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An extension of Assad-Kirk's fixed point theorem for multivalued nonself mappings

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ABSTRACT. In the present paper, taking into account the recent developments on the theory of fixed point, we give some fixed point results for multivalued nonself mappings on complete metrically convex metric spaces. Our main result properly includes the famous Assad-Kirk fixed point theorem for nonself mappings. Also, we provide a nontrivial example which shows the motivation for such investigations of multivalued nonself contraction mappings.

1. INTRODUCTION

Metric fixed point theory is one of the most rapidly growing research areas in nonlinear functional analysis and a very powerful tool in solving existence and uniqueness problems in many branches of mathematical analysis, e.g., operator theory and variational analysis, especially, differential, integral and functional equations as applications of fixed points of contractive mappings defined for different types of spaces. Nowadays, these problems require the search for more and better tools which is very remarkable in the literature. One of such tools was given by Jleli and Samet [18], introduced a new type of contractive mapping. Throughout this study we shall call the contraction defined in [18] as θ-contraction. Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then we say that T is θ-contraction if there exists $k \in (0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{1.1}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$, where $\theta : (0,\infty) \to (1,\infty)$ is a function satisfying the following conditions:

(θ_1) θ is nondecreasing.

(θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.

(θ_3) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

We denote by Θ be the class of all functions θ satisfying (θ_1)-(θ_3). Considering inequality (1.1), we obtain different types of nonequivalent contractions. For example, for $\theta(t) = e^{\sqrt{t}}$, (1.1) turns to

$$d(Tx,Ty) \leq k^2 d(x,y), \tag{1.2}$$

for all $x, y \in X$ with $Tx \neq Ty$. It is clear that the inequality $d(Tx,Ty) \leq k^2 d(x,y)$ also holds for $x, y \in X$ with $Tx = Ty$. Therefore, every Banach contraction mapping is a θ-contraction with $\theta(t) = e^{\sqrt{t}}$. Similarly, for $\theta(t) = e^{\sqrt{te^t}}$, (1.1) turns to

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2, \tag{1.3}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

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Let $\theta_1, \theta_2 \in \Theta$. If $\theta_1(t) \leq \theta_2(t)$ for all $t > 0$ and $\theta = \frac{\theta_2}{\theta_1}$ is nondecreasing, then every θ_1-contraction mapping is also θ_2-contraction. Therefore, if a mapping T satisfies (1.2), then it satisfies (1.3).

In addition, from (θ_1) and (1.1), it is easy to concluded that every θ-contraction T is a contractive mapping, i.e., $d(Tx, Ty) < d(x, y)$ for each $x, y \in X$ with $x \neq y$. Thus, every θ-contraction mapping on a metric space is continuous. For each such mapping T, the following theorem has been proved by Jleli and Samet [18], which is a proper generalization of Banach contraction principle.

Theorem 1.1 (Corollary 2.1 of [18])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. If T is an θ-contraction, then T has a (unique) fixed point.*

In the literature, one can find pivotal papers related to θ-contractions. For example, in [4, 17], the authors analyzed θ-contractions considering generalized contractive and almost contractive conditions for single valued mappings.

2. Preliminaries

In this section, we give some notational and terminological conventions which will be used throughout this paper for the sake of completeness.

2.1. **Multivalued Contraction.** Let (X, d) be a metric space. Denote $P(X)$ by the class of all nonempty subsets of X, $CB(X)$ by the class of all nonempty closed and bounded subsets of X and, $K(X)$ by the class of all nonempty compact subsets of X. Let H be the Pompeiu-Hausdorff metric with respect to d, for $A, B \in CB(X)$,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. We can find detailed information about the Pompeiu-Hausdorff metric in [1, 7, 10, 15]. Then, a map $T : X \to CB(X)$ is said to be multivalued contraction if there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$ (see [21]). In 1969, Nadler [21] initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and after, proved a fundamental fixed point theorem for multivalued mappings. This result states that every multivalued contraction mappings on complete metric spaces has at least one fixed point, that is, there exists $x \in X$ such that $x \in Tx$. Inspired by his result, there has been continuous, intense research activity for fixed point results concerning multivalued contractions, and by now, there are a number of results that extend this fixed point result in many ways over the years (see [8, 11, 13, 19, 20, 22, 23, 24]). One of the most interesting extension was given by Assad and Kirk [6] for multivalued nonself mappings defined on a closed subset of metrically convex metric spaces. They gave a sufficient condition for fixed point of such mappings considering specific boundary condition. We can find some significant generalization of Assad and Kirk's result in [2, 3, 5, 9, 12, 16] and references therein.

