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On the uniform convergence of a q-series

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ABSTRACT. The paper deals with a class of linear positive operators expressed by q-series. By using modulus of smoothness an upper bound of approximation error is determined. We identify functions for which these operators provide uniform approximation over noncompact intervals. A particular case is delivered.

1. INTRODUCTION

In the last years Quantum Calculus began to be widely used in the construction of linear positive approximation processes. The first step in this direction was made in 1987 by Lupaş [16] and through the work of Ostrovska [22] his research was internationally disseminated. Starting from q-Bernstein operators, other important classes of discrete operators have been reintroduced by using q-calculus. For example, we can refer to operators q-Meyer-König and Zeller [25], [13], q-Bleimann, Butzer and Hahn operators [5], q-Szász-Mirakjan operators [4], [18], q-Baskakov operators [20]. We quoted only a few of works that served as a model for investigating and generalizations classes of discrete q-operators. Integral extensions in q-Calculus of the above discrete operators have also been studied, see for example [1], [6], [14]. For a comprehensive view of the results obtained in this area, the recent monograph [7] can be consulted. The newest trend in this domain is the investigation of linear operators introduced by using (p, q) - integers [21], [26]. Originally, this type of integers has been introduced in order to generalize or to unify several forms of q-oscillator algebras. In the end we specify that over time the mentioned classes of operators have been extensively studied by many mathematicians and collective research. Among them, here we mention the papers of Barbosu and his collaborators [9] - [12], [23] - [24].

Our study aimed at a class of q-operators with two particular features: they are expressed through a series and acts on functions on unbounded interval. Since we take into account linear approximation processes, clearly the sequence of operators associated to a function f must converge to approximated element. For continuous functions defined on a compact, uniform convergence takes place. If we work with continuous functions defined on an unbounded interval, only pointwise convergence occurs. In this note we indicate sufficient conditions which ensure uniform convergence for our class of q-operators.

2. THE OPERATORS

First of all we recall some formulas in q-Calculus, see, e.g., [3], [17]. Throughout the paper we consider $q \in (0,1)$. For any $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the q-integer $[n]_q$ and the

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q-factorial $[n]_q!$ are respectively defined by

$$[n]_q = \sum_{j=0}^{n-1} q^j, \quad [n]_q! = \prod_{j=1}^{n} [j]_q, \ n \in \mathbb{N},$$

and $[0]_q = 0$, $[0]_q! = 1$. The q-binomial coefficients, also known as Gaussian coefficients, are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and are given as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \ k = 0, 1, \ldots, n.$$

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ x \neq 0, \quad D_q f(0) = \lim_{x \to 0} D_q f(x),$$

and the high q-derivatives are given recursively by

$$D_q^0 f = f, \ D_q^n f = D_q(D_q^{n-1} f), \ n \in \mathbb{N}.$$

A real function f is q-differentiable on a real interval I if for every $x \in I$ the q-derivative of f exists and it is finite.

The roots of our operators are in connection with a general class of operators introduced by Baskakov [8] and developed by Mastroianni [19]. Following [2], let $(\phi_n)_{n \geq 1}$ be a sequence of real valued functions defined on $\mathbb{R}_+$, continuously infinitely q-differentiable on $\mathbb{R}_+$ and satisfying the following conditions:

$$\phi_n(0) = 1, \ n \in \mathbb{N}; \tag{2.1}$$

$$(-1)^k D_q^k \phi_n(x) \geq 0, \ n \in \mathbb{N}, \ k \in \mathbb{N}_0, \ x \geq 0; \tag{2.2}$$

for all $(x, k) \in \mathbb{R}_+ \times \mathbb{N}_0$ there exists a positive integer i_k, $0 \leq i_k \leq k$, and a function $\beta_{n,k,i_k,q} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$D_q^{k+1} \phi_n(x) = (-1)^{i_k+1} D_q^{k-i_k} \phi_n(q^{i_k+1} x) \beta_{n,k,i_k,q}(x), \tag{2.3}$$

where

$$\lim_n \frac{\beta_{n,k,i_k,q}(0)}{[n]_q^{i_k+1} q^{k-i_k}} = 1. \tag{2.4}$$

We consider the operators

$$(T_{n,q} f)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{k(k-1)/2} D_q^k \phi_n(x) f\left(\frac{[k]_q}{[n]_q q^{k-1}} \right), \ x \geq 0, \tag{2.5}$$

where $f \in \mathcal{F}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R}$, the q-series in (2.5) is absolutely convergent for each $n \in \mathbb{N}\}$.

