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# The iterates of positive linear operators with the set of constant functions as the fixed point set

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ABSTRACT. Let  $\Omega \subset \mathbb{R}^p$ ,  $p \in \mathbb{N}^*$  be a nonempty subset and  $B(\Omega)$  be the Banach lattice of all bounded real functions on  $\Omega$ , equipped with *sup norm*. Let  $X \subset B(\Omega)$  be a linear sublattice of  $B(\Omega)$  and  $A: X \to X$  be a positive linear operator with constant functions as the fixed point set. In this paper, using the weakly Picard operators techniques, we study the iterates of the operator A. Some relevant examples are also given.

#### 1. INTRODUCTION

Let *X* be a real Banach space and  $A: X \to X$  be a linear operator. Let us denote  $(AD)_A(x^*) := \{x \in X | A^n(x) \to x^* \text{ as } n \to \infty\}$  the attraction domain of a fixed point  $x^*$  of the operator *A*. Let  $F_A := \{x \in X | A(x) = x\}$  be the fixed point set of *A*. By definition, the operator *A* is weakly Picard operator (WPO) if,  $X = \bigcup_{x^* \in F_A} (AD)_A(x^*)$ .

Let us denote by  $\theta$  the zero element of *X*. It is clear that  $(AD)_A(\theta)$  is a linear subspace of *X* and  $(AD)_A(x^*) = \{x^*\} + (AD)_A(\theta)$ , i.e., is an affine subspace of *X*. This remark gives rise to the following notion (see [20]).

A partition of X,  $X = \bigcup_{x^* \in F_A} X_{x^*}$ , is a linear fixed point partition (LFPP) of X with respect to a linear operator A iff:

- (i)  $X_{x^*} \cap F_A = \{x^*\}, \forall x^* \in F_A;$
- (ii)  $A(X_{x^*}) \subset X_{x^*}, \forall x^* \in F_A;$
- (iii)  $X_{\theta}$  is a linear subspace of X;
- (iv)  $X_{x^*} = \{x^*\} + X_{\theta}$ .

The aim of this paper is to study the iterates of a linear operator, in the case of function spaces, using the technique of LFPP of the space.

## 2. PRELIMINARIES

Let  $\Omega \subset \mathbb{R}^p$ ,  $p \in \mathbb{N}^*$  be a nonempty subset,  $B(\Omega)$  be the Banach lattice of all bounded real valued functions on  $\Omega$ , equipped with *sup norm*. Let  $X \subset B(\Omega)$  be a linear sublattice of *X* and *A*:  $X \to X$  be a linear operator with constant functions as the fixed point set.

Following [20], we consider some notions that will be used in the sequel.

**Definition 2.1.** The operator  $A: X \to X$  is a weakly Picard operator (WPO) if the sequence  $(A^n(f))_{n \in \mathbb{N}}$  converges, for all  $f \in B(\Omega)$ , and the limit (which may depend on f) is a fixed point of A.

**Definition 2.2.** If *A* is WPO, then we define the operator  $A^{\infty}$ ,  $A^{\infty}$ :  $X \to X$ , by

 $A^{\infty}(f) := \lim_{n \to \infty} A^n(f).$ 

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We remark that  $A^{\infty}(X) = F_A$ .

**Definition 2.3.** Let  $A: X \to X$  be a weakly Picard operator and  $\psi: \mathbb{R}_+ \to \mathbb{R}_+$  an increasing function, continuous in 0 and  $\psi(0) = 0$ . The operator A is said to be a  $\psi$ -weakly Picard operator ( $\psi$ -WPO) iff

$$d(f, A^{\infty}(f)) \le \psi d(f, A(f)), \, \forall f \in X.$$

**Definition 2.4.** The operator  $A: X \to X$  is a Picard operator (PO) if A is WPO and  $F_A$  is a unit set.

**Remark 2.1.** We observe that  $A^k(X)$  is an invariant subset of A for each  $k \in \mathbb{N}^*$  and  $F_A \subset A^k(X)$ . We suppose that

$$A_k := A|_{A^k(X)} \colon A^k(X) \to A^k(X)$$
 is WPO.

Then we have that

$$A_k^n(u) \to A_k^\infty(u) \text{ as } n \to \infty, \ \forall u \in A^k(X),$$

i.e.,  $A^n(A^k(f)) \to A^{\infty}_k(A^k(f)) = A^{\infty}(f)$ . So, if for some  $k \in \mathbb{N}^*$ ,  $A|_{A^k(X)}$  is WPO then,  $A: X \to X$  is WPO and  $A^{\infty}(f) = A^{\infty}_k(A^k(f))$ .