Let (X, d) be a metric space. Then X is said to be metrically convex if there is a point $z \in X$, $x \neq y \neq z$ such that

$$d(x, y) = d(x, z) + d(z, y),$$

for any $x, y \in X$ with $x \neq y$. For the sequel, we need the following very useful lemma.

Lemma 2.1 ([6])**.** *Let C be a nonempty closed subset of a complete and metrically convex metric space (X, d). Then, for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ such that*

$$d(x, z) + d(z, y) = d(x, y),$$

where ∂C denotes the boundary of C.

Theorem 2.2 (Assad and Kirk's fixed point theorem)**.** *Let (X, d) be a complete and metrically convex metric space, C be a nonempty closed subset of X, and $T : C \to CB(X)$ be a mapping such that, for all $x, y \in C$,*

$$H(Tx, Ty) \leq kd(x, y), \tag{2.4}$$

for some $k \in (0, 1)$. If $Tx \subseteq C$ for each $x \in \partial C$, then T has a fixed point in C.

2.2. **Multivalued θ-Contraction.** The concept of multivalued θ-contraction introduced by Hançer et al [14]. Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. If there exist $k \in (0, 1)$ and $\theta \in \Theta$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k \tag{2.5}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Considering $\theta(t) = e^{\sqrt{t}}$, we can say that every multivalued contraction is also multivalued θ-contraction. Thus, they established a fixed point theorem, which extended Nadler's result in a different way than the well-known methods in the literature.

Theorem 2.3 ([14])**.** *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a mapping. If T is a multivalued θ-contraction, then T has a fixed point.*

Note that Tx is compact for all $x \in X$ in Theorem 2.3. In the proof of this theorem, it has been used the fact that for any $x \in X$, there exists a point $s \in S$ such that $d(x, s) = d(x, S)$, where (X, d) is a metric space and S is a compact subset of X. Therefore the following problem arised: Can we replace $CB(X)$ instead of $K(X)$ in Theorem 2.3? Unfortunately, the answer is negative with the same conditions as shown in the following example (see [14]).

Example 2.1 ([14])**.** Consider the complete metric space (X, d), where $X = [0, 2]$ and $d(x, y) = 1 + |x - y|$ if $x \neq y$, $d(x, y) = 0$ if $x = y$. Define a mapping $T : X \to CB(X)$, $Tx = \mathbb{Q}$ if $x \in X \backslash \mathbb{Q}$ and $Tx = X \backslash \mathbb{Q}$ if $x \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers in X. Then, T is a multivalued θ-contraction with $k = \frac{1}{2}$ and $\theta \in \Theta$ defined by $\theta(t) = e^{\sqrt{t}}$ if $t \leq 1$ and $\theta(t) = 9$ if $t > 1$, but it has no fixed points.

However, it has been demonstrated that if the function θ satisfies the following (θ_4) condition in Theorem 2.3, then $K(X)$ can be replaced by $CB(X)$:

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Note that, if θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is right continuous. Let Ξ be the family of all functions θ satisfying (θ_1)-(θ_4) .

Theorem 2.4 ([14])**.** *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a mapping. If T is a multivalued θ-contraction with $\theta \in \Xi$, then T has a fixed point.*

The purpose of this paper is to give a new approach to Assad-Kirk fixed point theorem and a new real generalization of it, by using θ-contractiveness of a multivalued mapping.

