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Fixed points of mappings defined on spaces with distance

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ABSTRACT. In the present article we study distinct metrical structures guaranteeing the existence of fixed points for a given mapping (Propositions 3.4 and 5.9, Theorems 4.1 and 7.3, Corollaries 2.1, 3.3, 4.4, 4.7, 5.10, 6.12, 6.13). Some examples are proposed (Examples 1.4, 4.9, 6.12).

1. PRELIMINARIES

By a space we understand a topological T_0-space. We use the terminology from [23, 25, 38].

The problem of fixed points is one of the most investigated and consists in finding conditions under which for a given mapping $\varphi : X \longrightarrow X$ the set of fixed points $Fix(\varphi)$ = $\{x \in X : \varphi(x) = x\}$ of φ is non-empty. Still now were founded various conditions that use distinct structures on X: metrical structures [9, 10, 11, 12, 16, 17, 18, 20, 21, 25, 27, 28, 35, 36, 38]; ordering structures [8, 25, 36, 37, 38, 39]; structures of topological nature [25, 36, 38]; linear structures [8, 14, 25, 38, 36] etc.

Let X be a non-empty set and $d : X \times X \rightarrow \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:

(i_m) $d(x, y) \geq 0$;

(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$.

Then (X, d) is called a *distance space* and d is called a *distance* on X.

General problems of the distance spaces were studied in [1, 3, 12, 15, 24, 29, 30, 31, 32, 33, 34]. In [18] were proposed some reduction principles of fixed point theorems for metric spaces to the case of topological spaces with a continuous pseudometric. The similar reduction principles are true for distinct classes of distance spaces. The notion of a distance space is more general than the notion of o-metric spaces in sense of A. V. Arhangel'skii [3] and S. I. Nedev [29]. A distance d is an o-metric if from $d(x, y) = 0$ it follows that $x = y$. These notions coincide in the class of T_1-spaces.

Let d be a distance on X and $B(x, d, r) = \{y \in X : d(x, y) < r\}$ be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. A distance space is a *sequential space*, i.e. a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [23].

Let (X, d) be a distance space, $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1) is *convergent* to x if and only if $lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = lim_{n\to\infty} x_n$ (really, we may denote $x \in lims_{n\to\infty} x_n$);

2) is *convergent* if it converge to some point in X;

3) is *Cauchy* or *fundamental* if $lim_{n,m\to\infty} d(x_n, x_m) = 0$.

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A distance space (X, d) is *complete* if every Cauchy sequence in X converges to some point in X.

Remark 1.1. Let ρ be a pseudo-distance on a space X and $d(x, y) = \rho(x, y) + \rho(y, x)$ for all $x, y \in X$. Then: (X, d) is a pseudo-symmetric space; d is a symmetric if and only if ρ is a distance; $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence in (X, ρ) if and only if it is a Cauchy sequence in (X, d); $T(\rho) \subseteq T(d)$.

Lemma 1.1. *Let (X, d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping and for each point $x \in X$ there exist two positive numbers $c(x), k(x) > 0$ such that $d(\varphi(x), \varphi(y)) \leq k(x) \cdot d(x, y)$ provided $y \in X$ and $d(x, y) \leq c(x)$. Then the mapping φ is continuous.*

Proof. Let $\{x_n \in X : n \in \mathbb{N}\}$ be a convergent to $x \in X$ sequence. Then $lim_{n \to \infty} d(x, x_n) = 0$, $lim_{n \to \infty} d(\varphi(x), \varphi(x_n)) = 0$ and $lim_{n \to \infty} \varphi(x_n) = \varphi(x)$. Hence the mapping φ is continuous. □

Let X be a non-empty set and d be a distance on X. Then:

- (X, d) is called a *symmetric space* and d is called a *symmetric* on X if for all $x, y \in X$ we have

(iii_m) $d(x, y) = d(y, x)$;

- (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X if for all $x, y, z \in X$ we have

(iv_m) $d(x, z) \leq d(x, y) + d(y, z)$;

- (X, d) is called a *metric space* and d is called a *metric* if d is a symmetric and a quasimetric simultaneous.

Lemma 1.2. *Let (X, d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping and $d(\varphi(x), \varphi(y)) + d(\varphi(y), \varphi(x)) < d(x, y) + d(y, x)$ for all distinct points $x, y \in X$. Then:*

1. The mapping φ does not have two distinct fixed points.

2. The mapping φ does not have periodic non-fixed points.

Proof. Let $\rho(x, y) = d(x, y) + d(y, x)$ for all $x, y \in X$. Then ρ is a symmetric on X and $\rho(\varphi(x), \varphi(y)) < \rho(x, y))$ for all distinct points $x, y \in X$. From $\rho(\varphi(x), \varphi(y)) < \rho(x, y)$ it follows that at most one of the points x, y is not fixed. Hence the mapping φ does not have two distinct fixed points. Assume that the mapping φ has a periodic point, say z, of period $m \geq 2$, i.e. the points $z_1 = z$, $z_2 = \varphi(z_1)$, ..., $z_m = \varphi(z_{m-1})$ are distinct and $z_1 = \varphi(z_m)$. Then $\rho(z_1, z_2) = \rho(\varphi(z_m), \varphi(z_1) < \rho(z_m, z_1) = \rho(\varphi(z_{m-1}), \varphi(z_m) < \rho(z_{m-1}, z_m) ... < \rho(z_1, z_2)$, a contradiction. The proof is complete. □

Let X be a non-empty set and $d(x, y)$ be a distance on X with the following property:

(N) for each point $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$.

Then (X, d) is called an *N-distance space* and d is called an *N-distance* on X. If d is a symmetric, then we say that d is an N-symmetric.

Spaces with N-distances were studied by V. Niemyzki [33] and by S. I. Nedev [29].

If d satisfy the condition

(F) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$,

then d is called an *F-distance* or a *Fréchet distance* and (X, d) is called an *F-distance space*.

Any F-distance is an N-distance. If d is a symmetric and an F-distance on a space X, then we say that d is an F-symmetric.

Remark 1.2. If (X, d) is an F-symmetric space, then any convergent sequence is a Cauchy sequence. For N-symmetric spaces and for quasimetric spaces this assertion is not true.

Example 1.1. Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, $\rho(x,x) = d(x,x) = 0$, $d(x,y) = d(y,x)$ for all $x,y \in X$, $\rho(2^{-n}, 2^{-m}) = d(2^{-n}, 2^{-m}) = 1$ for all distinct $n,m \in \mathbb{N}$ and $\rho(2^{-n}, 0) = 1$, $\rho(0, 2^{-n}) = d(0, 2^{-n}) = 2^{-n}$ for each $n \in \mathbb{N}$. The distance d is an N-symmetric and it is not an F-distance. The topology $\mathcal{T}(d)$ generate by d is a compact metric topology on X. By construction, $\mathcal{T}(\rho) = \mathcal{T}(d)$. The distance ρ is a quasimetric. The sequence $\{2^{-n} : n \in \mathbb{N}\}$ is convergent and it is not a Cauchy sequence in the distance spaces (X, ρ) and (X, d) .

We say that a distance d on a space (X,d) is *balanced* if for every Cauchy sequence $\{x_n : n \in \mathbb{N}\}$ convergent to x in X and any point $y \in X$ we have $d(y,x) = lim_{n\to\infty} d(y, x_n)$.

Remark 1.3. Any metric is balanced. Moreover, assume that $x, y \in X$, (X,d) is a metric space and $\{x_n : n \in \mathbb{N}\}$ is a sequence convergent to x. Then $d(y,x) = lim_{n\to\infty} d(y, x_n)$.

Example 1.2. Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0, 2\}$, $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all $x, y \in X$, $d(0,2) = 2$, $d(2^{-n}, 2^{-m}) = |2^{-n} - 2^{-m}|$ for all $n, m \in \mathbb{N}$ and $d(2^{-n}, 2) = 3$, $d(2^{-n}, 0) = 2^{-n}$ for each $n \in \mathbb{N}$. By construction, (X,d) is an F-symmetric. The symmetric d is not balanced and the topology $\mathcal{T}(d)$ generate by d is a compact metric topology on X.