**Remark 2.2.** Let  $\phi: X \to \mathbb{R}$  be a linear functional and  $A: X \to X$  a linear operator. We suppose that  $\phi$  is an invariant functional of A, i.e.,  $\phi(A(f)) = \phi(f), \forall f \in X$ . Let us denote, for  $\lambda \in \mathbb{R}$ ,

$$X_{\lambda} := \{ f \in X | \phi(f) = \lambda \}.$$

Then  $X = \bigcup_{\lambda \in \mathbb{R}} X_{\lambda}$  is a partition of *X*. If  $X_{\lambda} \cap F_A = \{f_{\lambda}^*\}, \forall \lambda \in \mathbb{R}$ , then,  $X = \bigcup_{\lambda \in \mathbb{R}} X_{\lambda}$  is a LFPP of *X* with respect to *A*.

We also need the following result.

**Lemma 2.1.** ([20]) Let  $A: X \to X$  be a linear operator. We suppose that  $X = \bigcup_{f^* \in F_A} X_{f^*}$  is a

LFPP of X with respect to A. Then:

- (i) If  $A|_{X_{\theta}} \colon X_{\theta} \to X_{\theta}$  is PO, then A is WPO.
- (ii) If A is WPO, then  $A^{\infty}(f) = f^*, \forall f \in X_{f^*}, f^* \in F_A$ .

As a suggestion for finding a LFPP of the space, the following result is useful.

**Theorem 2.1.** (Characterization theorem, [15]) An operator A is a weakly Picard operator if and only if there exists a partition of  $X, X = \bigcup X_{\lambda}$ , such that

- (a)  $A(X_{\lambda}) \subset X_{\lambda}, \forall \lambda \in \Lambda;$
- (b)  $A|_{X_{\lambda}} : X_{\lambda} \to X_{\lambda}$  is a Picard operator,  $\forall \lambda \in \Lambda$ .

The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [2], [3], [8], [9], [12], [16], [18], [19].

## 3. BASIC RESULTS

Let  $\Omega \subset \mathbb{R}^p$ ,  $p \in \mathbb{N}^*$  be a nonempty subset,  $B(\Omega)$  be the Banach lattice of all bounded real valued functions on  $\Omega$ , equipped with *sup norm*. Let  $X \subset B(\Omega)$  a linear sublattice of  $B(\Omega)$  and  $A: X \to X$  be a linear operator. We have

## **Theorem 3.2.** We suppose that:

- (i) X contains all constant functions on  $\Omega$ ;
- (ii)  $F_A$  consists of all constant functions on  $\Omega$ ;

(iii)  $X = \bigcup_{\lambda \in \mathbb{R}} X_{\lambda}$  is a LFPP of X with respect to A such that, for some  $k \in \mathbb{N}$ , we have that:

$$||A(u)|| \le l ||u||, \ \forall u \in A^k(X_0), \ with \ some \ 0 < l < 1.$$

Then:

- (a) A is WPO and  $A^{\infty}(f) = \lambda, \forall f \in X_{\lambda}, \lambda \in \mathbb{R};$
- (b)  $||A^k(f) A^{\infty}(f)|| \le \frac{1}{1-l} ||A^k(f) A(A^k(f))||, \forall f \in X;$
- (c)  $||A^n(A^k(f)) A^\infty(f)|| \le \frac{l^n}{1-l} ||A^k(f) A(A^k(f))||, \forall f \in X, \forall n \in \mathbb{N};$
- (d) if M > 0:  $||A^k(f) A(A^k(f))|| \le M$ ,  $\forall f \in X$ , then  $A^n(A^k(f)) \xrightarrow{unif} A^\infty(f)$  as  $n \to \infty$ , on X.
- *Proof.* (a) Let  $f, g \in X_{\lambda}$ . By linearity of A and by the fact that  $f g \in X_0$ , it follows that

$$||A(f) - A(g)|| = ||A(f - g)|| \le l ||f - g||$$
, with  $0 < l < 1, \forall f, g \in X_{\lambda}, \lambda \in \mathbb{R}$ .