3. Main Results

The following theorem is our main result:

Theorem 3.5. *Let (X, d) be a complete and metrically convex metric space, C be a nonempty closed subset of X, $T : C \to CB(X)$ be a mapping such that, for all $x, y \in C$ with $H(Tx, Ty) > 0$,*

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k , \tag{3.6}$$

for some $k \in (0, 1)$ and $\theta \in \Xi$. If $Tx \subseteq C$ for each $x \in \partial C$, then T has a fixed point in C.

Proof. Suppose that T has no fixed points. Thus $d(x, Tx) > 0$ for all $x \in C$. Now, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in C in the following way. Let $x_0 \in C$ and $y_1 \in Tx_0$.

If $y_1 \in C$, let $x_1 = y_1$.

If $y_1 \notin C$, then from Lemma 2.1, there exists a point $x_1 \in \partial C$ such that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

Thus $x_1 \in C$. Now, we claim that $d(y_1, Tx_1) > 0$. Suppose $d(y_1, Tx_1) = 0$. If $y_1 \in C$, then x_1 is a fixed point of T, which is a contradiction. If $y_1 \notin C$, then $x_1 \in \partial C$ and so $Tx_1 \subseteq C$. Therefore, $y_1 \notin Tx_1$, which is a contradiction. Thus, $d(y_1, Tx_1) > 0$. Now, since $d(y_1, Tx_1) \leq H(Tx_0, Tx_1)$, then we have

$$\theta(d(y_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^k . \tag{3.7}$$

On the other hand, from (θ_4) we get

$$\theta(d(y_1, Tx_1)) = \theta(\inf\{d(y_1, m) : m \in Tx_1\}) = \inf\{\theta(d(y_1, m)) : m \in Tx_1\},$$

and so, from (3.7) we get

$$\inf\{\theta(d(y_1, m)) : m \in Tx_1\} \leq [\theta(d(x_0, x_1))]^k .$$

Thus, there exists $y_2 \in Tx_1$ such that

$$\theta(d(y_1, y_2)) \leq [\theta(d(x_0, x_1))]^\gamma ,$$

where $0 < k < \gamma < 1$.

If $y_2 \in C$, let $x_2 = y_2$.

If $y_2 \notin C$, select a point $x_2 \in \partial C$ such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, $x_2 \in C$. We can show that $d(y_2, Tx_2) > 0$. As above, we can find a point $y_3 \in Tx_2$ such that

$$\theta(d(y_2, y_3)) \leq [\theta(d(x_1, x_2))]^\gamma .$$

Continuing the arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that for $n \in \mathbb{N}$,

i) $y_{n+1} \in Tx_n$

ii)

$$\theta(d(y_n, y_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^\gamma ,$$

where $y_{n+1} = x_{n+1}$ if $y_{n+1} \in C$ or

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}) \tag{3.8}$$

if $y_{n+1} \notin C$ and $x_{n+1} \in \partial C$.

Now, we set

$$P = \{x_i \in \{x_n\} : x_i = y_i, i \in \mathbb{N}\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i, i \in \mathbb{N}\}.$$

Observe that if $x_i \in Q$ for some i, then $x_{i+1} \in P$.

We wish to estimate the distance $d(x_n, x_{n+1})$ for $n \geq 2$. Note that $d(x_n, x_{n+1}) > 0$, otherwise, T has a fixed point. For this we have to consider three cases:

