Ricci solitons on CR submanifolds of maximal CR dimension in complex projective space

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ABSTRACT. We study parallel and symmetric second order tensor fields on CR submanifolds of maximal CR dimension of the complex projective space. Under some natural conditions, these tensors are scalar multiples of the metric; an example of submanifold satisfying these assumptions is the geodesic hypersphere in $P^{\frac{n+1}{2}}(\mathbb{C})$. The result is used for Ricci solitons on these CR submanifolds.

1. Introduction

In 1923, Eisenhart [11] proved that if a Riemannian manifold (M, g) admits a parallel and symmetric second order covariant tensor which is not a constant multiple of the metric tensor, then it is reducible. In 1926, Levy [15] proved that a parallel and symmetric second order non degenerate tensor α in a space form is proportional to the metric tensor. While both Eisenhart and Levy work locally, Ramesh Sharma gives in [18] a global approach based on Ricci identities. In addition to space-forms, Sharma considered this *Eisenhart problem* in contact geometry [19]-[21], for example for K-contact manifolds in [20]. Since then, several other studies appeared in various almost contact manifolds, see the bibliography of [2].

In this paper we propose a new framework for this problem, also from the almost contact geometry, namely CR submanifolds of maximal CR dimension in a complex projective space, introduced by M. Djoric and M. Okumura in [9] and intensively studied in [10]. In order to obtain a similar result with the previous studies, namely a constant multiple of metric, we impose a number of four hypothesis, two of them about the manifold and other two about the given tensor field. An example satisfying these first two assumptions is the geodesic hypersphere which is discussed as the end of Section 2.

Our main result is then connected with the recent theory of Ricci solitons ([7]), a subject included in the Hamilton-Perelman approach (and proof) of Poincaré Conjecture. So, in case of existence, a Ricci soliton on these submanifolds having the structural vector field as soliton generator must be shrinking. A study on eta-Ricci solitons on Hopf hypersurfaces (a setting similar but different to the present one) is done in [3], while for the notion of $*$-Ricci solitons of real hypersurfaces in non-flat complex space forms see [14].

Our work is structured as follows. The first section is a very brief review of CR submanifolds of maximal CR dimension and Ricci solitons. The next section is devoted to the (symmetric case of) Eisenhart problem in this framework, while the relationship with the Ricci solitons is pointed out in the last section. Let us remark that there are correspondences between our present results in Ricci solitons and some previous papers from Bibliography.

2. A REVIEW OF CR SUBMANIFOLDS OF MAXIMAL CR DIMENSION AND RICCI SOLITONS

Let p and n be two integer numbers with $p \geq 1$ and $n > 2p - 1$ and let us consider $(P^{\frac{n+p}{2}}(\mathbb{C}), J)$ the complex projective space equipped with the Fubini-Study metric of constant holomorphic sectional curvature $4k$, $k > 0$. In all what follows we use the notions and notations of [10].

Fix M an oriented CR submanifold of $P^{\frac{n+p}{2}}(\mathbb{C})$ of maximal CR dimension $\frac{n-1}{2}$; then M has dimension n. From [10, p. 95] n is necessarily odd and therefore there exists a unit vector field ξ normal to M (i.e. $\xi \in T^{\perp}M$) such that for every $x \in M$ we have $JT_xM \subset T_xM \oplus span(\xi_x)$. Let g be the induced Riemannian metric on M and A be the shape operator corresponding to ξ; let us also consider on M the tangent vector field $U = -J\xi$ and the associated 1-form $u = U^{\#}$. U is the *structural vector field* of M.

From Lemma 15.1. of [10, p. 96] an orthonormal basis of $T^{\perp}M$ is:
$\xi, \xi_1, ..., \xi_q, J\xi_1, ..., J\xi_q$ where $q = \frac{p-1}{2}$. Denotes A_a and A_{a^*} the shape operator of ξ_a respectively $J\xi_a$ for $1 \leq a \leq q$. From [10, p. 98] we define for a vector field X:

$$s_a(X) = -g(A_{a^*}U, X) \quad s_{a^*}(X) = g(A_aU, X)$$

and then the normal connection D of M is [10, p. 97]:

$$D_X\xi = \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\}.$$

At this moment we need two supplementary assumptions:
P1) ξ is parallel with respect to the normal connection; therefore $A_a(U) = A_{a^*}(U) = 0$,
P2) the shape operator A has two distinct eigenvalues; then these eigenvalues are constant, [10, p. 126], and $A = \sigma I + (\rho - \sigma)U \oplus u$ as formula (19.17) of [10, p. 129], ρ and σ being different to zero.
Let us remark that from Theorem 19.2. of [10, p. 131] the properties P1 and P2 implies for $p \geq 2$ the existence of a geodesic hypersphere S of $P^{\frac{n+p}{2}}(\mathbb{C})$ such that M lies in S.