Example 1.3. Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0, 2\}$, $d(x,x) = 0$ for any $x \in X$, $d(0,2) = 2$, $d(2,0) = 3$, $d(2^{-n}, 2^{-m}) = |2^{-n} - 2^{-m}|$ for all $n, m \in \mathbb{N}$ and $d(2^{-n}, 2) = 3$, $d(2, 2^{-n}) = 2$, $d(2^{-n}, 0) = 1$ for each $n \in \mathbb{N}$. By construction, (X,d) is a quasimetric. By construction, $3 = d(2,0) > 2 = lim_{n\to\infty} d(2, 2^{-n})$ and $\{2^{-n} : n \in \mathbb{N}\}$ is a Cauchy sequence convergent to 0. Hence the quasimetric d is not balanced and the topology $\mathcal{T}(d)$ generate by d is a compact metric topology on X.

Fix a mapping $\varphi : X \longrightarrow X$. For any point $x \in X$ we put $\varphi^0(x) = x$, $\varphi^1(x) = \varphi(x), ..., \varphi^n(x) = \varphi(\varphi^{n-1}(x))$,... . The sequence $O(\varphi, x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ is called the *orbit of* φ with respect to the point x or the *Picard sequence* of the point x.

Fix a distance space (X,d) and a mapping $\varphi : X \longrightarrow X$. We say that the mapping φ:

- is *contractive* if $d(\varphi(x), \varphi(y)) < d(x,y)$ provided $d(x,y) > 0$;
- is a *contraction* if there exists $\lambda \in [0,1)$ such that $d(\varphi(x), \varphi(y)) \leq \lambda d(x,y)$ for all $x, y \in X$;
- is *strongly asymptotically regular* if $lim_{n\to\infty}(d(\varphi^n(x), \varphi^{n+1}(x) + d(\varphi^{n+1}(x), \varphi^n(x)))) = 0$ for each $x \in X$.

Any contraction is strongly asymptotically regular.

Proposition 1.1. *Let (X,d) be a symmetric space with the following property:*

(AF) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that from $d(x,y) \geq \varepsilon$ it follows that $\rho(x,y) = inf\{\Sigma\{d(z_i, z_{i+1}) : i \leq n\} : z_1, z_2, ..., z_n \in X, n \in \mathbb{N}, x = z_1, y = z_n\} \geq \delta$. Then:

1. d is a symmetric with the condition (F).

2. ρ is a metric on X and $\rho(x,y) \leq d(x,y)$ for all $x, y \in X$.

3. $\mathcal{T}(\rho) = \mathcal{T}(d)$.

4. The distance space (X,d) is complete if and only if the metric space (X,ρ) is complete.

5. If $\varphi : X \longrightarrow X$ is a mapping, λ is a positive number and $d(\varphi(x), \varphi(y)) \leq \lambda d(x,y)$ for all $x, y \in X$, then $\rho(\varphi(x), \varphi(y)) \leq \lambda \rho(x,y)$ for all $x, y \in X$. In particular, if the space (X,d) is complete and $\lambda < 1$, then φ is strongly asymptotically regular, any Picard sequence is a Cauchy sequence, and φ has a unique fixed point.

Proof. Obviously from $d(x,y) < \delta(\varepsilon)$ and $d(y,z) < \delta(\varepsilon)$ it follows that $d(x,z) < \varepsilon$. Hence d is a symmetric with the condition (F).

By construction, $\rho(u,v) \leq d(u,v)$, $\rho(x,y) = 0$ if and only if $x = y$ and $\rho(u,w) \leq \rho(u,v) + \rho(v,w)$ for all $u, v, w \in X$. Hence ρ is a metric on X. Fix $\varepsilon > 0$ and $\delta = \delta(\varepsilon)$. Then $B(x,d,\varepsilon) \subseteq B(x,\rho,\varepsilon)$ and $B(x,\rho,\delta) \subseteq B(x,d,\varepsilon)$. Therefore:

- $\mathcal{T}(\rho) = \mathcal{T}(d)$;

- the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same convergent sequences;
- the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same Cauchy sequences;
- the space (X, d) is complete if and only if the space (X, ρ) is complete.

Let $\varphi : X \longrightarrow X$ be a mapping, λ be a positive number and $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$. Fix $\mu > 0$ and $x = z_1$, z_2, ..., z_n, $z_{n+1} = y$ in X such that $\rho(x, y) \leq \Sigma\{d(z_i, z_{i+1}) : i \leq n\} \leq \rho(x, y) + \mu$. Then $\rho(\varphi(x), \varphi(y)) \leq \Sigma\{d(\varphi(z_i), \varphi(z_{i+1})) : i \leq n\} \leq \Sigma\{\lambda d(z_i, z_{i+1}) : i \leq n\} \leq \lambda\rho(x, y) + \lambda\mu$. Hence $\rho(\varphi(x), \varphi(y)) \leq \lambda\rho(x, y)$ for all $x, y \in X$. The Banach Contraction Principle [25, 38, 36] completes the proof. □

Example 1.4. Let $X = \{2^{-n} : n \in \mathbb{N}\}$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(2^{-n}, 2^{-m}) = min\{2^{-n}, 2^{-m}\}$ for all distinct $n, m \in \mathbb{N}$. The topology $\mathcal{T}(d)$ generated by d is a compact T_1-topology on X, $\{2^{-n} : n \in \mathbb{N}\}$ is a Cauchy sequence convergent to any point $x \in X$. On X consider the continuous mapping $\varphi : X \longrightarrow X$, where $\varphi(2^{-n}) = 2^{-n-1}$ for any $n \in \mathbb{N}$. Hence:
- d is not an N-distance on X;
- d is not a balanced distance on X;
- $\mathcal{T}(d) = \{\emptyset\} \cup \{X \setminus F : F$ *is a finite subset of* $X\}$;
- $d(\varphi(x), \varphi(y)) = 2^{-1}d(x, y)$ for all $x, y \in X$;
- $Fix(\varphi) = \emptyset$.

2. Spaces with H-distances

A distance space (X, d) is called an *H-distance space* if for any two distinct points $x, y \in X$ there exists $\delta = \delta(x, y) > 0$ such that $B(x, d, \delta) \cap B(y, d, \delta) = \emptyset$.

Remark 2.4. Let (X, d) be a distance space. Then (X, d) is an H-distance space if and only if any convergent sequence has a unique limit point.

Lemma 2.3. *Let (X, d) be a distance space and the space $(X, \mathcal{T}(d))$ is Hausdorff. Then d is an H-distance.*

Proof. Fix two distinct points $x, y \in X$. Then there exist two d-open sets $U, V \in \mathcal{T}(d)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. By definition of d-open sets, there exists $r > 0$ such that $B(x, d, r) \subseteq U$ and $B(y, d, r) \subseteq V$. Hence $B(x, d, r) \cap B(y, d, r) = \emptyset$. □

Example 2.5. Let $X = [0, 1] \cup \{s\}$, where $s \notin [0, 1]$, and $D = \{n^{-1} : n \in \mathbb{N}\}$. Consider on X the symmetric d, where $d(x, y) = |x-y|$ if $0, s \notin \{x, y\}$, $d(0, n^{-1}) = d(0, s) = 1$ and $d(s, n^{-1}) = n^{-1}$ for each $n \in \mathbb{N}$, $d(0, x) = x$ if $x \in [0, 1] \setminus D$, and $d(s, x) = 1$ if $x \in [0, 1] \setminus D$. The set $B = B \cup \{s\}$ is a metrizable compact closed subset of the space (X, d). Let $U, V \in \mathcal{T}(d)$, $0 \in U$ and $s \in V$. There exists $n \in \mathbb{N}$ such that $B(0, d, (n-1)^{-1}) \subseteq U$ and $B(s, d, (n-1)^{-1}) \subseteq V$. Then $((m+1)^{-1}, m^{-1}) \subseteq U$ for each $m \geq n$. For each $m \geq n$ we have $m^{-1} \in V$ and there exists $\delta_m \in (0, m^{-1} - (m+1)^{-1})$ such that $(m^{-1} - \delta_m, m^{-1} + \delta_m) \subseteq V$. Hence $U \cap V \neq \emptyset$ and the space $(X, \mathcal{T}(d))$ is not Hausdorff. Since $B(0, d, 1) \cap B(s, d, 1) = \emptyset$ and the subspaces $X \setminus \{0\}$, $X \setminus \{s\}$ of $(X, \mathcal{T}(d))$ are open and Hausdorff, d is an H-distance. The space (X, d) is a compact T_1-space in which any convergent sequence has a unique limit.