Case 1. If $x_n \in P$ and $x_{n+1} \in P$, then, we get

$$\theta(d(x_n, x_{n+1})) = \theta(d(y_n, y_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^\gamma .$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$. then, from (3.8), we get

$$\theta(d(x_n, x_{n+1})) \leq \theta(d(x_n, y_{n+1})) = \theta(d(y_n, y_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^\gamma .$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$, then, since

$$\theta(d(y_n, y_{n+1})) \leq [\theta(d(x_{n-1}, x_n))]^\gamma \Rightarrow d(y_n, y_{n+1}) < d(x_{n-1}, x_n),$$

we get

$$\begin{aligned}\theta(d(x_n, x_{n+1})) &\leq \theta(d(x_n, y_n) + d(y_n, x_{n+1})) = \theta(d(x_n, y_n) + d(y_n, y_{n+1}))\\ &\leq \theta(d(x_n, y_n) + d(x_{n-1}, x_n)) = \theta(d(x_{n-1}, y_n))\\ &= \theta(d(y_{n-1}, y_n)) \leq [\theta(d(x_{n-2}, x_{n-1}))]^{\gamma}.\end{aligned}$$

The only other possibility $x_n \in Q$ and $x_{n+1} \in Q$ can not occur. Thus, we get

$$\theta(d(x_n, x_{n+1})) \leq \begin{cases} [\theta(d(x_{n-1}, x_n))]^{\gamma} \\ \\ [\theta(d(x_{n-2}, x_{n-1}))]^{\gamma} \end{cases} \tag{3.9}$$

for $n \geq 2$. Now we claim that

$$\theta(d(x_n, x_{n+1})) \leq \delta^{\left(\gamma^{\frac{n-1}{2}}\right)} \tag{3.10}$$

for all $n \in \mathbb{N}$, where

$$\delta = \max\{\theta(d(x_0, x_1)), \theta(d(x_1, x_2))\}.$$

Let us prove (3.10) by induction method.

For $n = 1$, it is clear that (3.10) is satisfied.

For $n = 2$, we use (3.9) and taking each case separately, we get

$$\theta(d(x_2, x_3)) \leq [\theta(d(x_1, x_2))]^{\gamma} \leq \delta^{\gamma} \leq \delta^{\left(\gamma^{\frac{1}{2}}\right)};$$

$$\theta(d(x_2, x_3)) \leq [\theta(d(x_0, x_1))]^{\gamma} \leq \delta^{\gamma} \leq \delta^{\left(\gamma^{\frac{1}{2}}\right)}.$$

For $n = 3$, we use (3.9) and taking each case separately, we get

$$\theta(d(x_3, x_4)) \leq [\theta(d(x_2, x_3))]^{\gamma} \leq \delta^{\left(\gamma^{\frac{1}{2}}\right)\gamma} = \delta^{\left(\gamma^{\frac{3}{2}}\right)} \leq \delta^{\gamma};$$

$$\theta(d(x_3, x_4)) \leq [\theta(d(x_1, x_2))]^{\gamma} \leq \delta^{\gamma}.$$

For $n = 4$, we use (3.9) and taking each case separately, we get

$$\theta(d(x_4, x_5)) \leq [\theta(d(x_3, x_4))]^{\gamma} \leq \delta^{(\gamma^2)} \leq \delta^{\left(\gamma^{\frac{3}{2}}\right)};$$

$$\theta(d(x_4, x_5)) \leq [\theta(d(x_2, x_3))]^{\gamma} \leq \delta^{\left(\gamma^{\frac{1}{2}}\right)\gamma} = \delta^{\left(\gamma^{\frac{3}{2}}\right)}.$$

Now, assume that (3.10) holds for $1 \leq n \leq m$. Observe that for $m > 2$,

$$\begin{aligned}\theta(d(x_{m+1}, x_{m+2})) &\leq [\theta(d(x_m, x_{m+1}))]^{\gamma}\\ &\leq \delta^{\left(\gamma^{\frac{m-1}{2}}\right)\gamma} = \delta^{\left(\gamma^{\frac{m+1}{2}}\right)} \leq \delta^{\left(\gamma^{\frac{m}{2}}\right)};\end{aligned}$$

$$\theta(d(x_{m+1}, x_{m+2})) \leq [\theta(d(x_{m-1}, x_m))]^{\gamma} \leq \delta^{\left(\gamma^{\frac{m-2}{2}}\right)\gamma} = \delta^{\left(\gamma^{\frac{m}{2}}\right)}.$$