In particular $C_B(\mathbb{R}_+) \subset \mathcal{F}(\mathbb{R}_+)$, where $C_B(\mathbb{R}_+)$ stands for the space of all continuous and bounded real-valued functions defined on $\mathbb{R}_+$. For each $n \in \mathbb{N}$, $T_{n,q}$ is a linear positive operator.

For our study we use the modulus of smoothness associated to any bounded function $h : \mathbb{R}_+ \to \mathbb{R}$ and given by

$$\omega_h(\delta) \equiv \omega(h; \delta) = \sup\{|h(x') - h(x'')| : \ x', x'' \in \mathbb{R}_+, \ |x' - x''| \leq \delta\}, \ \delta \geq 0. \tag{2.6}$$

Its relevant properties are presented, e.g., in [15, *pp. 43-46*]. Among them we recall that ω_h is a non-decreasing function and

$$\omega(h; \lambda\delta) \leq (1 + \lambda)\omega(h; \delta), \ \delta \geq 0 \text{ and } \lambda \geq 0. \tag{2.7}$$

Also, if h is uniformly continuous function on $\mathbb{R}_+$, then

$$\lim_{\delta\to 0^+} \omega(h;\delta) = 0. \tag{2.8}$$

Finally we set $e_0(t) = 1$ and $e_j(t) = t^j$, $t \geq 0$.

3. Results

Primarily we indicate a relation satisfied by our operators. The second central moment of $T_{n,q}$, $n \in \mathbb{N}$, is given by the formula

$$T_{n,q}(\varphi_x^2; x) = a_{n,q}x^2 + b_{n,q}x,\ x \geq 0, \tag{3.9}$$

where $\varphi_x(t) = t - x$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$, and

$$a_{n,q} = 1 + 2\frac{D_q\phi_n(0)}{[n]_q} + \frac{D_q^2\phi_n(0)}{q[n]_q^2}, \quad b_{n,q} = -\frac{D_q\phi_n(0)}{[n]_q^2}, \tag{3.10}$$

see [2, Eqs. (21)-(22)]. Relation (2.2) ensures $b_{n,q} \geq 0$.

Lemma 3.1. *Let the operators $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.5). The following inequality*

$$(T_{n,q}h_x)(x) \leq \sqrt{c_{n,q}},\ x \geq 0, \tag{3.11}$$

holds, where

$$h_x(t) = |w(t) - w(x)|,\ t \geq 0,\ w = e_1/(e_1 + e_0),\ c_{n,q} = |a_{n,q}| + b_{n,q} \tag{3.12}$$

and $a_{n,q}$, $b_{n,q}$ are given at (3.10).

Proof. For $x = 0$ the relation is evident because $T_{n,q}$ enjoys the interpolatory property, i.e. $(T_{n,q}f)(0) = f(0)$ for any function $f \in \mathcal{F}(\mathbb{R}_+)$.