We observe that for $\delta < 2^{-1}$, $n^{-1} < \delta$ and $x \in (0, n^{-1}) \setminus D$ we have $d(0, x) < \delta$, $d(x, n^{-1}) < \delta$, $d(0, n^{-1}) = 1$ and $d(s, n^{-1}) < \delta$, $d(n^{-1}, x) < \delta$, $d(s, x) = 1$. Therefore on X,$R = [0, 1]$ and $S = (0, 1] \cup \{s\}$ the symmetric d is not an N-symmetric and is not a balanced distance.

The subspace S of (X, d) is a normal Lindelöf non-metrizable space. The subspace R is Hausdorff and not regular. The space R is the first example of H-closed non-compact space which was constructed by P. Alexandroff and P. Urysohn ([2], Chapter 1, Section

1.5). A Hausdorff space Y is called an *H-space* or *an absolutely closed space* if Y is a closed subspace of every Hausdorff space in which it is contained [2, 23].

Proposition 2.2. *Let (X, d) be an H-distance space, $\varphi : X \longrightarrow X$ be a continuous mapping. Then:*

1. The set $Fix(\varphi)$ of fixed points of φ is closed.

2. If for some point $x \in X$ the Picard sequence $O(\varphi, x)$ is convergent, then the set of fixed points $Fix(\varphi)$ of the mapping φ is non-empty.

Proof. Assume that $\{x_n \in Fix(\varphi) : n \in \mathbb{N}\}$, $b \in X$ and $x_n \to b$. Then $b = lim_{n\to\infty}x_n = lim_{n\to\infty}\varphi(x_n) = \varphi(b)$. Hence $b \in Fix(\varphi)$ and Assertion 1 is proved.

Let $\{x_n = \varphi^n(x) \in X : n \in \mathbb{N}\}$ be the Picard sequence of the given point $x \in X$ which is a convergent to a point $a \in X$. Then, since the mapping φ is continuous and $lim_{n\to\infty}d(a, x_n) = 0$, we have $lim_{n\to\infty}d(\varphi(a), \varphi(x_n)) = lim_{n\to\infty}d(\varphi(a), x_n) = 0$ and $lim_{n\to\infty}x_n = \varphi(a)$. Hence $\varphi(a) = a$. □

Example 2.6. Let $A = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and $X = \{0, 1\} \times A$. Consider on X the metric d, where $d((x, y),(u, v)) = |x - u| + |y - v|$, and the mapping $\varphi : X \longrightarrow X$, where $\varphi(x, 0) = (x, 0)$ and $\varphi(x, 2^{-n}) = (x, 2^{-n-1})$ for each $x \in \{0, 1\}$ and $n \in \mathbb{N}$. The space $(X, \mathcal{T}(d))$ is a metric compact space. Any Picard sequence is a convergent Cauchy sequence and $Fix(\varphi) = \{(0, 0),(1, 0)\}$. The mapping φ is not contractive. It is a contraction along each Picard sequence with its limit. The article [11] contained some applications of such mappings.

Proposition 2.3. *Let (X, d) be a balanced distance space. Then:*

1. $d(x, y) > 0$ for any two distinct points $x, y \in X$.

2. If $\{x_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence convergent to $a \in X$, then:

- a is the unique limit point of the sequence $\{x_n \in X : n \in \mathbb{N}\}$;

- for each point $y \in X$ there exists the limit $lim_{n\to\infty}d(y, x_n) = d(y, a)$.

3. If each convergent sequence is a Cauchy sequence, then (X, d) is an H-distance space.

Proof. Assume that a, b are two distinct points of X and $d(a, b) = 0$. Since $d(a, b) + d(b, a) > 0$, we have $d(b, a) > 0$. We put $b_n = b$ for each $n \in \mathbb{N}$. Then $b = lim_{n\to\infty}b_n$, $a = lim_{n\to\infty}b_n$ and $\{b_n : n \in \mathbb{N}\}$ is a Cauchy sequence. Since $a = lim_{n\to\infty}b_n$, we have $d(b, a) = lim_{n\to\infty}d(b, b_n) = 0$, a contradiction. Hence $d(x, y) > 0$ for any two distinct points $x, y \in X$. Assertion 1 is proved.

Let $\{x_n \in X : n \in \mathbb{N}\}$ be a Cauchy sequence convergent to $a \in X$ and $y \neq a$. Then $lim_{n\to\infty}d(y, x_n) = d(y, a) > 0$ and y is not a limit of the sequence $\{x_n : n \in \mathbb{N}\}$. Assertions 2 are proved.

Assertion 3 follows from Assertions 2. The proof is complete. □

Corollary 2.1. *Let (X, d) be a balanced complete distance space and $\varphi : X \longrightarrow X$ be a mapping with properties:*

- there exists $\lambda > 0$ such that $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all $x, y \in X$;

- if $x \in X$, then the Picard sequence $\{x_n \in X : n \in \mathbb{N}\}$, generated by the point x, is a Cauchy sequence.

Then:

1. The mapping φ is continuous.

2. The set $Fix(\varphi)$ of fixed points of φ is closed and non-empty.

3. If $d(\varphi(x), \varphi(y)) < d(x, y)$ for all distinct points $x, y \in X$, then φ has a unique fixed point.

Remark 2.5. By virtue of Example 1.4, the requirement in Proposition 2.3 that d is an H-distance is essential. The assertions of Corollary 2.1 remains true if the conditions "d is an balanced distance" is replaced by the condition "d is an H-distance". Moreover, the assertions of Corollary 2.1 remains true for the distance spaces (X, d) with property:

(UFL): Any convergent Cauchy sequence has a unique limit.

3. On bounded distance spaces

Fix a distance space (X, d) and a mapping $\varphi : X \longrightarrow X$. We say that the space (X, d) is *φ-bounded* if for each $x \in X$ there exists a positive number $\lambda(x)$ such that $d(\varphi^n(x), x) + d(x, \varphi^n(x)) \leq \lambda(x)$ for each $n \in \mathbb{N}$. The space (X, d) is *weakly φ-bounded* if for each $x \in X$ there exist a positive number $\lambda(x)$ and $p = p(x) \in \mathbb{N}$ such that $d(\varphi^n(x), \varphi^{p-1}(x) + d(\varphi^{p-1}(x), \varphi^n(x)) \leq \lambda(x)$ for each $n \geq p$. Some orbital conditions involved in common fixed point theorems were examined in [6] and [11].

We say that a subset L of a distance space (X, d) is *bounded* if there exists a positive number λ such that $d(x, y) \leq \lambda$ for all $x, y \in L$. If the set X is bounded, then we say that (X, d) is a bounded distance space.

Example 3.7. Let $X = \{0, 1\} \cup \{2^{-n} : n \in \mathbb{N}\}$. Consider on X the F-symmetric d, where $d(0, x) = x$, $d(1, 2^{-n}) = n$ and $d(2^{-m}, 2^{-n}) = |2^{-m} - 2^{-n}|$ for all $n, m \in \mathbb{N}$. Now consider the mapping $\varphi : X \longrightarrow X$, where $\varphi(0) = 0$, $\varphi(1) = 2^{-1}$ and $\varphi(2^{-n}) = 2^{-n-1}$ for each $n \in \mathbb{N}$. The space $(X, \mathcal{T}(d))$ is a metric compact space and $lim_{n\to\infty} d(1, 2^{-n}) = \infty$. Any Picard sequence is a convergent Cauchy sequence and $Fix(\varphi) = \{0\}$. The mapping φ is a contraction and $d(\varphi(x), \varphi(y)) \leq 2^{-1}d(x, y)$. The space (X, d) is weakly φ-bounded and is not φ-bounded.

If (X, d) is a distance space, $f : X \longrightarrow X$ is a mapping and any Picard sequence $O(\varphi, x)$, $x \in X$, is a Cauchy sequence, then the space (X, d) is weakly φ-bounded.

Proposition 3.4. *Let (X, d) be a distance space and the mapping $\varphi : X \longrightarrow X$ be a contraction. If the space (X, d) is weakly φ-bounded, then:*

1. For each point $x \in X$ the Picard sequence $O(\varphi, x)$ is Cauchy.

2. The mapping φ has a unique fixed point provided (X, d) is a complete H-distance space.

3. The mapping φ has a unique fixed point provided (X, d) is a complete balanced distance space.

Proof. Fix a point $x \in X$ and the numbers $k \in (0, 1)$, $p \in \mathbb{N}$ and $\lambda > 0$ such that:

- $d(\varphi(z), \varphi(y)) \leq kd(z, y)$ for all $z, y \in X$;
- $d(\varphi^n(x), \varphi^{p-1}(x) + d(\varphi^{p-1}(x), \varphi^n(x)) \leq \lambda$ for each $n \geq p$.

