\mathcal{O}(U, X)$ denote the Fréchet algebra of all X-valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $T \in \mathcal{B}(X)$ is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{O}(U, X)$, $\lim_{n\to\infty}(T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$ implies $\lim_{n\to\infty} f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. We denote by $\sigma_\beta(T)$ the set where T fails to satisfy (β) and we say that T satisfies *Bishop's property* (β) if $\sigma_\beta(T) = \emptyset$.

3. Main results

We begin by the following local version of [5, Lemma 2.2.1].

Lemma 3.1. *Under the hypothesis* (1.1), *we have*
(1) $\mathcal{S}(A) \subseteq \mathcal{S}(B) \subseteq \mathcal{S}(A) \cup \mathcal{S}(C)$;
(2) $\sigma_\beta(A) \subseteq \sigma_\beta(B) \subseteq \sigma_\beta(A) \cup \sigma_\beta(C)$.

Proof. The first inclusions of (1) and (2) are evident, while the second inclusions of (1) and (2) can be seen along the lines of the argument in [5, Lemma 2.2.1]. □

In the next theorem, we relate the local spectral properties of B to those of A and C by using Gleason's theorem and Allan-Leiterer's theorem.

Theorem 3.1. *Under the hypothesis* (1.1), *we have*
(1) $\mathcal{S}(B) \cup \sigma_{su}(A) = \mathcal{S}(A) \cup \mathcal{S}(C) \cup \sigma_{su}(A)$;
(2) $\sigma_\beta(B) \cup \sigma_{su}(A) = \sigma_\beta(A) \cup \sigma_\beta(C) \cup \sigma_{su}(A)$.

Proof. We will prove (2), omitting the similar proof of (1).
(2) By Lemma 3.1, it suffices to show that

$$\sigma_\beta(C) \subseteq \sigma_\beta(B) \cup \sigma_{su}(A).$$

Let $\lambda \notin \sigma_\beta(B) \cup \sigma_{su}(A)$. Then there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$ and for any sequence $\{g_n\}_{n=1}^\infty \subseteq \mathcal{O}(U, Y)$, $\lim_{n\to\infty}(B-\mu)g_n(\mu) = 0$ in $\mathcal{O}(U, Y)$ implies $\lim_{n\to\infty} g_n(\mu) = 0$ in $\mathcal{O}(U, Y)$. We can take r sufficiently small such that $D(\lambda, r) \cap \sigma_{su}(A) = \varnothing$.

Let $U \subseteq D(\lambda, r)$ be open and $\{h_n\}_{n=1}^\infty \subseteq \mathcal{O}(U, Z)$ such that $\lim_{n\to\infty}(C-\mu)h_n(\mu) = 0$ in $\mathcal{O}(U, Z)$. By the version of Gleason's theorem for exact sequences ([5, Proposition 2.1.5]), we can find a sequence $\{g_n\}_{n=1}^\infty$ of analytic Y-valued functions such that

$$Qg_n(\mu) = h_n(\mu), \text{ for all } \mu \in U.$$

Hence $Q(B-\mu)g_n(\mu) = (C-\mu)Qg_n(\mu) = (C-\mu)h_n(\mu) \to 0$ in $\mathcal{O}(U, Z)$. By [5, Proposition 1.2.1], there exists $\{u_n\}_{n=1}^\infty$ of analytic Y-valued functions such that $Qu_n(\mu) = 0$ for all $\mu \in U$ and

$$(B-\mu)g_n(\mu) + u_n \to 0 \text{ in } \mathcal{O}(U, Y).$$

Again by the version of Gleason's theorem for exact sequences, we can find a sequence $\{v_n\}_{n=1}^\infty$ of analytic X-valued functions such that

$$Jv_n(\mu) = u_n(\mu), \text{ for all } \mu \in U.$$

It follows from Allan-Leiterer's theorem ([5, Theorem 3.2.1]) that we can find a sequence $\{f_n\}_{n=1}^\infty$ of analytic X-valued functions such that

$$(A-\mu)f_n(\mu) = v_n(\mu), \text{ for all } \mu \in U.$$

Thus $u_n(\mu) = Jv_n(\mu) = J(A-\mu)f_n(\mu) = (B-\mu)Jf_n(\mu)$ for all $\mu \in U$, and hence

$$(B-\mu)g_n(\mu) + (B-\mu)Jf_n(\mu) \to 0 \text{ in } \mathcal{O}(U, Y).$$

Since B satisfies Bishop's property (β) at λ,

$$g_n(\mu) + Jf_n(\mu) \to 0 \text{ in } \mathcal{O}(U, Y).$$

Therefore,

$$h_n(\mu) = Qg_n(\mu) = Qg_n(\mu) + QJf_n(\mu) \to 0 \text{ in } \mathcal{O}(U, Z).$$