Further, let $x > 0$ be arbitrarily fixed. For the sake of brevity we denote

$$\alpha_{n,k,q}(x) = \frac{(-x)^k}{[k]_q!}q^{k(k-1)/2}D_q^k\phi_n(x) \quad \text{and} \quad x_{n,k,q} = \frac{[k]_q}{[n]_q q^{k-1}}. \tag{3.13}$$

We can write

$$\begin{aligned}
(T_{n,q}h_x)(x) &= \sum_{k=0}^{\infty}\alpha_{n,k,q}(x)\left|\frac{x_{n,k,q}}{x_{n,k,q}+1} - \frac{x}{x+1}\right| \\
&\leq \frac{1}{x+1}\sum_{k=0}^{\infty}\alpha_{n,k,q}(x)|x_{n,k,q} - x| \\
&\leq \frac{1}{x+1}\left(\sum_{k=0}^{\infty}\alpha_{n,k,q}(x)\right)^{1/2}\left(\sum_{k=0}^{\infty}\alpha_{n,k,q}(x)\varphi_x^2(x_{n,k,q})\right)^{1/2} \\
&= \frac{1}{x+1}((T_{n,q}e_0)(x))^{1/2}((T_{n,q}\varphi_x^2)(x))^{1/2} \\
&\leq \sqrt{|a_{n,q}| + b_{n,q}}.
\end{aligned}$$

We used above Cauchy-Schwarz inequality, the identity $T_{n,q}e_0 = e_0$ and relations (3.9)-(3.10). Inequality (3.11) follows. □

Based on the properties (2.1)-(2.4), we deduce

$$\lim_n \frac{D_q\phi_n(0)}{[n]_q} = -1, \quad \lim_n \frac{D_q^2\phi_n(0)}{q[n]_q^2} = \frac{1}{q^{i_1}},\ i_1 \in \{0,1\}, \tag{3.14}$$

consequently

$$0 \le \lim_n c_{n,q} = \left| \frac{1}{q^{i_1}} - 1 \right| + 1 - q \le \frac{1-q^2}{q}. \tag{3.15}$$

Theorem 3.1. *Let the operators $T_{n,q}$, $n \in \mathbb{N}$, be defined by (2.5). Let $f \in C_B(\mathbb{R}_+)$ and the function*

$$f^*(t) = f(w^{-1}(t)),\ t \ge 0. \tag{3.16}$$

For all $x \in \mathbb{R}_+$ the following inequality

$$|(T_{n,q}f)(x) - f(x)| \le 2\omega(f^*; \sqrt{c_{n,q}}) \tag{3.17}$$

holds. In the above $w : \mathbb{R}_+ \to [0,1)$ and $c_{n,q}$ are given at (3.12).

Proof. For $x = 0$ the first member of the inequality is null, consequently (3.17) takes place. Further, we consider $x > 0$ arbitrarily fixed and we use the notation introduced by relation (3.13).

In view of definitions (2.5) and (2.7) we can write

$$\begin{aligned} |(T_{n,q}f)(x) - f(x)| &= |T_{n,q}(f^* \circ w; x) - (f^* \circ w)(x)| \\ &\le \sum_{k=0}^{\infty} \alpha_{n,k,q}(x)|f^*(w(x_{n,k,q})) - f^*(w(x))| \\ &\le \sum_{k=0}^{\infty} \alpha_{n,k,q}(x)\omega(f^*; |w(x_{n,k,q}) - w(x)|). \end{aligned} \tag{3.18}$$

In the next step we use (3.12) and the property (2.8) in which is chosen

$$\lambda = |w(x_{n,k,q} - w(x)|/(T_{n,q}h_x)(x).$$

We get

$$\begin{aligned} \omega(f^*; |w(x_{n,k,q}) - w(x)|) &\le \left(1 + \frac{|w(x_{n,k,q}) - w(x)}{(T_{n,q}h_x)(x)} \omega(f^*; T_{n,q}h_x)(x)\right) \\ &\le \left(1 + \frac{|w(x_{n,k,q}) - w(x)|}{(T_{n,q}h_x)(x)}\right) \omega(f^*; \sqrt{c_{n,q}}), \end{aligned}$$

see (3.11). Returning at (3.18), and knowing both the form of the function $T_{n,q}h_x$ and the identity $T_{n,q}e_0 = e_0$, we arrive at the desired result. □