Obviously, $d(\varphi^{n+p}(x), \varphi^{n+p+m}(x)) + d(\varphi^{n+p+m}(x), \varphi^{n+p}(x)) \leq k^{n+1} \cdot \lambda$ for all $n, m \in \mathbb{N}$. Hence $lim_{n,m\to\infty}(d(\varphi^n(x), \varphi^m(x)) = 0$. Assertion 1 is proved. Corollary 2.1 completes the proof. □

Corollary 3.2. *Let (X, d) be a bounded complete H-distance space or a bounded complete balanced distance space. Then any contraction $\varphi : X \longrightarrow X$ has a unique fixed point. Moreover, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(\varphi^n(x), \varphi^m x) < \varepsilon$ for all $x \in X$ and $n, m \geq n_0$.*

A function $\lambda : [0, \infty) \longrightarrow [0, \infty)$ is called a *comparison function* ([38], Section 3.0.3) if it satisfies the following conditions:

(i) λ is is increasing;

(ii) $lim_{n\to\infty}\lambda^n(t) = 0$ for each $t \in [0, \infty)$.

Remark 3.6. If $\lambda : [0, \infty) \longrightarrow [0, \infty)$ is a comparison function, then satisfies the following conditions: $\lambda(0) = 0$ and $\lambda(t) < t$ for each $t \in (0, \infty)$.

The following assertions for complete metric spaces were proved by J. Matkowski ([38], p. 31).

Proposition 3.5. *Let (X,d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping and the space (X,d) is weakly φ-bounded. If there exists a comparison function λ such that $d(\varphi(x),\varphi(y)) \leq \lambda(d(x,y))$ for all $x,y \in X$, then:*

1. For each point $x \in X$ the Picard sequence $O(\varphi,x)$ is Cauchy.

2. The mapping φ has a unique fixed point provided (X,d) is a complete H-distance space.

3. The mapping φ has a unique fixed point provided (X,d) is a complete balanced distance space.

Proof. The proof for a weakly φ-bounded space is as for a φ-bounded space. Assume that the space (X,d) is φ-bounded. Fix a point $x \in X$ and the number $k > 0$ such that $d(\varphi^n(x),x) + d(x,\varphi^n(x)) \leq k$ for each $n \in \mathbb{N}$. Obviously, $d(\varphi^n(x),\varphi^{n+m}(x) + d(\varphi^{n+m}(x), \varphi^n(x)) \leq \lambda^n(d(x,\varphi^m x)) + \lambda^n(d(\varphi^m x,x)) \leq 2\lambda^n(k)$ for all $n,m \in \mathbb{N}$. Hence $lim_{n,m\to\infty}(d(\varphi^n(x) + \varphi^m(x)) = 0$. Assertion 1 is proved. Proposition 2.3 completes the proof. □

Corollary 3.3. *Let (X,d) be a bounded complete H-distance space or a bounded balanced distance space, $\varphi : X \longrightarrow X$ be a mapping and there exists a comparison function λ such that $d(\varphi(x),\varphi(y)) \leq \lambda(d(x,y))$ for all $x,y \in X$. Then φ has a unique fixed point. Moreover, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(\varphi^n(x),\varphi^m x) < \varepsilon$ for all $x \in X$ and $n,m \geq n_0$.*

4. On N-distances

Theorem 4.1. *Let (X,d) be an N-symmetric space and $\varphi : X \longrightarrow X$ be a mapping with properties:*

- $d(\varphi(x),\varphi(y)) < d(x,y)$ for all distinct points $x,y \in X$;

- for each point $x \in X$ the Picard sequence $O(\varphi,x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ has an accumulation point and the mapping φ is strongly asymptotically regular: $lim_{n\to\infty} d(x_n,x_{n+1}) = 0$.

Then the mapping φ has a unique fixed point. Moreover, d is an H-distance and any Picard sequence has a unique accumulation point.

Proof. For each $\varepsilon > 0$ and every $x \in X$ there exists $\delta = \delta(x,\varepsilon) > 0$ such that from $d(x,y) \leq \delta$ and $d(y,z) \leq \delta$ it follows $d(x,z) \leq \varepsilon$. We assume that $2\delta(x,\varepsilon) < \varepsilon$.

Fix two distinct points $x,y \in X$. We put $2\varepsilon = d(x,y) = d(y,x)$. Since $\varepsilon > 0$, there exists $\delta > 0$ such that $3\delta < \varepsilon$ and for $u \in \{x,y\}$ and $v,w \in X$ from $d(u,v) < \delta$ and $d(v,w) < \delta$ it follows that $d(u,w) < \varepsilon$. Then $B(x,d,\delta) \cap B(y,d,\delta) = \emptyset$. Hence d is an H-symmetric.

From the condition $d(\varphi(x),\varphi(y)) < d(x,y)$ for all distinct points $x,y \in X$ it follows that:

- the mapping φ is continuous;
- the mapping φ does not have two distinct fixed points;
- the mapping φ does not have periodic non-fixed points.

Fix $x \in X$. Let $O(\varphi,x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ be the Picard sequence generated by the point x.

If $a \in X$ and $a = x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then a is the unique fixed point of the mapping φ and $O(\varphi,x)$ is a Cauchy sequence with the unique accumulation point a.

Assume now that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$. Then $x_n \neq x_{n+m}$ for all $n,m \in \mathbb{N}$. In this case the set $O(\varphi,x)$ is infinite and non-closed in the sequential space $(X,\mathcal{T}(d))$. Then there exist a point $b \in X$ and a sequence $\{n_k \in \mathbb{N} : k \in \mathbb{N}\}$ such that $b = lim_{k\to\infty} x_{n_k}$, $n_k < n_{k+1}$ and $d(b,x_{n_{k+1}}) < d(b,x_{n_k}) < 2^{-k}$ for each $k \in \mathbb{N}$.

We put $c = \varphi(b)$, $y_k = x_{n_k}$ and $z_k = \varphi(y_k)$. Then $b = lim_{k\to\infty} y_k$ and, since the mapping φ is continuous, $c = lim_{k\to\infty} z_k$.

Claim 1. $b = c$.

Assume that $b \neq c$ and $d(b,c) = 4\varepsilon > 0$. Let $\varepsilon_1 = min\{\delta(b,\varepsilon), \delta(c,\varepsilon)\}$ and $\delta = \delta(b,\varepsilon_1)$. Since $lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, there exists $m_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \delta$, $d(b, y_n) < \delta$ and $d(c, z_n) < \delta$ for each $n \geq m_0$. Since $k \leq n_k$, for any $k \geq m_0$ we have $d(c, z_k) < d(b, y_k) < \delta$ and $d(y_k, z_k) < \delta$. From $d(b, y_k) < \delta$ and $d(y_k, z_k) < \delta$ it follows that $d(b, z_k) \leq \varepsilon_1$. From $d(b, z_k) \leq \varepsilon_1$ and $d(z_k, c) \leq \delta \leq \varepsilon_1$ it follows that $d(b, c) \leq \varepsilon$, a contradiction. Therefore $b = c$.

Claim 2. $b \in Fix(\varphi)$.

It follows from Claim 1.

Claim 3. $b = lim_{n\to\infty} x_n$.

Fix $\varepsilon > 0$. There exists $m_0 = n_k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. Then $d(b, x_n) < d(b, x_{m_0}) < \varepsilon$ for each $n > m_0$. Hence $b = lim_{n\to\infty} x_n$.

Since b is the unique fixed point of the mapping φ, the proof is complete. □

Corollary 4.4. *Let (X, d) be a N-symmetric compact space, $\varphi : X \longrightarrow X$ be a mapping, $d(\varphi(x), \varphi(y)) < d(x, y)$ for all distinct points $x, y \in X$ and the mapping φ is strongly asymptotically regular: $lim_{n\to\infty} d(\varphi^n(x), \varphi^{n+1}(x)) = 0$ for each point $x \in X$.*

Then the mapping φ has a unique fixed point. Moreover, any Picard sequence is convergent to the fixed point.

