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Univalence conditions and properties of a new general integral operator

ROBERTA BUCUR and DANIEL BREAZ

ABSTRACT. In this paper, we obtain univalence conditions and the order of convexity of a new general integral operator defined on the space of normalized analytic functions in the open unit disk U. Also, we give some other properties on the class $N(\varphi)$. Results presented in this paper may motivate further reserch in this fascinating field.

1. INTRODUCTION, DEFINITIONS AND P RELIMINARIES

Let $U = \{z : |z| < 1\}$ denote the open unit disk of the complex plane and $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1.1}$$

which are analytic in U and satisfy the condition $f(0) = f'(0) - 1 = 0$.
Consider $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

A function $f \in \mathcal{A}$ is said to be starlike of order δ, $0 \leq \delta < 1$, that is $f \in S^*(\delta)$, if and only if

$$Re\left[\frac{zf'(z)}{f(z)}\right] > \delta \quad (z \in U).$$

A function $f \in \mathcal{A}$ is said to be convex of order δ, $0 \leq \delta < 1$, that is $f \in K(\delta)$, if and only if

$$Re\left[\frac{zf''(z)}{f'(z)} + 1\right] > \delta \quad (z \in U).$$

It is well known that $S^*(0) \equiv S^*$ (see [1]) and $K(0) \equiv K$ (see [18]) are the classes of starlike and convex functions in U, respectively.

Also, let $N(\varphi)$, $\varphi > 1$ denote the class of functions $f \in A$ which satisfy

$$Re\left[\frac{zf''(z)}{f'(z)} + 1\right] < \varphi \quad (z \in U).$$

The class $N(\varphi)$ was studied in [3, 13, 21].

Frasin and Jahangiri [8] defined the family $B(\mu, \lambda), \mu \geq 0, 0 \leq \lambda < 1$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left|f'(z)\left[\frac{z}{f(z)}\right]^{\mu} - 1\right| < 1 - \lambda, \quad (z \in U). \tag{1.2}$$

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Corresponding author: Roberta Bucur; roberta_bucur@yahoo.com

The family $B(\mu,\lambda)$ is a comprehensive class of analytic functions. For instance, we have $B(1,\lambda)=S^*(\lambda)$ and $B(2,\lambda)=B(\lambda)$ (see Frasin and Darus [9]).

In the present paper, we define a new integral operator given by

$$F(z)=\int_0^z\prod_{i=1}^n\left[\frac{tf_i'(t)}{g_i(t)}e^{h_i(t)}\right]^{\alpha_i}dt, \tag{1.3}$$

where parameters $\alpha_i\in\mathbb{C}$ and the functions $f_i,g_i,h_i\in\mathcal{A}$, $i\in\{1,\dots,n\}$, are so constrained that the integral (1.3) exists.

The operator F generalizes certain integral operators:

(i) For $e^{h_i(t)}=1, g_i(t)=t$ and $\alpha_i>0$ we have $F(z)=\int_0^z\prod_{i=1}^n\left[f_i'(t)\right]^{\alpha_i}dt$, that was defined by Breaz, Owa and Breaz in [4], and this operator is a generalization of the integral operator $F(z)=\int_0^z\left[f'(t)\right]^{\alpha}dt$, discussed in [14, 15, 17].

(ii) For $g_i(t)=t$ we get $F(z)=\int_0^z\prod_{i=1}^n\left[f_i'(t)e^{h_i(t)}\right]^{\alpha_i}dt$ which was studied by Oprea and Breaz in [12] and this operator is a generalization of the integral operator $F(z)=\int_0^z\left[f'(t)e^{h(t)}\right]^{\alpha}dt$, defined and studied by Ularu and Breaz in [19, 20].

(iii) For $n=1$ and $e^{h_1(t)}=1$ we obtain $F(z)=\int_0^z\left[\frac{tf'(t)}{g(t)}\right]^{\alpha}dt$, introduced and studied by Bucur, Andrei and Breaz in [5].