This shows that $\lambda \notin \sigma_\beta(C)$. □

We remark that the key ingredient of the following lemma is the index equality

$$\text{ind}B = \text{ind}A + \text{ind}C.$$

Lemma 3.2. ([8]) *Under the hypothesis* (1.1), *then*

(1) *if both A and C are upper semi-Fredholm, so does B. In this case,*

$$\alpha(A) \leq \alpha(B) \leq \alpha(A) + \alpha(C)$$

and
$$\text{ind}B = \text{ind}A + \text{ind}C.$$

(2) *if both A and C are lower semi-Fredholm, so does B. In this case,*

$$\beta(C) \leq \beta(B) \leq \beta(A) + \beta(C)$$

and
$$\text{ind}B = \text{ind}A + \text{ind}C.$$

The next lemma concerns the ascent and descent of A, B and C.

Lemma 3.3. *Under the hypothesis* (1.1), *we have*

(1) $asc(A) \leq asc(B) \leq asc(A) + asc(C)$;

(2) $dsc(C) \leq dsc(B) \leq dsc(A) + dsc(C)$.

Proof. (1) Clearly, $asc(A) \leq asc(B)$. To show $asc(B) \leq asc(A) + asc(C)$, we may suppose that $asc(A) = p < \infty$ and $asc(C) = q < \infty$. Let $y \in \mathcal{N}(B^{p+q+1})$. Then $C^{p+q+1}Qy = QB^{p+q+1}y = 0$ and, since $asc(C) = q$, $C^qQy = 0$. Hence $QB^qy = C^qQy = 0$, so $B^qy \in \mathcal{N}(Q) = \mathcal{R}(J)$. Choose $x \in X$ for which $B^qy = Jx$. Thus $JA^{p+1}x = B^{p+1}Jx = B^{p+1}B^qy = 0$. Therefore, $A^{p+1}x = 0$. Since $asc(A) = p$, $A^px = 0$. Consequently, $B^{p+q}y = B^pJx = JA^px = 0$. This shows that $asc(B) \leq p + q$.

(2) Clearly, $dsc(C) \leq dsc(B)$. To show $dsc(B) \leq dsc(A) + dsc(C)$, we may suppose that $dsc(A) = p < \infty$ and $dsc(C) = q < \infty$. Let $y \in \mathcal{R}(B^{p+q})$. Then there exists $y_1 \in Y$ such that $y = B^{p+q}y_1$ and since $dsc(C) = q$ and Q is surjective,

$$QB^qy_1 = C^qQy_1 = C^{q+1}Qy_2 = QB^{q+1}y_2$$

for some $y_2 \in Y$. Thus $B^qy_1 - B^{q+1}y_2 \in \mathcal{N}(Q) = \mathcal{R}(J)$, and so

$$B^qy_1 - B^{q+1}y_2 = Jx_1$$

for some $x_1 \in X$. Because $dsc(A) = p < \infty$, we conclude that

$$\begin{aligned} B^{p+q}y_1 - B^{p+q+1}y_2 &= B^p(B^qy_1 - B^{q+1}y_2) \\ &= B^pJx_1 \\ &= JA^px_1 \\ &= JA^{p+q+1}x_2 \\ &= B^{p+q+1}Jx_2 \end{aligned}$$

for some $x_2 \in X$. Consequently, $y = B^{p+q}y_1 = B^{p+q+1}y_2 + B^{p+q+1}Jx_2 \in \mathcal{R}(B^{p+q+1})$. This shows that $dsc(B) \leq p + q$. □

Recall that an operator $T \in \mathcal{B}(X)$ is called *generalized Drazin invertible* if $0 \notin \text{acc}\sigma(T)$, where for a subset $K \subseteq \mathbb{C}$, $\text{acc}K$ stands for accumulation points of K.