Corollary 4.5. *Let (X, d) be a N-symmetric compact space, $0 < \lambda < 1$ and $\varphi : X \longrightarrow X$ be a mapping such that $d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y)$ for all points $x, y \in X$.*

Then the mapping φ has a unique fixed point. Moreover, any Picard sequence is convergent to the fixed point.

Corollary 4.6. *Let (X, d) be an F-symmetric space and $\varphi : X \longrightarrow X$ be a mapping with properties:*

- $d(\varphi(x), \varphi(y)) < d(x, y)$ for all distinct points $x, y \in X$;

- for each point $x \in X$ the Picard sequence $O(\varphi, x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ has an accumulation point and the mapping φ strongly asymptotically regular mapping: $lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ for any point $x \in X$.

Then the mapping φ has a unique fixed point. Moreover, d is an H-distance and any Picard sequence is a Cauchy sequence and has a unique accumulation point.

Example 4.8. Let $X = \{b\} \cup \{b_n : n \in \mathbb{N}\}$, $d(x, x) = 0$, $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(b, b_n) = 2^{-n}$ for each $n \in \mathbb{N}$ and $d(b_n, b_{n+m}) = 2^{-n}m$ for all $n, m \in \mathbb{N}$. On X consider the continuous mapping $\varphi : X \longrightarrow X$, where $\varphi(b) = b$ and $\varphi(b_n) = b_{n+1}$ for any $n \in \mathbb{N}$. Hence:

- d is an N-symmetric on X;
- d is not an F-symmetric on X;
- the topology $\mathcal{T}(d)$ generated by d is a compact metrizable topology on X;
- $O(\varphi, b_1) = \{b_n : n \in \mathbb{N}, n \geq 2\}$ is convergent to the point b and is not a Cauchy sequence;
- $d(\varphi(x), \varphi(y)) \leq 2^{-1} d(x, y)$ for all points $x, y \in X$ and φ is a contraction;
- $Fix(\varphi) = \{b\}$.

The following notion do to P. Alexandroff and P. Urysohn [1], A. H. Frink [24], S. Czerwik [22], I. A. Bakhtin [4], V. Berinde [7] (see [38]).

Let $s, q > 0$. We say that d is an (s, q)-*distance* on a space X if $d(x, y) \leq s(d(x, z) + d(z, y))$ and $d(y, x) \leq q d(x, y)$ for all points $x, y, z \in X$. If $d(x, y) \leq s(d(x, z) + d(z, y))$ for all points $x, y, z \in X$, then we say that d is an s-*distance*.

Any s-distance is an F-distance.

E. W. Chittenden [15] proved that a space with F-symmetric is metrizable. Then P. Alexandroff and P. Urysohn [1], using Chittenden's theorem, introduced a 2-symmetric.

The Chittenden's proof is complicated. A simple and elegant proof of Chittenden's theorem was found by A. H. Frink [24]. A. H. Frink [24] observed that a 2-symmetric has Property (AF) and proved that a space with an F-symmetric has a 2-symmetric. These facts were applied by J. W. Tukey in the theory of uniform spaces (see [23], Theorem 8.1.10).

Lemma 4.4. *Let (X, d) be an (s, q)-distance space. Then d is a H-distance.*

Proof. Assume that $x, y \in X$ and $x \neq y$. Obviously, $s \geq 1$ and $q \geq 1$. Let $b = min\{d(x, y), d(y, x)\}$. We put $2r = b : (s+q)$. Suppose that $z \in B(x, dr) \cap B(y, d, r)$. Then $b \leq d(x, y) \leq s(d(x, z) + d(z, y)) < (r + qd(y, z)) < r(1 + q) \leq b(1 + q)/2(s + q) \leq b/2$, a contradiction. Thus $B(x, d, r) \cap B(y, d, r) = \emptyset$. The proof is complete. □

The following assertion for symmetric spaces was proved by S. Czerwik [22] and I. A. Bakhtin [4] (see [38]).

Lemma 4.5. *Let (X, d) be an s-distance space, $0 \leq s\lambda < 1$, $\varphi : X \longrightarrow X$ and $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all points $x, y \in X$. Then any Picard sequence is a Cauchy sequence.*

Proof. Assume that $\rho(x, y) = d(x, y) + d(y, x)$ for all $x, y \in X$. Obviously, ρ is a symmetric on X and $\rho(\varphi(x), \varphi(y)) \leq \lambda\rho(x, y)$ for all $x, y \in X$.

Fix $x \in X$ and put $k = s\lambda < 1$. Let $O(\varphi, x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$ be the Picard sequence generated by the point x. We put $b = d(x, x_1) + d(x_1, x) = \rho(x, x_1)$. Then $\rho(x_n, x_{n+1}) \leq \lambda^n b$ and $\rho(x_n, x_{n+m}) \leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + ... + s^{m-1}\rho(x_{n+m-2}, x_{n+m-1}) + s^{m-1}\rho(x_{n+m-1}, x_{n+m}) \leq b(s\lambda^n + s^2\lambda^{n+1} + ... + s^{m-1}\lambda^{n+m-1} + s^{m-1}\lambda^{n+m}) \leq bs\lambda^n(1 - k^m) : (1 - k) < bs\lambda^n : (1 - k)$. Hence $O(\varphi, x)$ is a Cauchy sequence. □

The problem of existence of fixed points for contracting mappings of F-symmetric spaces was arised in [12]. The following statement improved the fixed point theorem of S. Czerwik [22] and I. A. Bakhtin [4] (see [38]).

Theorem 4.2. *Let (X, d) be a s-distance space, $0 \leq \lambda < 1$, $\varphi : X \longrightarrow X$ be a mapping and $d(\varphi(x), \varphi(y)) \leq \lambda d(x, y)$ for all points $x, y \in X$. Then:*

1. *Any Picard sequence $O(\varphi, x)$ is a Cauchy sequence.*
2. *The space (X, d) is φ-bounded.*
3. *If d is a complete H-distance, then the mapping φ has a unique fixed point.*
4. *If d is a balanced complete distance, then the mapping φ has a unique fixed point.*
5. *If d is a complete symmetric, then the mapping φ has a unique fixed point.*
6. *If any Cauchy sequence has a unique limit, then the mapping φ has a unique fixed point.*

Proof. Since $lim_{n\to\infty} s\lambda^n = 0$, there exists a number $k \in \mathbb{N}$ such that $s\lambda^k < 1$. We put $\mu = \lambda^k$ and $\psi = \varphi^k$. By construction, $s\mu < 1$ and $d(\psi(x), \psi(y)) \leq \mu d(x, y)$ for all points $x, y \in X$.

Fix $x \in X$ and $c = s\mu < 1$. Let $O(\psi, x) = \{x_n = \psi^n(x) : n \in \mathbb{N}\}$ be the Picard sequence generated by the point x. Then, by virtue of Lemma 4.5, $O(\psi, x)$ is a Cauchy sequence. There exists $p \in \mathbb{N}$ such that $p \geq k$ and $d(\psi^m(x), \psi^n(x)) < 1$ for all $n, m \geq p$. We put $A_1 = \{x, x_1, x_2, ..., x_{k+p}\}$ and $q = max\{d(x, y) + 1 : x, y \in A_1\}$. Let $A_{n+1} = \psi^n(A_1)$. Then $d(x, y) < q$ for all $n \in \mathbb{N}$ and $x, y \in A_n$. Let $y_n = \psi^n(x)$. Then $y_n \in A_n$ and $d(u, v) \leq s(d(u, y_n) + s(d(y_n, y_m) + d(y_m, v)) \leq s(q + s(1 + q))$ for all $n, m \in \mathbb{N}$, $u \in A_n$ and $v \in A_m$. Hence the space (X, d) is φ-bounded. From Proposition 3.4 it follows that any Picard sequence $O(\varphi, x)$ is a Cauchy sequence. Assume that any Cauchy sequence has a unique limit. Let b be the limit of the sequence $O(\varphi, x)$. Then $\varphi(b) = lim_{n\to\infty}\varphi(x_n) = lim_{n\to\infty} x_n = b$. Thus $b \in Fix(\varphi)$. By virtue of Lemma 1.2, the fixed point is unique. Assertions 1, 2, 6 are proved. Assertions 3, 4 and 5 follows from Assertion 6. The proof is complete. □