(iv) For $n=1$ and $f_1(t)=t$ we have $F(z)=\int_0^z\left[\frac{te^{h(t)}}{g(t)}\right]^{\alpha}dt$ defined and disscused by Bucur, Andrei and Breaz in [6].

Recently, many authors studied the sufficient conditions for the univalence and convexity of certain families of integral operators in the open unit disk and some of them motivated our work (see [7, 16]).

In order to derive our main results, we have to recall here the following:

Lemma 1.1. (Mocanu and Şerb [11]) *Let $M_0=1,5936...$, the positive solution of equation*

$$(2-M)e^M=2. \tag{1.4}$$

If $f\in\mathcal{A}$ and

$$\left|\frac{f''(z)}{f'(z)}\right|\le M_0\quad(z\in U),$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1\quad(z\in U). \tag{1.5}$$

The edge M_0 is sharp.

Lemma 1.2. (Becker [2]) *If the function f is regular in the unit disk U, $f(z)=z+a_2z^2+\dots$ and*

$$(1-|z|^2)\cdot\left|\frac{zf''(z)}{f'(z)}\right|\le 1\quad(z\in U), \tag{1.6}$$

then the function f is univalent in U.

Lemma 1.3. (The General Schwarz Lemma [10]) *Let f be regular function in the disk $U_R=\{z\in\mathbb{C}:|z|<R\}$ with $|f(z)|<M$, for M fixed. If f has at $z=0$ one zero with multiplicity*

bigger than m, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad (z \in U_R).$$

The equality holds if and only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

Lemma 1.4. (Wilken and Feng [22]) *If $0 \leq \delta < 1$ and $f \in K(\delta)$, then $f \in S^*(\nu(\delta))$, where*

$$\nu(\delta) = \begin{cases} \frac{1-2\delta}{2^{2(1-\delta)}-2}, & \text{if } \delta \neq \frac{1}{2}, \\ \frac{1}{2\log 2}, & \text{if } \delta = \frac{1}{2}. \end{cases} \tag{1.7}$$

2. Main results

In the following theorem we give sufficient conditions for the univalence of the operator F defined in (1.3), by using Mocanu and Şerb Lemma and Becker univalence criterion.

Theorem 2.1. *Let $\alpha_1, \alpha_2, \dots \alpha_n$ be complex numbers, M_0 the positive solution of the equation (1.4), $M_0 = 1,5936...$, functions $f_i, g_i \in \mathcal{A}$ and $h_i \in B(\mu_i, \lambda_i)$, $\mu_i \geq 0$, $0 \leq \lambda_i < 1$, $N_i \geq 1$ and $M_i \geq 1$, $i \in \{1, \dots, n\}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq N_i, \quad \left|\frac{g''(z)}{g'(z)}\right| \leq M_0, \quad |h_i(z)| < M_i \quad (z \in U;\ i \in \{1, \dots, n\}) \tag{2.8}$$

and

$$\sum_{i=1}^{n} |\alpha_i| \cdot \frac{9a_i^2 - 1 + (3a_i^2 + 1)\sqrt{3a_i^2 + 1}}{a_i^2} \leq \frac{27}{2}, \tag{2.9}$$

where $a_i = N_i + (2 - \lambda_i)M_i^{\mu_i}$, then the function F given by (1.3) is in the class S.

Proof. After some computations, we obtain that

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^{n} \alpha_i \left[\frac{zf_i''(z)}{f_i'(z)} - \left(\frac{zg_i'(z)}{g_i(z)} - 1\right) + zh_i'(z)\right], \tag{2.10}$$

which readily yields

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i| \left\{|z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + |zh_i'(z)|\right\}$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \left\{|z| \cdot \left|\frac{f_i''(z)}{f_i'(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \left(\left|h_i'(z)\left(\frac{z}{h_i(z)}\right)^{\mu_i} - 1\right| + 1\right) \cdot \frac{|h_i(z)|_i^{\mu}}{|z|^{\mu_i - 1}}\right\}. \tag{2.11}$$