The constant function  $\lambda \in X_{\lambda}$  is a fixed point of A. Cosequently, we have that A is a Picard operator, and taking into account (iii), by Theorem 2.1, it follows that the operator A is a weakly Picard operator, with  $A^{\infty}(f) = \lambda, \forall f \in X_{\lambda}, \lambda \in \mathbb{R}$ .

(3.1)

$$\|A^{k}(f) - A^{\infty}(f)\| = \|A^{k}(f) - \lambda\| \le \|A^{k}(f) - A(A^{k}(f))\| + \|A(A^{k}(f)) - \lambda\|, \forall f \in X.$$
By (iii) it follows

By (iii), it follows

(3.2) 
$$\|A(A^k(f)) - \lambda\| \le l \|A^k(f) - \lambda\|, \text{ with } f - \lambda \in A(X_0).$$

From (3.1) and (3.2) we get

$$||A^{k}(f) - \lambda|| \le ||A^{k}(f) - A(A^{k}(f))|| + l ||A^{k}(f) - \lambda||.$$

Then

$$||A^{k}(f) - A^{\infty}(f)|| \le \frac{1}{1-l} ||A^{k}(f) - A(A^{k}(f))||, \forall f \in X.$$

(c) We have

$$\begin{split} \left\| A^{n}(A^{k}(f)) - A^{n+p}(A^{k}(f)) \right\| \\ &\leq \left\| A^{n}(A^{k}(f)) - A^{n+1}(A^{k}(f)) \right\| + \left\| A^{n+1}(A^{k}(f)) - A^{n+2}(A^{k}(f)) \right\| \\ &\quad + \dots + \left\| A^{n+p-1}(A^{k}(f)) - A^{n+p}(A^{k}(f)) \right\| \\ &\leq l^{n} \left\| A^{k}(f) - A(A^{k}(f)) \right\| + l^{n+1} \left\| A^{k}(f) - A(A^{k}(f)) \right\| \\ &\quad + \dots + l^{n+p-1} \left\| A^{k}(f) - A(A^{k}(f)) \right\| \\ &= (l^{n} + \dots + l^{n+p-1}) \left\| A^{k}(f) - A(A^{k}(f)) \right\| \\ &\leq \frac{l^{n}}{1-l} \left\| A^{k}(f) - A(A^{k}(f)) \right\|, \text{for } n \in \mathbb{N}, p \in \mathbb{N}^{*}. \\ &\text{So, } \left\| A^{n}(A^{k}(f)) - A^{\infty}(f) \right\| \leq \frac{l^{n}}{1-l} \left\| A^{k}(f) - A(A^{k}(f)) \right\|, \forall f \in X, \forall n \in \mathbb{N}. \\ \end{split}$$
(d) By (c) we obtain  $\left\| A^{n}(A^{k}(f)) - A^{\infty}(f) \right\| \leq \frac{l^{n}}{1-l} M \stackrel{\text{unif}}{\to} 0. \end{split}$ 

Let us give a class of operators for which the condition (iii) in Theorem 3.2 is satisfied. Let  $X := C([0,1]^p), p \in \mathbb{N}^*, a_k \in [0,1]^p, k = \overline{0,m}, p, m \in \mathbb{N}^*$ , are distinct points such that  $I := co\{a_k | k = \overline{0,m}\}$  has a nonempty interior, and  $\psi_k \in C([0,1]^p, \mathbb{R}_+), k = \overline{0,m}$ . We suppose that:

(1)  $\{\psi_k | k = \overline{0, m}\}$  is linearly independent; (2)  $\sum_{k=0}^{m} \psi_k(x) = 1, \forall x \in [0, 1]^p.$ 

Now we consider the following positive linear operator,  $A: C([0,1]^p) \to C([0,1]^p)$ ,  $A(f) := \sum_{k=0}^{m} f(a_k)\psi_k.$ 

Condition (2) implies that the constant functions are the fixed points of A. On the other hand, the condition

(3)  $rank[\psi_i(a_k) - I_{m+1}] = m$ 

implies that  $F_A$  is the set of constant functions.

To study the iterates of A, we shall give conditions in which the operator A has an invariant functional of the following form

$$\phi(f) := \sum_{i=0}^{m} c_i f(a_i),$$

with some  $c_i \ge 0, \ i = \overline{0, m}$ .

We have the following result.