Lemma 3.4. *Under the hypothesis* (1.1), *then if any two of A, B and C are generalized Drazin invertible, so is the third one.*

Proof. Suppose that B and C are generalized Drazin invertible. Then $0 \notin \text{acc}\sigma(B)$ and $0 \notin \text{acc}\sigma(C)$. Hence there exists a deleted neighborhood $D = \{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon\}$ of 0 for which $D \subseteq \rho(B) \cap \rho(C)$. Thus by Lemma 3.2, we know that $\alpha(A-\lambda) \leq \alpha(B-\lambda) = 0$ and that $\text{ind}(A-\lambda) = \text{ind}(B-\lambda) - \text{ind}(C-\lambda) = 0$ for all $\lambda \in D$. Hence $\alpha(A-\lambda) = \beta(A-\lambda) = 0$ for all $\lambda \in D$, and thus $D \subseteq \rho(A)$. Consequently $0 \notin \text{acc}\sigma(A)$, that is A is generalized Drazin invertible.

The other conclusions follow by a similar argument. □

Remark 3.1. It is interesting to note that similar results to Lemma 3.4 hold for invertibility, Fredholmness, Weylness, Browderness. Indeed, this can be seen by the proof given in Lemma 3.4 with minor changes.

The following lemma shows that similar result to Lemma 3.4 also holds for Drazin invertibility. But for this, some notations and fundamental facts are needed.

Associated with $T \in \mathcal{B}(X)$, two important subspaces of X are the *analytic core* of T defined by

$$\begin{aligned} K(T) :=\{x \in X : &\text{ there exist a sequence } \{x_n\}_{n\geq 1} \text{ in } X \text{ and a constant } \delta > 0 \\ &\text{such that } Tx_1 = x, Tx_{n+1} = x_n \text{ and } ||x_n|| \leq \delta^n ||x|| \text{ for all } n \geq 1\}, \end{aligned}$$

and the *quasi-nilpotent part* of T defined by

$$H_0(T) := \{x \in X : \lim_{n\to\infty} ||T^n x||^{1/n} = 0\}.$$

These subspaces which are introduced and studied by Mbekhta in [6], play an important role in local spectral theory. Some basic facts about these two subspaces we will need later are collected as follows (see [1, 6, 7]):

(i) $\mathcal{N}(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N}$;

(ii) $TK(T) = K(T)$;

(iii) $0 \notin \text{acc}\sigma(T)$ if and only if $X = H_0(T) \oplus K(T)$, where the direct sum is topological. In this case, $T|_{H_0(T)}$ is quasi-nilpotent and $T|_{K(T)}$ is invertible.

Lemma 3.5. *Under the hypothesis* (1.1), *then if any two of A, B and C are Drazin invertible, so is the third one.*

Proof. Suppose that A and C are Drazin invertible. Then Lemma 3.3 implies that B is Drazin invertible.

Suppose that A and B are Drazin invertible. Then $dsc(C) < \infty$ and by Lemma 3.1, we know that C is generalized Drazin invertible, and so $0 \notin \text{acc}\sigma(C)$. Thus by [1, Theorem 3.81], we infer that C is Drazin invertible.

Now suppose that B and C are Drazin invertible. Observe that Drazin invertibility of B implies that there exists $p \in \mathbb{N}$ such that $H_0(B) = \mathcal{N}(B^p)$. It follows form

$$JH_0(A) \subseteq H_0(B) \cap \mathcal{R}(J) = \mathcal{N}(B^p) \cap \mathcal{R}(J) = J\mathcal{N}(A^p)$$

and injectivity of J that $H_0(A) = \mathcal{N}(A^p)$. By Lemma 3.1, we infer that A is generalized Drazin invertible, and so $0 \notin \text{acc}\sigma(A)$. Hence $X = H_0(T) \oplus K(T) = \mathcal{N}(A^p) \oplus K(T)$, from which it follows that

$$X = \mathcal{N}(A^p) \oplus \mathcal{R}(A^p).$$

Consequently, A is Drazin invertible. □

In the next theorem, with the help of the single valued extension property, we relate the spectral properties of B to those of A and C.

Theorem 3.2. *Under the hypothesis* (1.1), *we have*

(1) $\sigma(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) = \sigma(A) \cup \sigma(C)$;

(2) $\sigma_b(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) = \sigma_b(A) \cup \sigma_b(C)$;

(3) $\sigma_d(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) = \sigma_d(A) \cup \sigma_d(C)$;

(4) $\sigma_e(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) = \sigma_e(A) \cup \sigma_e(C) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$;

(5) $\sigma_w(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) = \sigma_w(A) \cup \sigma_w(C) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$.