Corollary 4.7. *Let (X,d) be a complete (s,q)-distance space, $0 \leq \lambda < 1$, $\varphi : X \longrightarrow X$ be a mapping and $d(\varphi(x), \varphi(y)) \leq \lambda d(x,y)$ for all points $x, y \in X$. Then the mapping φ has a unique fixed point.*

Example 4.9. Let $X = \mathbb{N}$, $d(x,x) = 0$, $\rho(x,x) = 0$, $d(x,y) = d(y,x)$ and $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$. If $n, m \in X$ and $n < m$, then $d(n,m) = (m-n)2^{-n}$ and $\rho(n,m) = (m-n) + (n^{-1} - m^{-1})$. On X consider the continuous mapping $\varphi : X \longrightarrow X$, where $\varphi(n) = n+1$ for any $n \in \mathbb{N}$. Then:

- d is a complete N-symmetric on X;
- d is not an F-symmetric on X;
- ρ is a complete metric on X;
- the topology $\mathcal{T}(d) = \mathcal{T}(\rho)$ is the discrete topology on X;
- $O(\varphi, n) = \{n+i : i \in \mathbb{N}\}$ is not a Cauchy sequence of the distance spaces (X,d) and (X,ρ);
- $\rho(\varphi(x), \varphi(y)) < \rho(x,y)$ for all distinct points $x, y \in X$, i.e. φ is a contractive mapping of the metric space (X,ρ);
- $d(\varphi(x), \varphi(y)) = 2^{-1}d(x,y)$ for all points $x, y \in X$, i.e. φ is a contraction of the distance space (X,d);
- $Fix(\varphi) = \emptyset$.

5. Berinde's transformation of distances

As in [5, 26], we denote by $\mathcal{F}$ the non-empty set of functions $f : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(i) f is non-decreasing and $f(t) = 1$ if and only if $t = 0$;

(ii) for each sequence $\{t_n \in (0,\infty) : n \in \mathbb{N}\}$ we have $lim_{n\to\infty} t_n = 0$ if and only if $lim_{n\to\infty} f(t_n) = 1$;

(iii) there exist $r \in (0,1)$ and $l \in (1,\infty]$ such that $lim_{t\to 0^+}((f(t)-1) : t^r) = l$.

If $f \in \mathcal{F}$, then we say that f is a *logarithmic comparison function*. This denomination was suggested by the following three statements.

Proposition 5.6. *Let d be a distance on X, $f \in \mathcal{F}$ and $\rho(x,y) = ln(f(d(x,y)))$ for all $x, y \in X$. Then:*

1. *ρ is a distance on X and $\mathcal{T}(\rho) = \mathcal{T}(d)$.*
2. *The space (X,d) is complete if and only if the space (X,ρ) is complete.*
3. *If d is a symmetric, then ρ is a symmetric too.*
4. *If f is continuous and d is balanced, then ρ is balanced too.*

Proof. Let $x \in X$ and $\{x_n \in X : n \in \mathbb{N}\}$ be a sequence. Then:

1. $lim_{n\to\infty} d(x, x_n) = 0$ if and only if $lim_{n\to\infty} \rho(x, x_n) = 0$.

Hence the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same convergent sequences. Thus $\mathcal{T}(\rho) = \mathcal{T}(d)$.

2. $lim_{n,m\to\infty} d(x_n, x_m) = 0$ if and only if $lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.

Hence the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same Cauchy sequences. Therefore the space (X,d) is complete if and only if the space (X,ρ) is complete. Assertion 3 is obvious. Assertion 4 follows from the continuity of the functions f and ln. The proof is complete. □

Proposition 5.7. *(V. Berinde* [5]*) Let (X,d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping, $f \in \mathcal{F}$, $k \in (0,\infty)$, $\rho(x,y) = ln(f(d(x,y)))$ and $f(d(\varphi(x), \varphi(y))) \leq (f(d(x,y))^k$ for all $x, y \in X$. Then:*

1. *$\rho(\varphi(x), \varphi(y)) \leq k\rho(x,y))$ for all $x, y \in X$.*

2. The mapping φ is a ρ-contraction provided $k < 1$.

Proof. Really, let $x, y \in X$ and $d(\varphi(x), \varphi(y)) > 0$. Then $\rho(\varphi(x), \varphi(y)) > 0$ and $f(d(\varphi(x), \varphi(y))) \leq (f(d(x, y))^k$. Hence $\rho(\varphi(x), \varphi(y)) = lnf(d(\varphi(x), \varphi(y))) \leq ln(f(d(x, y))^k = k\rho(x, y)$. The proof is complete. □

Proposition 5.8. *Let (X, d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping, $f \in \mathcal{F}$, $k \in (0, \infty)$ and $\rho(x, y) = ln(f(d(x, y)))$, $f(d(\varphi(x), \varphi(y))) \leq (f(d(x, y))^k$ for all $x, y \in X$. Then:*

1. The distance space (X, d) is bounded if and only if the distance space (X, ρ) is bounded.

2. The distance space (X, d) is φ-bounded if and only if the distance space (X, ρ) is φ-bounded.

Proof. Let $q > 0$ and $p = lnf(q)$. Since the mapping φ is non-decreasing ($f(u) \leq f(v)$ provided $u \leq v$ and $u, v \in (0, \infty)$), we have $\rho(x, y) \leq p$ if and only if $d(x, y) \leq q$. The proof is complete. □

Remark 5.7. Let d be a distance on X and $f \in \mathcal{F}$. The distance $\rho(x, y) = ln(f(d(x, y)))$ for all $x, y \in X$ is called the *Berinde transformation* of the distance d.

In [5] V. Berinde has proved:

1. If d is a metric and $f(u + v) \leq f(u) \cdot f(v)$, then ρ is a metric too.
2. If d is a quasimetric and $f(u + v) \leq f(u) \cdot f(v)$, then ρ is a quasimetric too.

The next concept was examined by M. Jleli and B. Samet [26] for special distance spaces.

Let (X, d) be a distance space. A mapping $\varphi : X \longrightarrow X$ is called a *log-contraction* if there exist $f \in \mathcal{F}$ and $k \in (0, 1)$ such that $f(d(\varphi(x), \varphi(y))) \leq (f(d(x, y))^k$ for all $x, y \in X$.

In [5] V. Berinde arose the the following problems:

Problem 1. *Let ρ be the Berinde transformation of the distance d on X. Under which conditions the distance space (X, ρ) is complete?*

Problem 2. *Let d be a complete distance on X. Under which conditions on d the log-contraction $\varphi : X \rightarrow X$ has fixed points?*

Proposition 5.6 contains a complete solution of the Problem 1. Obviously, the Problem 2 is large and general. The following results highlight some positive responses to the Problem 2.

Proposition 5.9. *Let (X, d) be a distance space and $\varphi : X \rightarrow X$ be a given mapping. Suppose that there exist $f \in \mathcal{F}$ and $\lambda \in (0, 1)$ such that $f(d(\varphi(x), \varphi(y))) \leq (f(d(x, y))^\lambda$ for all $x, y \in X$. We put $\rho(x, y) = lnf(d(x, y)$. Then:*

(1) $\rho(\varphi(x), \varphi(y)) \leq \lambda\rho(x, y)$ for all $x, y \in X$ and $\rho(\varphi(x), \varphi(y)) < \rho(x, y)$ provided $\rho(x, y) > 0$;

(2) $lim_{n\to\infty} d(x_n, x_{n+k}) = lim_{n\to\infty} d(x_{n+k}, x_n) = 0$ and $lim_{n\to\infty} \rho(x_n, x_{n+k}) = lim_{n\to\infty} \rho(x_{n+k}, x_n) = 0$ for any $x \in X$ and each $k \in \mathbb{N}$.

(3) If $x \in X$, $p \in \mathbb{R}$, $p > 0$ and $max\{d(x, \varphi^n(x), d(\varphi^n(x), x)\Re\} \leq p$ for each $n \in \mathbb{N}$, then the Picard sequence $O(x, \varphi)$ is a Cauchy sequence.

(4) If the distance d is φ-bounded, then any Picard sequence of the mapping φ is a Cauchy sequence.

(5) The mapping φ does not have two distinct fixed points.

(6) The mapping φ does not have periodic non-fixed points.

Proof. By virtue of Proposition 5.7, we have $\rho(\varphi(x), \varphi(y)) \leq \lambda\rho(x, y)$ for all $x, y \in X$. Assertion (1) is proved. From Propositions 1.1 and 1.2 it follows that:

- the mapping φ is continuous;
- the mapping φ does not have two distinct fixed points;
- the mapping φ does not have periodic non-fixed points.