**Theorem 3.3.** In the conditions (1), (2) and (3), there exists  $c^* \in \mathbb{R}^{m+1}$  such that:

- (a)  $c^* \ge 0, \sum_{i=0}^m c_i^* = 1;$
- (b) the functional,  $\phi: C([0,1]^p) \to \mathbb{R}, \ \phi(f) := \sum_{i=0}^m c_i^* f(a_i)$ , is an invariant functional of the operator A;
- (c) if  $C([0,1]^p) = \bigcup_{\lambda \in \mathbb{R}} X_\lambda$  is a LFPP corresponding to  $\phi$ , and  $\psi_i(x) > 0, \forall x \in I, i = \overline{0,m},$ then  $||A|_{X_0 \cap C(I)}|| = l < 1$ , and so,  $A|_{C(I)}$  is WPO and  $A^{\infty}(f) = \sum_{i=1}^m c_i^* f(a_i), f \in C(I).$

*Proof.* First, we remark that a functional,  $\phi(f) = \sum_{i=0}^{m} c_i f(a_i), c_i \neq 0$ , is invariant for A iff:

$$\sum_{i=0}^{m} c_i \psi_k(a_i) = c_k, \ k = \overline{0, m}.$$

Let us consider the subset  $K \subset \mathbb{R}^{m+1}$ ,  $K := \{c \in \mathbb{R}^{m+1} | c_i \ge 0, \sum_{i=0}^m c_i = 1\}$ . We take in K the following function:

$$T: K \to \mathbb{R}^{m+1}, T(c) := \left(\sum_{i=0}^{m} c_i \psi_0(a_i), \dots, \sum_{i=0}^{m} c_i \psi_m(a_i)\right).$$

Since the matrix  $[\varphi_i(a_k)]$  is a stochastic matrix, it follows that  $T(K) \subset K$ . From the Brouwer fixed point theorem, there exists  $c^* \in K$  such that  $T(c^*) = c^*$ . From the condition (3), it follows that there exists such an unique fixed point. So, we have (a) and (b). Let  $f \in X_0$ , i.e.,  $\sum_{i=0}^{m} c_i^* f(a_i) = 0$ , and  $c_i^* > 0$ ,  $i = \overline{0, m}$ . For  $f \in C(I) \cap X_0$  we have  $|A(f)(x)| = \left|\sum_{k=0}^{m} f(a_k)\psi_k(x)\right| \le \max_{0\le k\le m} (1-\psi_k(x)) ||f|| = |l||f||$ , with l < 1.

So,  $||A(f)|| \le l ||f||, \forall f \in C(I)$ , with  $\phi(f) = 0$ , and we have (*c*), from Theorem 3.2.

**Remark 3.3.** ([18]) If  $\psi_i$ ,  $i = \overline{0,m}$  are polynomial functions, then the operator  $A: C([0,1]^p) \to C([0,1]^p)$  is WPO and,  $A^{\infty}(f) = \sum_{i=0}^m c_i^* \psi_i(a_i).$ 

For presenting the next result we need the following definition.

**Definition 3.5.** Let  $Y \subset X$  be a linear subspace of X. By definition, an element  $e \in Y$  is an order unit element if, for any  $f \in Y$  there exists  $M_f > 0$  such that  $|f| \leq M_f e$ . (See, e.g., [6], [7], [13].)

In this case we have on Y the Minkowski norm,  $\|\cdot\|_{e}$ , and it follows:

- $|f| \le ||f||_{e} e;$
- $||f|| \le ||f||_e ||e||$ .

**Theorem 3.4.** *We suppose that:* 

- (i) *A* is a linear positive operator:
- (ii) X contains all constant functions on  $\Omega$ ;
- (iii)  $F_A$  consists of all constant functions on  $\Omega$ ;
- (iv)  $X = \bigcup_{\lambda \in \mathbb{R}} X_{\lambda}$  is a LFPP of X with respect to A such that, for some  $k \in \mathbb{N}^*$ ,  $A^k(X_0)$  has an order unit element e.