Proof. (1) By Remark 3.1, we infer that $\sigma(B) \subseteq \sigma(A) \cup \sigma(C)$, and hence

$$\sigma(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) \subseteq \sigma(A) \cup \sigma(C).$$

Conversely, let $\lambda \notin \sigma(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$. Then $A - \lambda$ is bounded below and $C - \lambda$ is surjective. We claim that $\lambda \notin \sigma(A) \cup \sigma(C)$. Indeed:

(a) if $\lambda \notin \mathcal{S}(A^*)$, then by [1, Corollary 2.50], $A - \lambda$ is invertible. Hence by Lemma 3.2,

$$0 = \text{ind}(B - \lambda) = \text{ind}(A - \lambda) + \text{ind}(C - \lambda) = \text{ind}(C - \lambda),$$

and therefore $C - \lambda$ is invertible.

(b) if $\lambda \notin \mathcal{S}(C))$, then by [1, Corollary 2.50], $C - \lambda$ is invertible. Hence by Lemma 3.2,

$$0 = \text{ind}(B - \lambda) = \text{ind}(A - \lambda) + \text{ind}(C - \lambda) = \text{ind}(A - \lambda),$$

and therefore $A - \lambda$ is invertible.

(2) From Remark 3.1 we know that $\sigma_b(B) \subseteq \sigma_b(A) \cup \sigma_b(C)$, and therefore

$$\sigma_b(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) \subseteq \sigma_b(A) \cup \sigma_b(C).$$

Conversely, let $\lambda \notin \sigma_b(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$. Then $A - \lambda$ is upper semi-Browder and $C - \lambda$ is lower semi-Browder. We claim that $\lambda \notin \sigma_b(A) \cup \sigma_b(C)$. Indeed:

(a) if $\lambda \notin \mathcal{S}(A^*)$, then by [4, Corollary 16], $A - \lambda$ is Browder. Hence by Lemma 3.2,

$$0 = \text{ind}(B - \lambda) = \text{ind}(A - \lambda) + \text{ind}(C - \lambda) = \text{ind}(C - \lambda),$$

and therefore by [1, Theorem 3.4] $C - \lambda$ is Browder.

(b) if $\lambda \notin \mathcal{S}(C))$, then by [4, Theorem 15], $C - \lambda$ is Browder. Hence by Lemma 3.2,

$$0 = \text{ind}(B - \lambda) = \text{ind}(A - \lambda) + \text{ind}(C - \lambda) = \text{ind}(A - \lambda),$$

and therefore by [1, Theorem 3.4] $A - \lambda$ is Browder.

(3) To show $\sigma_d(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) \subseteq \sigma_d(A) \cup \sigma_d(C)$, it suffices to prove that $\sigma_d(B) \subseteq \sigma_d(A) \cup \sigma_d(C)$. Let $\lambda \notin \sigma_d(A) \cup \sigma_d(C)$. By Lemma 3.3, we conclude that $\lambda \notin \sigma_d(B)$.

Conversely, let $\lambda \notin \sigma_d(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$. We claim that $\lambda \notin \sigma_d(A) \cup \sigma_d(C)$. Indeed:

(a) if $\lambda \notin \mathcal{S}(C))$, then by the fact that $dsc(C - \lambda)$ is finite and [1, Theorem 3.81], $C - \lambda$ is Drazin invertible. Consequently, by Lemma 3.5 we infer that $A - \lambda$ is Drazin invertible.

(b) if $\lambda \notin \mathcal{S}(A^*)$, since we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^* & \xrightarrow{Q^*} & Y^* & \xrightarrow{J^*} & X^* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle C^*} & & \Big\downarrow{\scriptstyle B^*} & & \Big\downarrow{\scriptstyle A^*} & & \\ 0 & \longrightarrow & Z^* & \xrightarrow[Q^*]{} & Y^* & \xrightarrow[J^*]{} & X^* & \longrightarrow & 0 \end{array} \tag{3.2}$$

with exact rows, by the preceding argument in (a), $A^* - \lambda$ is Drazin invertible. Hence $A - \lambda$ is Drazin invertible. By Lemma 3.5, we get that $C - \lambda$ is Drazin invertible.