Then, our claim is true. Using (3.10), we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1. \tag{3.11}$$

From (θ_2), $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+$ and so from (θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left|\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - l\right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,,$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,,$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\,[\theta(d(x_n, x_{n+1})) - 1]\,.$$

Using (3.10), we obtain, for all $n \geq n_0$,

$$n\,[d(x_n, x_{n+1})]^r \leq An\left[\delta^{\left(\gamma^{\frac{n-1}{2}}\right)} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\,[d(x_n, x_{n+1})]^r = 0.$$

Thus, there exits $n_1 \in \mathbb{N}$ such that $n\,[d(x_n, x_{n+1})]^r \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_0$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}. \tag{3.12}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (3.12), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, passing to limit $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This implies that the sequence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed, the sequence $\{x_n\}$ converges to some point $z \in C$. By our choice of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in P$, that is, $x_{n_k} = y_{n_k}$, $k \in \mathbb{N}$. Note that $x_{n_k} \in Tx_{n_k-1}$ for $k \in \mathbb{N}$ and $x_{n_k-1} \to z$ as $k \to \infty$. Also note that from (3.6) and (θ_1) we get

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$ and so, we have

$$d(x_{n_k}, Tz) \leq H(Tx_{n_k-1}, Tz) \leq d(x_{n_k-1}, z),$$

which on letting $k \to \infty$ implies that $d(z, Tz) = 0$, which is a contradiction. Therefore, T has a fixed point in C. $\square$

In the following theorem we remove the condition (θ_4) on θ, but we have to restrict the set of values of T.

Theorem 3.6. *Let (X,d) be a complete and metrically convex metric space, C be a nonempty closed subset of X, $T: C \to K(X)$ be a mapping such that, for all $x, y \in C$ with $H(Tx,Ty) > 0$,*

$$\theta(H(Tx,Ty)) \leq [\theta(d(x,y))]^k,$$

for some $k \in (0,1)$ and $\theta \in \Theta$. If $Tx \subseteq C$ for each $x \in \partial C$, then T has a fixed point in C.

Proof. Suppose that T has no fixed points. Thus $d(x,Tx) > 0$ for all $x \in C$. Now, we construct two sequence $\{x_n\}$ and $\{y_n\}$ in C in the following way. Let $x_0 \in C$ and $y_1 \in Tx_0$.

If $y_1 \in C$, let $x_1 = y_1$.

If $y_1 \notin C$, select a point $x_1 \in \partial C$ such that

$$d(x_0,x_1) + d(x_1,y_1) = d(x_0,y_1).$$

Thus $x_1 \in C$. Now, we claim that $d(y_1,Tx_1) > 0$. Suppose $d(y_1,Tx_1) = 0$. If $y_1 \in C$, then x_1 is a fixed point of T, which is a contradiction. If $y_1 \notin C$, then $x_1 \in \partial C$ and so $Tx_1 \subseteq C$. Therefore, $y_1 \notin Tx_1$, which is a contradiction. Thus, $d(y_1,Tx_1) > 0$. Now, since $d(y_1,Tx_1) \leq H(Tx_0,Tx_1)$, then we get

$$\theta(d(y_1,Tx_1)) \leq \theta(H(Tx_0,Tx_1)) \leq [\theta(d(x_0,x_1))]^k. \tag{3.13}$$

On the other hand, since Tx_1 is compact there exists $y_2 \in Tx_1$ such that

$$d(y_1,Tx_1) = d(y_1,y_2).$$

Thus, from (3.13) we get

$$\theta(d(y_1,y_2)) \leq [\theta(d(x_0,x_1))]^k.$$

The rest of the proof can be completed as in the proof of Theorem 3.5. □

The provided nontrivial example shows that Theorem 3.5 is a proper generalization of Theorem 2.4.