From Proposition 5.6 it follows that:

- ρ is a distance on X and $\mathcal{T}(\rho) = \mathcal{T}(d)$;
- the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same convergent sequences;
- the sequential spaces $(X, \mathcal{T}(\rho))$ and $(X, \mathcal{T}(d))$ have the same Cauchy sequences.

Let $x \in X$ be the given point. We put $x_1 = \varphi(x)$ and $x_{n+1} = \varphi(x_n)$ for each $n \in \mathbb{N}$. Then $O(x, \varphi) = \{x_n : n \in \mathbb{N}\}$ is the Picard sequence of the point x.

Fix $k \in \mathbb{N}$. We put $q_k = max\{d(x_k, x), d(x, x_k)\}$ and $p_k = f(q_k)$. By construction, $max\{\rho(x_k, x), \rho(x, x_k)\} \leq p_k$. Hence $\rho(x_n, x_{n+k}) \leq p_k\lambda^n$ and $lim_\rho(x_{n+k}, x_n) \leq p_k\lambda^n$ for each $n \in \mathbb{N}$. Therefore $lim_{n\to\infty}\rho(x_n, x_{n+k}) = 0$ and $lim_{n\to\infty}\rho(x_{n+k}, x_n) = 0$. Assertion (2) is proved.

Assume that $p \in \mathbb{R}$, $p > 0$ and $max\{d(x, \varphi^n(x), d(\varphi^n(x), x)\} \leq p$ for each $n \in \mathbb{N}$. Then $\rho(x_n, x_m) \leq pk^{min\{n,m\}}$. Therefore $lim_{n,m\to\infty}\rho(x_n, x_m) = 0$. Hence the Picard sequence $O(x, \varphi)$ is a Cauchy sequence of the distance spaces (X, d) and (X, ρ). Assertion (3) is proved. Assertion (4) follows from Assertion (3). The proof is complete. □

The following assertion is well known and elementary.

Lemma 5.6. *Let $p > 1$, and $k, c \in (0, 1)$. Then there exists $n(p, r, c) \in \mathbb{N}$ such that $0 < n(p^{k^n} - 1) < c$ for each $n \geq n(p, k, c)$.*

Proof. Denote by $g(t)'$ the derivative of the real-valued function $g(t)$. In the first we observe that $lim_{n\to\infty}n(p^{k^n} - 1) = lim_{t\to 0^+}(lnt/lnk)(p^t - 1) = (1/lnk)lim_{t\to 0^+}((p^t - 1)/t) \cdot t \cdot lnt$ $= (lnp/lnk) \cdot lim_{t\to 0^+}(lnt/(1 : t)) = (lnp/lnk) \cdot lim_{t\to 0^+}((lnt)'/(1 : t)') = (lnp/lnk) \cdot lim_{t\to 0^+}(1 : t)/(-1 : t^2) = -(lnp/lnk) \cdot lim_{t\to 0^+}t = 0$. Hence for each $c > 0$ there exists $n(p, k, c) \in \mathbb{N}$ such that $0 < n(p^{k^n} - 1) < c$ for each $n \geq n(p, k, c)$. □

In [26] for log-contraction of special symmetric spaces were proposed special estimation of the distance $d(\varphi^n(x), \varphi^{n+m}(x))$. The following is a more general result.

Proposition 5.10. *Let (X, d) be a distance space and $\varphi : X \to X$ be a given mapping. Suppose that there exist $f \in \mathcal{F}$ and $k \in (0, 1)$ such that $f(d(\varphi(x), \varphi(y))) \leq (f(d(x, y))^k$ for all $x, y \in X$. Then for each positive number $q \in (0, \infty)$ there exist $r \in (0, 1)$ and $n(f, q) \in \mathbb{N}$ such that from $x, y \in X$ and $d(x, y) \leq q$ it follows that $d(\varphi^n(x), \varphi^n(y)) < 1/n^{1/r}$ for each $n \geq n(f, q)$.*

Proof. Fix two distinct points $a, b \in X$ for which $d(a, b) \leq q$. Let $p = f(q)$ and $a_n = \varphi^n(a)$, $b_n = \varphi^n(b)$ for any $n \in \mathbb{N}$.

Claim 1. $1 \leq f(d(a_n, b_n)) \leq p^{k^n}$ *for each* $n \in \mathbb{N}$.

The assertion of Claim 1 is true for $n = 1$. Assume that $n \geq 1$ and $f(d(a_n, b_n)) \leq p^{k^n}$. Then $f(d(a_{n+1}, b_{n+1})) = f(d(\varphi(a_n), \varphi(b_n))) \leq f(d(a_n, b_n))^k \leq p^{k^{n+1}}$. Claim is proved.

Claim 2. $lim_{n\to\infty}f(d(a_n, b_n)) = 1$ *and* $lim_{n\to\infty}d(a_n, b_n) = 0$.

The equality $lim_{n\to\infty}f(d(a_n, b_n)) = 1$ follows from Claim 1. The equality $lim_{n\to\infty}d(a_n, b_n) = 0$ follows from the proprieties of the functions $\mathcal{F}$.

Claim 3. *There exist a number $r \in (0, 1)$, a number $c = c(f, q) > 0$ and a natural number $m(f, q) \in \mathbb{N}$ such that $(d(a_n, b_n))^r \leq c(p^{k^n} - 1)$ for each $n \geq m(f, q)$, $n \in \mathbb{N}$.*

Since $f \in \mathcal{F}$, there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $lim_{t\to 0^+}((f(t) - 1) : t^r) = l$. Thus, there exist two positive numbers $c, t_0 > 0$ such that $((f(t) - 1) : t^r) > c^{-1}$ for each $t \in (0, t_0]$. Hence $t^r < c(f(t) - 1)$ for each $t \in (0, t_0]$. Since $f(t_0) > 1$, there exists $m(f, q) \in \mathbb{N}$ such that $p^{k^n} \leq f(t_0)$ for each $n \geq m(f, q)$. Therefore for $n \geq m(f, q)$ we have $(d(a_n, b_n))^r \leq c(f(d(a_n, b_n)) - 1) \leq c(p^{k^n} - 1)$.

Claim 4. *There exists a natural number $n(f, q) \in \mathbb{N}$ such that $d(\varphi^n(x), \varphi^n(y))^r < 1/n$ for each $n \geq n(f, q)$.*

From Claim 3 it follows that there exists $m(f, q) \in \mathbb{N}$ such that $(d(a_n, b_n))^r \leq c(p^{k^n} - 1)$ for each $n \geq m(f, q)$. By virtue of Lemma 5.6, there exists $n(p, k, c^{-1}) \in \mathbb{N}$ such that

$0 < n(p^{k^n} - 1) < c^{-1}$ for each $n \geq n(p, k, c^{-1})$. Let $n(f, q) = max\{m(f, q), n(p, k, c^{-1})\}$. For $n \geq n(f, q)$ we have $(d(a_n, b_n))^r \leq c(p^{k^n} - 1) < c \cdot c^{-1} \cdot n^{-1} = 1/n$. Claim 4 and Proposition 5.10 are proved. □

Corollary 5.8. *Let $\varphi : X \to X$ be a given mapping and (X, d) be a φ-bounded complete H-distance space. If the mapping φ is log-contractive, then:*
1. The mapping φ has a unique fixed point.
2. Any Picard sequence $O(\varphi, x)$ is a Cauchy sequence.

Corollary 5.9. *Let $\varphi : X \to X$ be a given mapping and (X, d) be a φ-bounded complete balanced distance space. If the mapping φ is log-contractive, then:*
1. The mapping φ has a unique fixed point.
2. Any Picard sequence $O(\varphi, x)$ is a Cauchy sequence.

Corollary 5.10. *Let (X, d) be an N-symmetric compact space and $\varphi : X \longrightarrow X$ be a log-contractive mapping. Then:*
1. $d(\varphi(x), \varphi(y)) < d(x, y)$ for all distinct points $x, y \in X$.
2. $lim_{n\to\infty} d(\varphi^n(x), \varphi^{n+1}(x)) = 0$ for each point $x \in X$.
3. The mapping φ has a unique fixed point.
4. Any Picard sequence $O(\varphi, x)$ is a Cauchy sequence.