(4) By Remark 3.1, we infer that $\sigma_e(B) \subseteq \sigma_e(A) \cup \sigma_e(C)$, and hence

$$\sigma_e(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)) \subseteq \sigma_e(A) \cup \sigma_e(C) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C)).$$

Conversely, let $\lambda \notin \sigma_e(B) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$. Then $A - \lambda$ is upper semi-Fredholm and $C - \lambda$ is lower semi-Fredholm. We claim that $\lambda \notin \sigma_e(A) \cup \sigma_e(C)$. Indeed:

(a) if $\lambda \notin \mathcal{S}(A^*)$, then by [4, Corollary 12], $A - \lambda$ is Fredholm. Hence by Lemma 3.2,

$$\text{ind}(C - \lambda) = \text{ind}(B - \lambda) - \text{ind}(A - \lambda) < \infty,$$

and therefore $C - \lambda$ is Fredholm.

(b) if $\lambda \notin \mathcal{S}(C))$, then by [4, Corollary 11], $C - \lambda$ is Fredholm. Hence by Lemma 3.2,

$$\text{ind}(A - \lambda) = \text{ind}(B - \lambda) - \text{ind}(C - \lambda) < \infty,$$

and therefore $A - \lambda$ is Fredholm.

(5) By Remark 3.1, we infer that $\sigma_w(B) \subseteq \sigma_w(A) \cup \sigma_w(C)$, and hence

$$\sigma_w(B)\cup(\mathcal{S}(A)\cap\mathcal{S}(C^*))\cup(\mathcal{S}(A^*)\cap\mathcal{S}(C)) \subseteq \sigma_w(A)\cup\sigma_w(C)\cup(\mathcal{S}(A)\cap\mathcal{S}(C^*))\cup(\mathcal{S}(A^*)\cap\mathcal{S}(C)).$$

Conversely, let $\lambda \notin \sigma_w(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup (\mathcal{S}(A^*) \cap \mathcal{S}(C))$. Then $A - \lambda$ is upper semi-Fredholm and $C - \lambda$ is lower semi-Fredholm. We claim that $\lambda \notin \sigma_w(A) \cup \sigma_w(C)$. Indeed:

(a) if $\lambda \notin \mathcal{S}(A) \cup \mathcal{S}(A^*)$ (resp. $\lambda \notin \mathcal{S}(C) \cup \mathcal{S}(C^*)$), then by [4, Corollary 13], $A - \lambda$ is Weyl (resp. $C - \lambda$ is Weyl). Hence by Lemma 3.2 we infer that $C - \lambda$ (resp. $A - \lambda$) is Weyl.

(b) if $\lambda \notin \mathcal{S}(A) \cup \mathcal{S}(C))$ (resp. $\lambda \notin \mathcal{S}(A^*) \cup \mathcal{S}(C^*)$), then by [4, Corollary 11] (resp. [4, Corollary 12]), $\text{ind}(A-\lambda) \leq 0$ and $\text{ind}(C-\lambda) \leq 0$ (resp. $\text{ind}(A-\lambda) \geq 0$ and $\text{ind}(C-\lambda) \geq 0$). Hence by Lemma 3.2,

$$0 = \text{ind}(B - \lambda) = \text{ind}(A - \lambda) + \text{ind}(C - \lambda),$$

from which it follows that both $A - \lambda$ and $C - \lambda$ are Weyl. □

Recall that an operator $T \in \mathcal{B}(X)$ is called $Riesz$ if $\sigma_e(T) = \{0\}$. We say that an operator is $polynomially\ Riesz$ (resp. $polynomially\ quasi\text{-}nilpotent$) if there exists a nonzero complex polynomial p such that $p(T)$ is Riesz (resp. quasi-nilpotent).

Corollary 3.1. *Under the hypothesis* (1.1), *if X, Y and Z are infinite-dimensional complex Banach spaces, then*

(1) *B is quasi-nilpotent if and only if A and C are quasi-nilpotent;*

(2) *B is polynomially quasi-nilpotent if and only if A and C are polynomially quasi-nilpotent;*

(3) *B is Riesz if and only if A and C are Riesz;*

(4) *B is polynomially Riesz if and only if A and C are polynomially Riesz.*

Proof. (1) Suppose that A and C are quasi-nilpotent. By Theorem 3.2, we have B is quasi-nilpotent.