Example 3.2. Consider the sequence $\{s_n\}_{\mathbb{N}\cup\{0\}}$ as $s_0 = 0$ and

$$s_n = s_{n-1} + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for } n \geq 1.$$

Let $X = \mathbb{R}$ and $C = (-\infty,0) \cup \{s_n : n \in \mathbb{N}\cup\{0\}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete and metrically convex metric space and C is a closed subset of X. Let $T: C \to CB(X)$ be given by

$$Tx = \begin{cases} \{-\frac{x}{4}\} & , \quad x \in (-\infty,0] \\ \\ \{0, s_{n-1}\} & , \quad x = s_n, n \geq 1 \end{cases}.$$

Note that $\partial C = \{s_n : n \in \mathbb{N}\cup\{0\}\}$ and, for each $x \in \partial C$, $Tx \subset C$. Now, we show that the contractive condition (3.6) of Theorem 3.5 is satisfied with $\theta(t) = e^{\sqrt{te^t}}$ and $k = e^{-0.5}$. So, the contractive condition (3.6) turns into

$$\frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} \leq e^{-1},$$

for each $x, y \in C$ with $H(Tx,Ty) > 0$. Note that, if $H(Tx,Ty) > 0$, then $(x,y) \notin \{(0,s_1),(s_1,0)\}$ and $x \neq y$. In the following cases, without lost of generality we may assume $x > y$:

Case 1. Consider $x, y \in (-\infty, 0]$. Then, we obtain

$$\frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} = \frac{\frac{1}{4}d(x,y)}{d(x,y)} e^{\frac{1}{4}d(x,y)-d(x,y)} = \frac{1}{4}e^{-\frac{3}{4}d(x,y)} < \frac{1}{4} < e^{-1}.$$

Case 2. Consider $x, y \in \{s_n : n \in \mathbb{N}\}$. Then for $x = s_m$ and $y = s_n$, we obtain

$$\begin{aligned} \frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} &= \frac{s_{m-1} - s_{n-1}}{s_m - s_n} e^{s_{m-1}-s_{n-1}-s_m+s_n} \\ &= \frac{2(m^3-n^3) - 3(m^2-n^2) + (m-n)}{2(m^3-n^3) + 3(m^2-n^2) + (m-n)} e^{-(m^2-n^2)} \\ &< e^{-(m^2-n^2)} < e^{-1}. \end{aligned}$$

Case 3. Consider $x \in \{s_n : n \in \mathbb{N}\}$ and $y \in (-\infty, 0]$. Then,

$$H(Tx,Ty) = H(\{0, s_{n-1}\}, \{-\frac{y}{4}\}) \leq \max\{\frac{|y|}{4}, s_{n-1}\}$$

and $d(x,y) = s_n + |y|$. Thus

$$\begin{aligned} \frac{H(Tx,Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} &\leq \frac{\max\{\frac{|y|}{4}, s_{n-1}\}}{s_n + |y|} e^{\max\{\frac{|y|}{4}, s_{n-1}\} - s_n - |y|} \\ &< e^{\max\{\frac{|y|}{4}, s_{n-1}\} - s_n - |y|} < e^{-1}. \end{aligned}$$

Consequently, by summarizing all cases, we conclude that contractive condition (3.6) is satisfied. Hence, all assumptions in Theorem 3.5 are satisfied and so, T has a fixed point in C.

On the other hand, it is easy to show that Theorem 2.2 cannot be applied to this example. Indeed, suppose there exists $k \in (0, 1)$ such that condition (2.4) holds. If we take $x = s_n$ for $n \geq 2$ and $y = 0$, then $H(Tx, Ty) = H(\{0, s_{n-1}\}, \{0\}) = s_{n-1}$ and $d(x, y) = s_n$. Then, $\lim_{n\to\infty} \frac{H(Ts_n, T0)}{d(s_n, 0)} = \lim_{n\to\infty} \frac{s_{n-1}}{s_n} = 1$, which contradict to $k < 1$.

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