6. On B-symmetric spaces

Let X be a non-empty set. A distance d on X is called a *Branciari metric* or a *B-symmetric* and (X, d) is called a *B-symmetric space*, if:
(i) d is a symmetric;
(iii) $d(x, y) \leq d(x, u)+d(u, v)+d(v, y)$ for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y.

The concept of a B-symmetric was introduced by A. Branciari [13] as a generalized metric. We called them B-symmetrics, since there are many distinct distances with that name and, in general, any distance function is a generalized metric (see [38, 36, 30, 31, 32]).

Example 6.10. Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(2^{-n}, 2^{-m}) = 1$ for all distinct $n, m \in \mathbb{N}$ and $d(2^{-n}, 0) = 2^{-n}$ for each $n \in \mathbb{N}$. The symmetric d is a B-symmetric and an N-distance on X, the topology $\mathcal{T}(d)$ generate by d is a compact metric topology on X, $\{2^{-n} : n \in \mathbb{N}\}$ is a convergent to 0 not Cauchy sequence. Hence d is not an F-distance on X.

Lemma 6.7. *Let d be a B-symmetric on X. Then d is balanced.*

Proof. Assume that $\{x_n : n \in \mathbb{N}\}$ convergent to $x \in X$ Cauchy sequence and $y \in X$. We can suppose that $x \neq y$, $x \neq x_n$, $x_n \neq y$ and $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. By assumptions, we have $d(x, y) \leq d(x, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y)$ and $d(x_{n+1}, y) \leq d(x_{n+1}, x_n) + d(x_n, x) + d(x, y)$. Hence for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x, y) < d(x_n, y) + \varepsilon$ and $d(x_n, y) < d(x, y) + \varepsilon$ for every $n > k$. Thus $d(x, y) = lim_{n\to\infty} d(x_n, y)$. The proof is complete. □

Corollary 6.11. *Let d be a B-symmetric on X. Then any convergent Cauchy sequence has a unique limit point in X.*

Example 6.11. Let $X = \{2^{-n} : n \in \mathbb{N}\} \cup \{0, 2\}$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(0, 2) = 1$, $d(2^{-n}, 2^{-m}) = 1$ for all distinct $n, m \in \mathbb{N}$ and $d(2^{-n}, 0) = d(2^{-n}, 2) = 2^{-n}$ for each $n \in \mathbb{N}$. The symmetric d is a balanced B-symmetric and the topology $\mathcal{T}(d)$ generate by d is a compact T_1-topology and is not a T_2-topology. Moreover, d is not an H-distance, since $B(0, d, r) \cap B(2, d, r) \neq \emptyset$ for any $r > 0$. Consider the mapping $\varphi : X \longrightarrow X$, where

$\varphi(0) = 2$, $\varphi(2) = 0$ and $\varphi(2^{-n}) = 2^{-n-1}$ for each $n \in \mathbb{N}$. Then $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in X$, $\{0, 2\}$ is the set of periodic points of φ and the set of fixed points is empty. By virtue of Proposition 5.6, φ is not a log-contraction.

Proposition 6.11. *Let d be a B-symmetric on X and $\varphi : X \longrightarrow X$ be a log-contraction of the distance space (X, d). Then:*

1. The distance space (X, d) is φ-bounded.

2. Any Picard sequence $O(\varphi, x)$ is a Cauchy sequence.

Proof. Let $x \in X$ be the given point. We put $x_1 = \varphi(x)$ and $x_{n+1} = \varphi(x_n)$ for each $n \in \mathbb{N}$. Then $O(x, \varphi) = \{x_n : n \in \mathbb{N}\}$ is the Picard sequence of the point x. We put $q = max\{d(x, x_1), d(x, x_2), d(x, x_3)\}$. By virtue of Proposition 5.10, there exist $r \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that from $y \in X$ and $d(x, y) \leq q$ it follows that $d(\varphi^n(x), \varphi^n(y)) < 1/n^{1/r}$ for each $n \geq n_0$. In this case we have $b = d(x, x_1) + \Sigma\{d(x_n, x_{n+1}) : n \in \mathbb{N}\} < \infty$.

We have two possible cases.

Case 1. $x_m = x_{m+1}$ for some $m \in \mathbb{N}$.

In this case x_m is a fixed point, the Picard sequence $O(\varphi, x)$ is a Cauchy sequence and $sup\{d(x, x_n) : n \in \mathbb{N}\} = sup\{d(x, x_n) : n \leq m\} < \infty$.

Case 2. $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$.

By virtue of Proposition 5.9, the mapping φ does not have two distinct fixed points and the mapping φ does not have periodic non-fixed points. Hence $x \neq x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. In this case $d(x, x_{2n+1}) \leq d(x, x_1) + \Sigma\{d(x_i, x_{i+1}) : i \leq 2n\} < b$ and $d(x, x_{2n+2}) \leq d(x, x_2) + \Sigma\{d(x_i, x_{i+1}) : 2 \leq i \leq 2n + 1\} < q + b$ for each $n \in \mathbb{N}$. Hence $sup\{d(x, x_n) : n \in \mathbb{N}\} < q + b < \infty$. Assertion 1 is proved.

By virtue of Proposition 5.9, any Picard sequence of the mapping φ is a Cauchy sequence. The proof is complete. □

Corollary 6.12. *Let (X, d) be a complete B-symmetric space and $\varphi : X \longrightarrow X$ be a log-contractive mapping. Then:*

1. The mapping φ has a unique fixed point.

2. Any Picard sequence of the mapping φ is a Cauchy sequence convergent to the fixed point of φ.

Corollary 6.13. *Let (X, d) be a complete metric space and $\varphi : X \longrightarrow X$ be a log-contractive mapping. Then:*

1. The mapping φ has a unique fixed point.

2. Any Picard sequence of the mapping φ is a Cauchy sequence convergent to the fixed point of φ.

Remark 6.8. Corollaries 6.12 and 6.13 were formulated in ([26], Theorem 2.1 and Corollary 2.1). We mention that the Lemma 2.1 from [26] is not true (see the following Example 6.12) and that lemma was using in the proof of Theorem 2.1 from [26]. Corollary 2.2 from [26] remain true to.

Example 6.12. Let $\mathbb{R}$ be the real line with the metric $d(x, y) = |x - y|$. Consider the points $x = -2$, $y = 2$ and the sequence $\{x_n = 2^{-n} : n \in \mathbb{N}\}$. By construction:

(i) $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$;

(ii) $x_n \neq x$ for each $n \in \mathbb{N}$;

(iii) $x_n \neq y$ for each $n \in \mathbb{N}$;

(iv) $lim_{n\to\infty} d(x_n, x) = lim_{n\to\infty} d(x_n, y) = 2$.

Lemma 2.1 [3] affirms that $x = y$, a contradiction. Hence Lemma 2.1 from [3] is not true.

7. Conditions of existence of distances on spaces

As in [3] we say that X is a space with a weak axiom of countability if there exists a family $\mathcal{B} = \{Q_n x : n \in \mathbb{N}, x \in X\}$ of subsets of X with the following properties:

- $x \in Q_{n+1}x \subseteq Q_n x$ for all $n \in \mathbb{N}$ and $x \in X$;

- for each open subset U of X and for any point $x \in U$ there exists $n \in \mathbb{N}$ such that $Q_n x \subseteq U$.

The family $\mathcal{B} = \{Q_n x : n \in \mathbb{N}, x \in X\}$ is called a weak base of the space. Every weak base is a network of the space.

A sequence $\{L_n : n \in \mathbb{N}\}$ of subsets of a space X is a sequential base of the space X at the point x if:

- $x \in L_{n+1} \subseteq L_n$ for each $n \in \mathbb{N}$;

- if $A = \{x_n : n \in \mathbb{N}\}$ is a sequence of points in X convergent to x, then the set $A \setminus L_n$ is finite for each $n \in \mathbb{N}$;

- for each open subset U of X for which $x \in U$ there exists $n \in \mathbb{N}$ such that $L_n \subseteq U$.

The proof of the following assertion is similar as for T_1-spaces.

Theorem 7.3. *(S. I. Nedev* [29]*, Theorem 5, for T_1-spaces). For a T_0-space X the following assertions are equivalent:*

1. There exists a distance d on X such that $T(d)$ is the topology of the space X.

2. X is a space with a weak axiom of countability.

3. The space X is sequential and for each point $x \in X$ there exists a sequential base of the space X at the point x.

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