Conversely, suppose that B is quasi-nilpotent. Then $\rho(B)$ is connected, and hence from [3, Theorem 1.29] we can infer that $\sigma(A) = \{0\}$, that is A is quasi-nilpotent. Dually, C is quasi-nilpotent too.

(2) Suppose that A and C are polynomially quasi-nilpotent. By the spectral mapping theorem for the ordinary spectrum, we infer that $\sigma(A)$ and $\sigma(C)$ are both finite. Thus by Theorem 3.2, we know that $\sigma(B)$ is finite. Hence again by the spectral mapping theorem for the ordinary spectrum, B is polynomially quasi-nilpotent.

Conversely, suppose that B is polynomially quasi-nilpotent. Then $\sigma(B)$ is finite, and hence $\rho(B)$ is connected. From [3, Theorem 1.29] we can infer that $\sigma(A)$ is finite, that is A is polynomially quasi-nilpotent. Dually, C is polynomially quasi-nilpotent too.

(3) Suppose that A and C are Riesz. By Theorem 3.2, we have B is Riesz.

Conversely, suppose that B is Riesz. By [1, Theorems 3.113 and 3.115], we can infer that A and C are Riesz.

(4) Suppose that A and C are polynomially Riesz. By the spectral mapping theorem for the essential spectrum, we infer that $\sigma_e(A)$ and $\sigma_e(C)$ are both finite. Thus by Theorem 3.2, we know that $\sigma_e(B)$ is finite. Hence again by the spectral mapping theorem for the essential spectrum, B is polynomially Riesz.

Conversely, suppose that B is polynomially Riesz. Then $\sigma_e(B) = K$ is finite. Hence for all $\lambda \in \mathbb{C}\backslash K$, $A - \lambda$ is upper semi-Fredholm. From [1, Theorem 3.36], we can infer that $A - \lambda$ is Fredholm for all $\lambda \in \mathbb{C}\backslash K$. Therefore $\sigma_e(A)$ is finite, that is A is polynomially Riesz. Dually, C is polynomially Riesz too. □

An operator $T \in \mathcal{B}(X)$ is called $meromorphic$ if $T - \lambda$ is Drazin invertible for every $\lambda \in \sigma(T)\backslash\{0\}$. We say that an operator is $algebraic$ if there exists a nonzero complex polynomial p such that $p(T) = 0$. It is easily seen that $T \in \mathcal{B}(X)$ is meromorphic if and only if $\sigma_d(T) \subseteq \{0\}$. It follows form [1, Theorem 3.83] that $T \in \mathcal{B}(X)$ is algebraic if and only if $\sigma_{dsc}(T) := \{\lambda \in \mathbb{C} : dsc(T-\lambda) = \infty\} = \emptyset$ if and only if $\sigma_d(T) = \emptyset$.

Corollary 3.2. *Under the hypothesis* (1.1), *we have*
(1) *B is meromorphic if and only if A and C are meromorphic;*
(2) *B is algebraic if and only if A and C are algebraic.*

Proof. (1) Suppose that A and C are meromorphic. Then $\sigma_d(A) \subseteq \{0\}$ and $\sigma_d(C) \subseteq \{0\}$. By Theorem 3.2, we can infer that $\sigma_d(B) \subseteq \{0\}$, that is B is meromorphic.

Conversely, suppose that B is meromorphic. By Lemma 3.3, we can infer that $\sigma_{dsc}(C) \subseteq \sigma_{dsc}(B) \subseteq \sigma_d(B) \subseteq \{0\}$. Hence form [2, Corollary 1.9] it follows that C is meromorphic. Dually, A is also meromorphic.

(2) Suppose that A and C are algebraic. Then $\sigma_d(A) = \emptyset$ and $\sigma_d(C) = \emptyset$. By Theorem 3.2, we can infer that $\sigma_d(B) = \emptyset$, that is B is algebraic.

Conversely, suppose that B is algebraic. By Lemma 3.3, we can infer that $\sigma_{dsc}(C) \subseteq \sigma_{dsc}(B) \subseteq \sigma_d(B) = \emptyset$. Hence C is algebraic. Dually, A is also algebraic. □

For the one-side spectral properties, we need the following key lemma.