Now, by using (2.8) and applying Lemma Mocanu and Şerb to the functions $g_1, \dots, g_n$, we obtain

$$\left|\frac{zg_i'(z)}{g_i(z)} - 1\right| < 1 \quad (z \in U). \tag{2.12}$$

Also, applying the General Schwarz Lemma to the functions $h_1, \dots, h_n$, we have

$$|h_i(z)| \leq M_i|z| \quad (z \in U). \tag{2.13}$$

Using the hypothesis of the theorem and replacing (2.12) and (2.13) in inequality (2.11), we find that

$$(1 - |z|^2) \cdot \left|\frac{zF''(z)}{F'(z)}\right| \leq (1 - |z|^2) \cdot \left\{\sum_{i=1}^{n} |\alpha_i| [1 + N_i|z| + (2 - \lambda_i)M_i^{\mu_i}|z|]\right\} = \sum_{i=1}^{n} |\alpha_i| G_i(x), \tag{2.14}$$

where function $G_i : [0,1) \to \mathbb{R}$,

$$G_i(x) = (1-x^2)(a_i x + 1),\ x = |z| \text{ and } a_i = N_i + (2-\lambda_i)M_i^{\mu_i}.$$

Since the maximum point of G_i is $x = \frac{\sqrt{1+3a_i^2}-1}{3a_i}$, it results

$$G_i(x) \le \frac{2[9a_i^2 - 1 + (3a_i^2+1)\sqrt{3a_i^2+1}]}{27a_i^2} \quad (x \in [0,1)),$$

which in conjunction with the inequality (2.14), leads us to

$$(1-|z|^2)\cdot\left|\frac{zF''(z)}{F'(z)}\right| \le 1 \quad (z \in U).$$

Finally by applying Lemma 1.2, it results that function F given by (1.3) is in the class S. □

Theorem 2.2. *Let $\alpha_1, \alpha_2, \dots \alpha_n$ be complex numbers, functions $f_i \in K(\delta_i), 0 \le \delta < 1$, $g_i \in S^*(k_i), 0 \le k_i < 1$ and $h_i \in B(\mu_i, \lambda_i),\ \mu_i \ge 0,\ 0 \le \lambda_i < 1,\ i \in \{1,\dots,n\}$. Suppose that for all $z \in U$, we have*

$$|h_i(z)| < M_i \quad (M_i \ge 1,\ i \in \{1,\dots,n\}). \tag{2.15}$$

If

$$\sum_{i=1}^{n} |\alpha_i| \cdot \frac{\left(3a_i^2 - b_i^2 + b_i\sqrt{3a_i^2+b_i^2}\right)\left(2b_i + \sqrt{3a_i^2+b_i^2}\right)}{a_i^2} \le \frac{27}{2}, \tag{2.16}$$

where $a_i = (2-\lambda_i)M_i^{\mu_i}$ and $b_i = 2 - k_i - \delta_i$, then the function F given by (1.3) is in the class S.

Proof. Just as in the proof of Theorem 2.1, we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left\{ \left|\frac{zf_i''(z)}{f_i'(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + |zh_i'(z)| \right\}.$$

(2.17)

Next, using the General Schwarz Lemma for functions h_i $(i = 1,\dots,n)$, together with the hypothesis of the Theorem 2.2, the last inequality implies

$$(1-|z|^2)\cdot\left|\frac{zF''(z)}{F'(z)}\right|$$

$$\le (1-|z|^2)\cdot\left\{\sum_{i=1}^{n}|\alpha_i|\left[2-\delta_i-k_i+\left(\left|h_i'(z)\left(\frac{z}{h_i(z)}\right)^{\mu_i}-1\right|+1\right)\cdot\frac{|h_i(z)|_i^{\mu}}{|z|^{\mu_i-1}}\right]\right\}$$

$$\le (1-|z|^2)\cdot\left\{\sum_{i=1}^{n}|\alpha_i|[2-\delta_i-k_i+(2-\lambda_i)M_i^{\mu_i}|z|]\right\} = \sum_{i=1}^{n}|\alpha_i|H_i(x), \tag{2.18}$$

where

$$H_i(x) = (1-x^2)(a_i x + b_i),\ x = |z|,\ a_i = (2-\lambda_i)M_i^{\mu_i} \text{ and } b_i = 2-\delta_i-k_i.$$