Then

- (a)  $A^n(e)(x) \to 0$  as  $n \to \infty, \forall x \in \Omega$  implies that  $A^n(f)(x) \to \lambda, \forall x \in \Omega, \forall f \in X_\lambda, \lambda \in \Omega$  $\mathbb{R}$ , *i.e.*, A is WPO with respect to pointwise convergence on X and  $A^{\infty}(f) = \lambda, \forall f \in X_{\lambda}$ ;
- (b)  $A^n(e) \xrightarrow{\|\cdot\|} 0 \Rightarrow A \text{ is WPO with respect to } \stackrel{\|\cdot\|}{\to} and A^{\infty}(f) = \lambda, \forall f \in X_{\lambda}, \lambda \in \mathbb{R};$
- (c) if there exists  $l \in ]0, 1[$  such that  $A(e) \le le$  then:
  - (1)  $||A(f)||_{e} \leq l ||f||_{e}, \forall f \in A^{k}(X_{0});$
  - (2)  $\|A(f) A(g)\|_e \le l \|f g\|_e, \forall f, g \in A^k(X_\lambda), \lambda \in \mathbb{R};$ (3)  $\|A(f) A^2(f)\|_e \le l \|f A(f)\|_e, \forall f \in A^k(X);$

  - (4)  $\|f A^{\infty}(f)\|_{e} \leq \frac{1}{1-l} \|f A(f)\|_{e}, \forall f \in A^{k}(X);$
  - (5) A is  $\psi$ -WPO on  $(X, \|\cdot\|_e)$  with  $\psi(t) = \frac{t}{1-t}, t \in \mathbb{R}_+$ .

*Proof.* (a), (b) By Lemma 2.1 it is sufficient to prove that  $A|_{A^k(X_0)}$  is Picard operator. Let  $f \in A^k(X_0)$ . Since *e* is an order unit for  $A^k(X_0)$  we have that:

$$|f| \le \lambda(f)e.$$

But *A* is a non-decreasing linear operator and we have that

$$0 \le |A^n(f)| \le A^n(|f|) \le \lambda(f)A^n e \to 0 \text{ as } n \to \infty,$$

i.e.,  $A|_{A^k(X_0)}$  is a Picard operator.

(c) (1) We have

$$\begin{split} |A(f)| &\leq A(|f|) \leq \|f\|_e \, A(e) \leq \|f\|_e \, le \\ &= l \, \|f\|_e \, e \end{split}$$

$$\|A(f)\| \le \|A\| \, \|f\| \le \|f\|_e \, A(e) \le \|f\|_e \, le,$$

whence it follows

$$||A(f)||_{e} \leq l ||f||_{e}, \forall f \in A^{k}(X_{0});$$

(2) We have

$$|f-g| \le \|f-g\|_e e$$

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$$\begin{split} |A(f-g)| &\leq A(|f-g|) \leq \|f-g\|_e \, A(e) \leq \|f-g\|_e \, le \\ &= l \, \|f-g\|_e \, e, \end{split}$$

so it follows

$$\|A(f-g)\|_{e} \leq l \|f-g\|_{e}, \, \forall f,g \in A^{k}(X_{\lambda}), \, \lambda \in \mathbb{R};$$

(3) We have

$$|f - A(f)| \le ||f - A(f)||_e e$$

$$\begin{aligned} |A(f - A(f))| &\leq A(|f - A(f)|) \leq ||f - A(f)||_e A(e) \leq ||f - A(f)||_e le \\ &= l ||f - A(f)||_e e, \end{aligned}$$

whence we obtain

$$\left|A(f) - A^{2}(f)\right|_{e} \leq l \left\|f - A(f)\right\|_{e}, \,\forall f \in A^{k}(X_{\lambda}), \,\lambda \in \mathbb{R};$$

(4) By Theorem 3.2, (b), we get the result.

(5) By (4) and Definition 2.3 we get the result.

 $\square$ 

### 4. APPLICATIONS

**Example 4.1.** Let  $\Omega = ([0,\infty[)^p, p \ge 1, X := C_B(\Omega), \phi: X \to \mathbb{R}, \phi(f) := f(0)$  and  $A_k: X \to X, k = \overline{1, m}$  a linear operator such that

- $A_k(\widetilde{1}) = \widetilde{1}, \ k = \overline{1, m}$ :
- $(A_k(f))(0) = f(0), \forall f \in X, k = \overline{1, m}.$
- Let  $X = \bigcup_{\lambda \in \mathbb{R}} X_{\lambda}$  be the linear partition corresponding to  $\phi$ .

Let  $c_k \in \mathbb{R} \setminus \{0\}, k = \overline{1, m}$  be such that

•  $c_1 + \dots + c_m = 0$ •  $|c_1| + \dots + |c_m| = l < 1.$ 

Let  $A: X \to X$  be defined by  $A(f) := \widetilde{f(0)} + \sum_{k=1}^{m} c_k A_k(f)$ . We remark that  $\phi$  is an

invariant functional for A.