Lemma 3.6. *Under the hypothesis* (1.1), *then*
(1) *if $\beta(B) < \infty$ and $\alpha(C) < \infty$, then $\beta(A) \leq \beta(B) + \alpha(C)$;*
(2) *if $dsc(B) = q < \infty$ and $asc(C) = p < \infty$, then $dsc(A) \leq p + q$.*

Proof. (1) Let $\widehat{J} : X \longrightarrow \mathcal{R}(J)$ and $\widehat{Q} : Y/\mathcal{N}(Q) \longrightarrow \mathcal{R}(Q)$ be defined by:

$$\widehat{J}x = Jx \quad \text{for all } x \in X$$

and

$$\widehat{Q}(y + \mathcal{N}(Q)) = Qy \quad \text{for all } y + \mathcal{N}(Q) \in Y/\mathcal{N}(Q),$$

respectively. From the hypothesis (1.1), we know that $\widehat{J}$ and $\widehat{Q}$ are isomorphic. Moreover, it is easily seen that $B|_{\mathcal{R}(J)} = \widehat{J}A\widehat{J}^{-1}$ and that $B_{\mathcal{R}(J)} = B_{\mathcal{N}(Q)} = \widehat{Q}^{-1}C\widehat{Q}$. Let $M = \mathcal{R}(J)$. Then it suffices to show that if $\beta(B) < \infty$ and $\alpha(B_M) < \infty$, then $\beta(B|_M) \leq \beta(B) + \alpha(B_M)$.

Now suppose that $\beta(B) < \infty$ and $\alpha(B_M) < \infty$. Let $\widetilde{B} : \frac{B^{-1}(M)}{M} \longrightarrow \frac{M}{B(M)}$ be an operator induced by B:

$$\widetilde{B}(y + M) = By + B(M) \quad \text{for all } y + M \in \frac{B^{-1}(M)}{M}.$$

It is easily seen that $R(\widetilde{B}) = \frac{M \cap \mathcal{R}(B)}{B(M)}$ and that $\frac{\frac{M}{B(M)}}{\frac{M \cap \mathcal{R}(B)}{B(M)}} \approx \frac{M}{M \cap \mathcal{R}(B)} \approx \frac{M + \mathcal{R}(B)}{\mathcal{R}(B)} \subseteq \frac{X}{\mathcal{R}(B)}$. But then we conclude that $\beta(B|_M) \leq \beta(B) + \alpha(B_M)$.

(2) Let $x \in \mathcal{R}(A^{p+q})$. Then there exists $x_1 \in Y$ such that $x = A^{p+q}x_1$ and since $dsc(B) = q$,

$$Jx = JA^{p+q}x_1 = B^{p+q}Jx_1 = B^{2p+q+1}y_1$$

for some $y_1 \in Y$. Thus $0 = QJx = QB^{2p+q+1}y_1 = C^{2p+q+1}Qy_1$, and because $asc(C) = p < \infty$, we conclude that

$$QB^p y_1 = C^p Q y_1 = 0,$$

so $B^p y_1 \in \mathcal{N}(Q) = \mathcal{R}(J)$. Choose $x_2 \in X$ for which $B^p y_1 = Jx_2$. Consequently, $Jx = B^{p+q+1}Jx_2 = JA^{p+q+1}x_2$, and so $x = A^{p+q+1}x_2 \in \mathcal{R}(A^{p+q+1})$. This shows that $dsc(A) \leq p + q$. □

Now, we will relate the one-side spectral properties of B to those of A and C.

Theorem 3.3. *Under the hypothesis* (1.1), *we have*
(1) $\sigma_{su}(B) \cup \mathcal{S}(C) = \sigma_{su}(A) \cup \sigma_{su}(C) \cup \mathcal{S}(C)$;
(2) $\sigma_{ap}(B) \cup \mathcal{S}(A^*) = \sigma_{ap}(A) \cup \sigma_{ap}(C) \cup \mathcal{S}(A^*)$;
(3) $\sigma_{lsf}(B) \cup \mathcal{S}(C) = \sigma_{lsf}(A) \cup \sigma_{lsf}(C) \cup \mathcal{S}(C)$;
(4) $\sigma_{usf}(B) \cup \mathcal{S}(A^*) = \sigma_{usf}(A) \cup \sigma_{usf}(C) \cup \mathcal{S}(A^*)$;
(5) $\sigma_{lsw}(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(C) = \sigma_{lsw}(A) \cup \sigma_{lsw}(C) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(C)$;
(6) $\sigma_{usw}(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(A^*) = \sigma_{usw}(A) \cup \sigma_{usw}(C) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(A^*)$;
(7) $\sigma_{lb}(B) \cup \mathcal{S}(C) = \sigma_{lb}(A) \cup \sigma_{lb}(C) \cup \mathcal{S}(C)$;
(8) $\sigma_{ub}(B) \cup \mathcal{S}(A^*) = \sigma_{ub}(A) \cup \sigma_{ub}(C) \cup \mathcal{S}(A^*)$.