A straightforward application of formula (15.28) of [10, p. 99] gives the curvature of M in the direction of U:

$$(2.1) \qquad R(X, Y)U = k\{u(Y)X - u(X)Y\} + \rho[u(Y)AX - u(X)AY].$$

Let us introduce also *the structural tensor field F* provided by the decomposition of JX into tangent and normal components: $JX = FX + u(X)\xi$.

In the last part of this section we recall the notion of Ricci solitons according to [22, p. 139]. On the manifold M, a *Ricci soliton* is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that:

$$(2.2) \qquad \mathcal{L}_V g + 2Ric_g + 2\lambda g = 0$$

where $\mathcal{L}_V$ is the Lie derivative with respect to the vector field V and Ric_g is the Ricci tensor field of g. The Ricci soliton is said to be *shrinking, steady* or *expanding* according as λ is negative, zero or positive.

3. PARALLEL AND SYMMETRIC SECOND ORDER TENSOR FIELDS

Fix α a symmetric covariant tensor field of order 2 on M. If we suppose α to be parallel on M, $\nabla\alpha = 0$, then:
i) $\alpha(U, U)$=constant; is a quickly consequence of the fact that U is unitary,
ii) for every vector field X, Y, Z, W of M we have [18]:

$$(3.3) \qquad \alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0.$$

With $Z = W = U$ in (1.1) we get:

(3.4) $k\{u(Y)\alpha(X, U) - u(X)\alpha(Y, U)\} + \rho[u(Y)\alpha(AX, U) - u(X)\alpha(AY, U)] = 0.$

In order to make α similar to g we consider two new hypothesis:
P3) the shape operator A is α-symmetric: $\alpha(AX, Y) = \alpha(X, AY)$,
P4) the structural tensor field F is α-skew-symmetric: $\alpha(FX, Y) = -\alpha(X, FY)$.

Theorem 3.1. *Let:*
i) M be a CR submanifold of maximal CR dimension in $P^{\frac{n+p}{2}}(\mathbb{C})$ with P1 and P2,
ii) α be a parallel and symmetric tensor field of $(0, 2)$-type on M satisfying P3 and P4.
Then α is a constant multiple of g:

(3.5) $$\alpha(X, Y) = \alpha(U, U)g(X, Y).$$

Proof. With P3 plugged into (3.4) we derive:

$$(k + \rho^2)[u(Y)\alpha(X, U) - u(X)\alpha(Y, U)] = 0$$

which means for $X = U$:

(3.6) $$\alpha(Y, U) = \alpha(U, U)u(Y) = \alpha(U, U)g(Y, U).$$

Let us apply ∇_X to the last relation and use the parallelism of g and α:

$$\alpha(\nabla_X Y, U) + \alpha(Y, \nabla_X U) = \alpha(U, U)[g(\nabla_X Y, U) + g(Y, \nabla_X U)]$$

which means, via (3.6) with $Y \to \nabla_X Y$ and the formula (15.27) of [10, p. 98]:

$$\alpha(Y, FAX) = \alpha(U, U)g(Y, FAX)$$

or, with P4:

$$\alpha(FY, AX) = \alpha(U, U)g(FY, AX).$$

In this last equation let $Y \to FY$ and use the formula $F^2 = -I + u \otimes U$ of [10, p. 96]; then:

(3.7) $-\alpha(Y, AX) + u(Y)\alpha(U, AX) = \alpha(U, U)[-g(Y, AX) + u(Y)g(U, AX)].$

From (3.7) we get:

$$\alpha(AX, Y) = \alpha(U, U)g(AX, Y)$$

and we use the expression of A given by P2 and the remark that $\sigma \neq 0$ in order to obtain (3.5). $\qquad\square$

In the following we search for an example concerning the above setting and a comparison with the papers [1]-[3] and [7] reveals the complexity of the present framework. Let $\mathbb{S}^{n+p+1}$ be the unit sphere defined by $\sum_{i=0}^{\frac{n+p}{2}} z^i \bar{z}^i = 1$ in $\mathbb{C}^{\frac{n+p}{2}+1} = \mathbb{C}^{2r+1} \oplus \mathbb{C}^{2s+1}$ with $2r + 2s = \frac{n+p}{2} - 1$. In $\mathbb{S}^{n+p+1}$ we choose two spheres, $\mathbb{S}^{4r+1}$ and $\mathbb{S}^{4s+1}$, in such a way that they lie respectively in the complex subspaces $\mathbb{C}^{2r+1}$ and $\mathbb{C}^{2s+1}$ of $\mathbb{C}^{\frac{n+p}{2}+1}$. Then the product $\mathbb{S}^{4r+1} \times \mathbb{S}^{4s+1}$ is a hypersurface of $\mathbb{S}^{n+p+1}$ and the quotient manifold $M_{2r,2s}^C = (\mathbb{S}^{4r+1} \times \mathbb{S}^{4s+1})/\mathbb{S}^1$ is a real hypersurface of $P^{\frac{n+p}{2}}(\mathbb{C})$.