After some computations, we obtain that $x = \frac{\sqrt{b_i^2+3a_i^2}-b_i}{3a_i}$ is the maximum point of G_i, so

$$G_i(x) \le \frac{2\left(3a_i^2 - b_i^2 + b_i\sqrt{3a_i^2+b_i^2}\right)\left(2b_i + \sqrt{3a_i^2+b_i^2}\right)}{27a_i^2} \quad (x \in [0,1)).$$

Thus,

$$(1-|z|^2)\cdot\left|\frac{zF''(z)}{F'(z)}\right|\le 1 \quad (z\in U), \tag{2.19}$$

and applying Lemma 1.2, we yield that F given by (1.3) is in the class S. □

Corollary 2.1. *Let $\alpha_1,\alpha_2,\dots\alpha_n$ be complex numbers, functions $f_i\in K, g_i\in S^*$ and $h_i\in S^*$, $i\in\{1,\dots,n\}$. Suppose that for all $z\in U$, we have*

$$|h_i(z)|<M_i \quad (M_i\ge 1,\ i\in\{1,\dots,n\}).$$

If

$$\sum_{i=1}^{n}|\alpha_i|\cdot\frac{\left(6M_i^2-1+\sqrt{3M_i^2+1}\right)\left(2+\sqrt{3M_i^2+1}\right)}{M_i^2}\le\frac{27}{2},$$

then the function F given by (1.3) is in the class S.

Corollary 2.2. *Let $\alpha_1,\alpha_2,\dots\alpha_n$ be complex numbers, functions $f_i\in K(\delta_i)$, $0\le\delta_i<1$, $g_i\in K(k_i)$, $0\le k_i<1$ and $h_i\in S^*(\lambda_i)$, $i\in\{1,\dots,n\}$). Suppose that for all $z\in U$, we have*

$$|h_i(z)|<M_i \quad (M_i\ge 1,\ i\in\{1,\dots,n\}).$$

If

$$\sum_{i=1}^{n}|\alpha_i|\cdot\frac{\left(3a_i^2-b_i^2+b_i\sqrt{3a_i^2+b_i^2}\right)\left(2b_i+\sqrt{3a_i^2+b_i^2}\right)}{a_i^2}\le\frac{27}{2},$$

where $a_i=(2-\lambda_i)M_i$, $b_i=2-\delta_i-\beta(k_i)$, and $\beta(k_i)$ is given by relation(1.7) , then the function F given by (1.3) is in the class S.

Proof. If $g_i\in K(k_i)$, applying Lemma 1.4, we obtain $g_i\in S^*(\beta(k_i))$. Further, using Theorem 2.2 with $\mu_i=1$, we deduce that function F given by (1.3) is in the class S. □

Theorem 2.3. *Let α_i be complex numbers, functions $f_i\in\mathcal{A}$, $g_i\in B(\eta_i,\nu_i)$, $\eta_i\ge 0$, $0\le\nu_i<1$, and $h_i\in B(\mu_i,\lambda_i)$, $\mu_i\ge 0$, $0\le\lambda_i<1$, $i\in\{1,\dots,n\}$. Suppose that*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right|<N_i\ (N_i\ge 1),\quad |g_i(z)|<P_i\ (P_i\ge 1)\quad \textit{and}\quad |h_i(z)|<M_i\ (M_i\ge 1), \tag{2.20}$$

for all $z\in U$ and $i\in\{1,\dots,n\}$. Then the function F given by (1.3) is in the class $K(\delta)$, where

$$\delta=1-\sum_{i=1}^{n}|\alpha_i|\cdot[1+N_i+(2-\nu_i)P_i^{\eta_i-1}+(2-\lambda_i)M_i^{\mu_i}] \tag{2.21}$$

and

$$0<\sum_{i=1}^{n}|\alpha_i|\cdot[1+N_i+(2-\nu_i)P_i^{\eta_i-1}+(2-\lambda_i)M_i^{\mu_i}]\le 1. \tag{2.22}$$