Proof. By (3.2), it suffices to show (1), (3), (5) and (7).

(1) By Lemma 3.2, we infer that $\sigma_{su}(B) \subseteq \sigma_{su}(A) \cup \sigma_{su}(C)$, and hence

$$\sigma_{su}(B) \cup \mathcal{S}(C) \subseteq \sigma_{su}(A) \cup \sigma_{su}(C) \cup \mathcal{S}(C).$$

Conversely, let $\lambda \notin \sigma_{su}(B) \cup \mathcal{S}(C)$. Then $dsc(B-\lambda) = 0$ and $C - \lambda$ is invertible. By Lemma 3.6, $dsc(A-\lambda) = 0$, that is $A - \lambda$ is surjective.

(3) By Lemma 3.2, we infer that $\sigma_{lsf}(B) \subseteq \sigma_{lsf}(A) \cup \sigma_{lsf}(C)$, and hence

$$\sigma_{lsf}(B) \cup \mathcal{S}(C) \subseteq \sigma_{lsf}(A) \cup \sigma_{lsf}(C) \cup \mathcal{S}(C).$$

Conversely, let $\lambda \notin \sigma_{lsf}(B) \cup \mathcal{S}(C)$. Then $\beta(B-\lambda) < \infty$ and $C - \lambda$ is Fredholm. By Lemma 3.6, $\beta(A-\lambda) < \infty$, that is $A - \lambda$ is lower semi-Fredholm.

(5) By Lemma 3.2, we infer that $\sigma_{lsw}(B) \subseteq \sigma_{lsw}(A) \cup \sigma_{lsw}(C)$, and hence

$$\sigma_{lsw}(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(C) \subseteq \sigma_{lsw}(A) \cup \sigma_{lsw}(C) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(C).$$

Conversely, let $\lambda \notin \sigma_{lsw}(B) \cup (\mathcal{S}(A) \cap \mathcal{S}(C^*)) \cup \mathcal{S}(C)$. Then $\beta(B-\lambda) < \infty$ and $C - \lambda$ is Fredholm. By Lemma 3.6, $\beta(A-\lambda) < \infty$. We claim that both $A - \lambda$ and $C - \lambda$ are lower semi-Weyl. Indeed:

(a) if $\lambda \notin \mathcal{S}(A) \cup \mathcal{S}(C))$, then by [4, Corollary 11], $\text{ind}(A-\lambda) \leq 0$ and $\text{ind}(C-\lambda) \leq 0$. Therefore by Lemma 3.2,

$$0 \leq \text{ind}(B-\lambda) = \text{ind}(A-\lambda) + \text{ind}(C-\lambda),$$

from which it follows that both $A - \lambda$ and $C - \lambda$ are lower semi-Weyl.

(b) if $\lambda \notin \mathcal{S}(C^*) \cup \mathcal{S}(C))$, then $C - \lambda$ is Weyl and by Lemma 3.2,

$$0 \leq \text{ind}(B-\lambda) = \text{ind}(A-\lambda) + \text{ind}(C-\lambda),$$

from which it follows that $A - \lambda$ is lower semi-Weyl.

(7) By Lemmas 3.2 and 3.3, we infer that $\sigma_{lb}(B) \subseteq \sigma_{lb}(A) \cup \sigma_{lb}(C)$, and hence

$$\sigma_{lb}(B) \cup \mathcal{S}(C) \subseteq \sigma_{lb}(A) \cup \sigma_{lb}(C) \cup \mathcal{S}(C).$$

Conversely, let $\lambda \notin \sigma_{lb}(B) \cup \mathcal{S}(C)$. Then $\beta(B-\lambda) < \infty$, $des(B-\lambda) < \infty$ and $C - \lambda$ is Browder. By Lemma 3.6, $\beta(A-\lambda) < \infty$ and $des(A-\lambda) < \infty$, that is $A - \lambda$ is lower semi-Browder. □

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