Now we suppose that:  $||A_k|_{X_0}|| \le 1$ . Then, from Theorem 3.2, we have:

- (i) A is WPO; (ii)  $F_A = \{ \widetilde{\lambda} \mid \lambda \in \mathbb{R} \};$
- (iii)  $F_A \cap X_\lambda = \{\widetilde{\lambda}\};$
- (iv)  $A^{\infty}(f) = f(0), \forall f \in X;$ (v)  $\|f A^{\infty}(f)\| \le \frac{1}{1-l} \|f A(f)\|, \forall f \in X;$

**Example 4.2.** Let  $\alpha, \beta \in \mathbb{R}, 0 \le \alpha < \beta$ . We consider the Stancu operator  $S_{m,\alpha,\beta}$ :  $C([0,1] \times C)$ [0,1])  $\rightarrow C([0,1] \times [0,1])$  defined by (see, e.g., [1], [4], [22])

$$(S_{m,\alpha,\beta}f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m}{i} \binom{m}{j} x^{i} y^{j} (1-x)^{m-i} (1-y)^{m-j} f\left(\frac{i+\alpha}{m+\beta}, \frac{j+\alpha}{m+\beta}\right)$$

We remark that the operator  $S_{m,\alpha,\beta}$  satisfies the conditions of Theorem 3.3. By this theorem we have the following properties:

(a) the operator  $S_{m,\alpha,\beta}$  is WPO;

(b)  $S_{m,\alpha,\beta}^{\infty}(f) = \sum_{i=0}^{m} \sum_{j=0}^{m} c_{ij}^* f\left(\frac{i+\alpha}{m+\beta}, \frac{j+\alpha}{m+\beta}\right)$ , where  $c_{ij}^*$  are the unique solutions in K of the following system

(4.3) 
$$\sum_{i=0}^{m} \sum_{j=0}^{m} {m \choose i} {m \choose j} \left(\frac{i+\alpha}{m+\beta}\right)^k \left(1 - \frac{i+\alpha}{m+\beta}\right)^{m-k} \left(\frac{j+\alpha}{m+\beta}\right)^l \left(1 - \frac{j+\alpha}{m+\beta}\right)^{m-l} c_{ij} = c_{k,l},$$
for  $k, l = \overline{0, m}$ .

For example, for m = 1 the system (4.3) implies:

 $\begin{cases} ((1+\beta-\alpha)^2 - (1+\beta)^2)c_{00} + (1+\beta-\alpha)(\beta-\alpha)c_{01} + (\beta-\alpha)(1+\beta-\alpha)c_{10} \\ +(\beta-\alpha)^2c_{11} = 0 \\ (1+\beta-\alpha)\alpha c_{00} + ((1+\beta-\alpha)(1+\alpha) - (1+\beta)^2)c_{01} + (\beta-\alpha)\alpha c_{10} \\ +(\beta-\alpha)(1+\alpha)c_{11} = 0 \\ \alpha(1+\beta-\alpha)c_{00} + \alpha(\beta-\alpha)c_{01} + ((1+\alpha)(1+\beta-\alpha) - (1+\beta)^2)c_{10} \\ +(1+\alpha)(\beta-\alpha)c_{11} = 0 \\ \alpha^2 c_{00} + \alpha(1+\alpha)c_{01} + \alpha(1+\alpha)c_{10} + ((1+\alpha)^2 - (1+\beta)^2)c_{11} = 0 \\ c_{00} + c_{01} + c_{10} + c_{11} = 1 \end{cases}$ 

and we get

$$S_{1,\alpha,\beta}^{\infty}(f) = \frac{(\alpha-\beta)^2}{\beta^2} f\left(\frac{\alpha}{1+\beta}, \frac{\alpha}{1+\beta}\right) - \frac{\alpha(\alpha-\beta)}{\beta^2} f\left(\frac{\alpha}{1+\beta}, \frac{1+\alpha}{1+\beta}\right) \\ - \frac{\alpha(\alpha-\beta)}{\beta^2} f\left(\frac{1+\alpha}{1+\beta}, \frac{\alpha}{1+\beta}\right) + \frac{\alpha^2}{\beta^2} f\left(\frac{1+\alpha}{1+\beta}, \frac{1+\alpha}{1+\beta}\right).$$

**Particular cases.** For  $\alpha = 0$ ,  $\beta > 0$  we have  $S^{\infty}_{1,\alpha,\beta}(f) = f(0,0)$ .

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