The Theorem 19.3. of [10, p. 132] assures that for $p = 1$ the real hypersurface M with condition P2 of our Theorem 3.1 is congruent with $M_{0,2s}^C$ for $s = \frac{n-1}{4}$. Obviously, the codimension $p = 1$ assures also the hypothesis P1 and then we have $M_{0,\frac{n-1}{2}}^C = (S^1 \times S^n)/S^1$ as real hypersurface in $P^{\frac{n+1}{2}}(\mathbb{C})$ satisfying both P1 and P2. Hence, every parallel and symmetric tensor field α of $(0, 2)$-type on $(M_{0,\frac{n-1}{2}}^C = (S^1 \times S^n)/S^1, g)$ satisfying P3 and P4 is a constant multiple of the metric g.

4. RICCI SOLITONS

Let $\alpha = \mathcal{L}_U g + 2Ric_g$. Obviously, α is symmetric and in order to obtain P3 and P4 we adapt the formula (23.1) of [10, p. 163]:

$$Ric_g(X,Y) = k\{(n+2)g(X,Y) - 3u(X)u(Y)\} + (traceA)g(AX,Y) - g(A^2X,Y) + ... \quad (3.1)$$

where the remaining terms are expressed with A_a and A_{a^*}. For P3 is necessary a commutation formula between A and A_a, A_{a^*} while for P4 we need a commutation formula between F and A, A_a, A_{a^*}.

The case $p = 1$ is more simple since $A_a = A_{a^*} = 0$ but in [5] it is proved that a real hypersurface in a non-flat complex space form does not admit a Ricci soliton with U as soliton vector field; the same problem but in terms of eta-Ricci solitons is treated in [12]. Other two results of non-existence with respect to the Ricci tensor are as follows:

i) ([13]) there are no real hypersurfaces M in $P^*(\mathbb{C})$ with recurrent Ricci tensor and such that the structure vector field of M is a principal curvature vector everywhere,

ii) ([16]) there are no real hypersurfaces with recurrent Ricci tensor in non-flat complex space forms of complex dimension ≥ 3.

It remains then $p \geq 2$ which yields $n > 3$.

We are interested in computing the scalar λ of a possible Ricci soliton for $p \geq 2$. From $\lambda = -\frac{1}{2}\alpha(U,U)$ and $A_a(U) = A_{a^*}(U) = 0$ we get:

$$(4.8) \qquad\qquad \lambda = -Ric_g(U,U) = -(n-1)(\rho\sigma + k)$$

since $traceA = \rho + (n-1)\sigma$. For $n > 3$ we have [10, p. 127]:

$$(4.9) \qquad\qquad \sigma = \frac{2k + \rho^2}{\rho}$$

which yields:

$$(4.10) \qquad\qquad \lambda = -(n-1)(3k + \rho^2) < 0.$$

Proposition 4.1. *Let M be a CR submanifold of maximal CR dimension in $P^{\frac{n+p}{2}}(\mathbb{C})$ with P1 and P2. If (M,g) admits a Ricci soliton with U as soliton vector field then this is shrinking with $\lambda < -2(3k + \rho^2)$.*

A result of non-existence is:

Proposition 4.2. *Let M_n be a compact, minimal CR submanifold of maximal CR dimension in $P^{\frac{n+p}{2}}(\mathbb{C})$ with the scalar curvature $r \geq (n+2)(n-1) > 10$. Then U is a Killing vector field on M but (g,U) is not a Ricci soliton on M which means that M is not an Einstein manifold.*

Proof. The first part of the conclusion results from Lemma 23.1 of [10, p. 163]. According to Theorem 23.1 of [10, p. 166] we get that M is a real hypersurface of $P^{\frac{n+1}{2}}(\mathbb{C})$ and then we apply the last result of the paper [5]. □

Let us remark that the Propositions 4.1 and 4.2 correspond to the Corollary 3.1 respectively Theorem 3.2 of [8]. Also, recall from [10] that a CR submanifold of maximal CR dimension is equipped with an almost contact structure which is naturally induced from the almost complex structure of the ambient manifold. Then, our Proposition 4.1 agrees with Theorem 1.1. of [4, p. 48] which states that a contact Ricci soliton is shrinking. Moreover, our Proposition 4.2 can be put in correspondence with Theorem 1.2 of [6, p. 1386] which states the non-existence of Ricci solitons on compact Hopf hypersurfaces of a non-flat complex space form. Also, concerning the example $M_{0,\frac{n-1}{2}}^C = (S^1 \times S^n)/S^1$ discussed in the previous Section we point out the non-Einstein ambient manifold $S^1 \times S^n$ for our $n > 3$, [17, p. 65].

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