Since $q \in (0,1)$ is fixed, we deduce $\lim_n T_{n,q}\varphi_x^2 \ne 0$, see (3.9), (3.10) and (3.14). Consequently, on the basis of Bohman-Korovkin criterion, $(T_{n,q})_{n\ge 1}$ does not form an approximation process. In order to transform it for satisfying this property, for each $n \in \mathbb{N}$ the constant q will be replaced by a number $q_n \in (0,1)$ such that $\lim_n q_n = 1$. Such a replacement should not be surprising because a q-analogue, also called q-extension of a mathematical object $\mathcal{M}$, is a family of objects $\mathcal{M}(q)$ such that $\lim_{q\to 1} \mathcal{M}(q) = \mathcal{M}$. Under these circumstances, the pointwise convergence of $(T_{n,q}g)(x)$ to $g(x)$ as $n \to \infty$ takes place, where $g \in \mathcal{F}(\mathbb{R}_+) \cap C(\mathbb{R}_+)$. Also, the uniform convergence of $T_{n,q}g$ to g as $n \to \infty$ holds for any compact $K \subset \mathbb{R}_+$.

As we stated at the beginning, our goal is to indicate sufficient conditions for which this class of operators provides uniform approximation on the whole interval $\mathbb{R}_+$.

Theorem 3.2. *Let $(q_n)_{n\ge 1}$, $0 < q_n < 1$, be a sequence such that $\lim_n q_n = 1$. Let the operators T_{n,q_n}, $n \in \mathbb{N}$, be defined as in (2.5). Let $f \in C_B(\mathbb{R}_+)$ and let f^* be defined by (3.16).*

If f^* is uniformly continuous on $\mathbb{R}_+$, then $T_{n,q_n} f$ converges uniformly to f on $\mathbb{R}_+$ as n tends to infinity.

Proof. Since $\lim_n q_n = 1$, using relation (3.15), we deduce

$$\lim_{n\to\infty} c_{n,q_n} = 0.$$

Further on, f^* being uniformly continuous on $\mathbb{R}_+$, property (2.8) guarantees

$$\lim_{n\to\infty} \omega(f^*; \sqrt{c_{n,q_n}}) = 0.$$

Considering this relation, the inequality established in (3.17) implies the conclusion of our theorem. □

4. Particular case

At first we recall the following expansion in q-Calculus of the exponential function

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!},\ x \in \mathbb{R}.$$

By using the notation $(1-a)_q^\infty = \prod_{j=0}^\infty (1-q^j a)$, we can write

$$E_q(x) = (1 + (1-q)x)_q^\infty,$$

see [17, *Eq. (9.10)*]. For obtaining a particular class of our sequence $(T_{n,q})_n$, we choose $\phi_n(x) = E_q(-[n]_q x)$, $x \geq 0$, $n \in \mathbb{N}$. By direct computation we get

$$D_q^k \phi_n(x) = (-1)^k [n]_q^k q^{k(k-1)/2} E_q(-[n]_q q^k x),\ x \geq 0.$$

Conditions (2.1) and (2.2) are evident fulfilled. By taking $i_k = 0$, (2.3) and (2.4) are also valid, where $\beta_{n,k,0,q}(x) = [n]_q q^k$, $k \in \mathbb{N}_0$, are constant functions. By simple calculations, in (3.9) we obtain $a_{n,q} = 0$ and $b_{n,q} = 1/[n]_q$. We arrived at a q-analogue of Szász-Mirakjan operator.

Relation (3.17) says: the order of approximation of f by this sequence of operators is $\mathcal{O}(1/\sqrt{[n]_{q_n}})$. Since $a_{n,q} = 0$, this time we can make a different choice most appropriate for the function w, i.e. $w(t) = \sqrt{t}$, $t \geq 0$. Both conclusions of Theorem 3.1 and Theorem 3.2 remain valid.

For another q-generalization of these particular operators, this time $q > 1$, a similar result has been obtained by Mahmudov [18, *Theorem 4.1*]. In this case was also used function $w = \sqrt{e_1}$.

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