Proof. From (2.10), we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right|\le\sum_{i=1}^{n}|\alpha_i|\cdot\left\{1+|z|\cdot\left|\frac{f_i''(z)}{f_i'(z)}\right|+\left|g_i'(z)\cdot\left(\frac{z}{g_i(z)}\right)^{\eta_i}\right|\cdot\left|\left(\frac{g_i(z)}{z}\right)\right|^{\eta_i-1}+\right.$$
$$\left.|z|\cdot\left|h_i'(z)\cdot\left(\frac{z}{h_i(z)}\right)^{\mu_i}\right|\cdot\left|\left(\frac{h_i(z)}{z}\right)\right|^{\mu_i}\right\}. \tag{2.23}$$

Using the hypothesis and the General Schwarz Lemma, inequation(2.23) implies

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i| \cdot \left\{1 + |z| \cdot N_i + \left(\left|g_i'(z) \cdot \left(\frac{z}{g_i(z)}\right)^{\eta_i} - 1\right| + 1\right) P_i^{\eta_i - 1} + \right.$$
$$\left.\left(\left|h_i'(z) \cdot \left(\frac{z}{h_i(z)}\right)^{\mu_i} - 1\right| + 1\right) M_i^{\mu_i}\right\}$$
$$\leq \sum_{i=1}^{n} |\alpha_i| \cdot [1 + N_i + (2 - \nu_i) P_i^{\eta_i - 1} + (2 - \lambda_i) M_i^{\mu_i}] = 1 - \delta.$$

This evidently completes the proof. □

Corollary 2.3. *Let α_i be complex numbers, functions $f_i \in \mathcal{A}$, $g_i \in S^*(\nu_i)$, $0 \leq \nu_i < 1$, and $h_i \in S^*(\lambda_i)$, $0 \leq \lambda_i < 1$, $i \in \{1, \dots, n\}$. Suppose that*

(2.24) $$\left|\frac{f_i''(z)}{f_i'(z)}\right| < N_i \ (N_i \geq 1), \quad |g_i(z)| < P_i \ (P_i \geq 1) \quad \text{and} \quad |h_i(z)| < M_i \ (M_i \geq 1),$$

for all $z \in U$ and $i \in \{1, \dots, n\}$. Then the function F given by (1.3) is in the class $K(\delta)$, where

(2.25) $$\delta = 1 - \sum_{i=1}^{n} |\alpha_i| \cdot [3 + N_i - \nu_i + (2 - \lambda_i) M_i]$$

and

(2.26) $$0 < \sum_{i=1}^{n} |\alpha_i| \cdot [3 + N_i - \nu_i + (2 - \lambda_i) M_i] \leq 1.$$

Proof. In Theorem 2.3, we consider $\mu_i = 1$ and $\eta_i = 1$ for all $i \in \{1, 2, \dots n\}$. □

Corollary 2.4. *Let α_i be complex numbers, functions $f_i \in \mathcal{A}$, $g_i \in S^*$ and $h_i \in S^*$, $i \in \{1, \dots, n\}$. Suppose that*

(2.27) $$\left|\frac{f_i''(z)}{f_i'(z)}\right| < N_i \ (N_i \geq 1), \quad |g_i(z)| < P_i \ (P_i \geq 1) \quad \text{and} \quad |h_i(z)| < M_i \ (M_i \geq 1),$$

for all $z \in U$ and $i \in \{1, \dots, n\}$. If

(2.28) $$\sum_{i=1}^{n} |\alpha_i| \cdot [3 + 2M_i + N_i] = 1,$$

then the function F given by (1.3) is convex in U.

Proof. In Corollary 2.3, we consider $\lambda_i = 1$, $\nu_i = 0$ for all $i \in \{1, 2, \dots n\}$ and $\delta = 0$. □

In the following theorem we give sufficient conditions such that the integral operator F is in the class $N(\varphi)$.

Theorem 2.4. *Let α_i be positive real numbers, functions $f_i \in N(\beta_i)$, $g_i \in S^*(k_i)$, $0 \leq k_i < 1$ and $h_i \in B(\mu_i, \lambda_i)$, $\mu_i \geq 0$, $0 \leq \lambda_i < 1$, $i \in \{1, \dots, n\}$. If*

$$|h_i(z)| < M_i \ (z \in U,\ M_i \geq 1,\ i \in \{1, \dots, n\}),$$

then the function F given by (1.3) is in the class $N(\varphi)$, where

$$\varphi = 1 + \sum_{i=1}^{n} \alpha_i [\beta_i - k_i + (2 - \lambda_i) M_i^{\mu_i}].$$

Proof. From (2.10), we obtain

(2.29)
$$\mathrm{Re}\left[\frac{zF''(z)}{F'(z)}+1\right]=\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left[\frac{zf_i''(z)}{f_i'(z)}+1\right]-\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left[\frac{zg_i'(z)}{g_i(z)}\right]+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left[zh_i'(z)\right]+1.$$

Letting $h_i\in B(\mu_i,\lambda_i)$ with $h_i(0)=0$ and $|h_i(z)|<M_i$ for all $i\in\{1,\dots,n\}$, from General Schwarz Lemma, we get

$$\mathrm{Re}\left[zh_i'(z)\right]\le\left|zh_i'(z)\right|\le\left(\left|h_i'(z)\cdot\left(\frac{z}{h_i(z)}\right)^{\mu_i}-1\right|+1\right)M_i^{\mu_i}\cdot|z|<(2-\lambda_i)M_i^{\mu_i}. \tag{2.30}$$

So, using the hypothesis and (2.30) in relation (2.29), we have

$$\mathrm{Re}\left[\frac{zF''(z)}{F'(z)}+1\right]<\sum_{i=1}^{n}\alpha_i(\beta_i-k_i+(2-\lambda_i)M_i^{\mu_i})+1.$$

Since $\sum_{i=1}^{n}\alpha_i(\beta_i-k_i+(2-\lambda_i)M_i^{\mu_i})>0$, we yield that F given by (1.3) is in the class $N(\varphi)$, where $\varphi=1+\sum_{i=1}^{n}\alpha_i[\beta_i-k_i+(2-\lambda_i)M_i^{\mu_i}]$. □

Corollary 2.5. *Let α_i be positive real numbers, functions $f_i\in N(\beta),\ \beta>1$ $g_i\in S^*(k),\ 0\le k<1$ and $h_i\in S^*(\lambda),\ 0\le\lambda<1,\ i\in\{1,\dots,n\}$. If*

$$|h_i(z)|\le M\ (z\in U, M\ge 1,\ i\in\{1,\dots,n\}),$$

then the function F given by (1.3) is in the class $N(\varphi)$, where

$$\varphi=1+\sum_{i=1}^{n}\alpha_i[\beta-k+(2-\lambda)M].$$

Proof. In Theorem 2.4, we consider $\mu_i=1$ and $\beta_i=\beta>1$ and $k_i=k$ for all $i\in\{1,\dots,n\}$. □

Remark 2.1. Many other interesting corollaries of Theorems 2.1 to 2.4 can be obtained by suitably specializing the parameters and the functions involved.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITEŞTI
STR. TÂRGUL DIN VALE 1, R-110040 PITEŞTI, ROMANIA
E-mail address: roberta_bucur@yahoo.com

DEPARTMENT OF MATHEMATICS
"1 DECEMBRIE 1918" UNIVERSITY OF ALBA IULIA
STR. N. IORGA 11-13, R-510009 ALBA IULIA, ROMANIA
E-mail address: dbreaz@uab.ro