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Crossed products of Hilbert pro-C^*-bimodules and associated pro-C^*-algebras

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ABSTRACT. An action (γ, α) of a locally compact group G on a Hilbert pro-C^*-bimodule (X, A) induces an action $\gamma \times \alpha$ of G on $A \times_X \mathbb{Z}$ the crossed product of A by X. We show that if (γ, α) is an inverse limit action, then the crossed product of $A \times_\alpha G$ by $X \times_\gamma G$ respectively of $A \times_{\alpha,r} G$ by $X \times_{\gamma,r} G$ is isomorphic to the full crossed product of $A \times_X \mathbb{Z}$ by $\gamma \times \alpha$ respectively the reduced crossed product of $A \times_X \mathbb{Z}$ by $\gamma \times \alpha$.

1. INTRODUCTION AND PRELIMINARIES

An automorphism α of a C^*-algebra A induces an action of the integers $\mathbb{Z}$ on A, and an action δ, called the dual action, of the circle group $\mathbb{T}$ on the crossed product $A \times_\alpha \mathbb{Z}$ of A by α. Given an action of $\mathbb{T}$ on an algebra A, it is natural to ask when A is isomorphic to the crossed product of a C^*-algebra by an automorphism. To answer this question, Abadie, Eilers and Exel [2] introduced the notion of crossed products by Hilbert C^*-bimodules, generalizing the notion of C^*-crossed products by automorphisms. In [11], Joiţa and Zarakas extended this construction in the context of pro-C^* -algebras. An action (γ, α) of a locally compact group G on a Hilbert C^*-bimodule (X, A) induces an action $\gamma \times \alpha$ of G on the crossed product $A \times_X \mathbb{Z}$ of A by X. It is natural to ask if $(A \times_X \mathbb{Z}) \times_{\gamma\times\alpha} G$ is isomorphic to the crossed product of a C^*-algebra by a Hilbert C^*-bimodule. In the case of amenable groups, Abadie [1] gave a positive answer to this question. In this paper, we show that the result of Abadie is valid for an arbitrary group and then we extend this result in the context of pro-C^*-algebras.

Throughout this paper all vector spaces are considered over the field $\mathbb{C}$ of complex numbers and all topological spaces are assumed to be Hausdorff.

A *pro-C^*-algebra* (alias locally C^* -algebra) is a complete Hausdorff topological $*$-algebra A whose topology is given by a directed family of C^*-seminorms $\{p_\lambda; \lambda \in \Lambda\}$.

Let A and B be pro-C^*-algebras with the topology given by the family of C^*-seminorms $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, and $\Gamma' = \{q_\delta; \delta \in \Delta\}$, respectively.

A *pro-C^*-morphism* is a continuous $*$ -morphism $\varphi : A \to B$ (that is, φ is linear, $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$, $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$ and for each $q_\delta \in \Gamma'$, there is $p_\lambda \in \Gamma$ such that $q_\delta(\varphi(a)) \leq p_\lambda(a)$ for all $a \in A$). An invertible pro-C^*-morphism $\varphi : A \to B$ is a *pro-C^*-isomorphism* if φ^{-1} is also a pro-C^*-morphism.

For $\lambda \in \Lambda$, $\ker p_\lambda$ is a closed $*$-bilateral ideal and $A_\lambda = A/\ker p_\lambda$ is a C^*-algebra in the C^*-norm $\|\cdot\|_{p_\lambda}$ induced by p_λ (that is, $\|a + \ker p_\lambda\|_{p_\lambda} = p_\lambda(a)$, for all $a \in A$). The canonical map π_λ^A from A to A_λ is defined by $\pi_\lambda^A(a) = a + \ker p_\lambda$ for all $a \in A$. For $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$ there is a surjective C^*-morphism $\pi_{\lambda\mu}^A : A_\lambda \to A_\mu$ such that $\pi_{\lambda\mu}^A(a + \ker p_\lambda) =$

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$a + \ker p_\mu$, and then $\{A_\lambda; \pi^A_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras. Moreover, pro-C^*-algebras A and $\lim\limits_{\leftarrow\lambda} A_\lambda$ are isomorphic (the Arens-Michael decomposition of A). For more details we refer the reader to [5, 13].

Here we recall some basic facts from [9] and [14] regarding Hilbert pro-C^*-modules and Hilbert pro-C^*-bimodules, respectively.

A right Hilbert pro-C^*-module over A (or just *Hilbert A-module*), is a linear space X that is also a right A-module equipped with a right A-valued inner product $\langle\cdot,\cdot\rangle_A$, that is $\mathbb{C}$ - and A-linear in the second variable and conjugate linear in the first variable, with the following properties:

- $\langle x,x\rangle_A \geq 0$ and $\langle x,x\rangle_A = 0$ if and only if $x = 0$;
- $(\langle x,y\rangle_A)^* = \langle y,x\rangle_A$

which is complete with respect to the topology induced by the family of seminorms $\{p^A_\lambda\}_{\lambda\in\Lambda}$, with $p^A_\lambda(x) = p_\lambda(\langle x,x\rangle_A)^{\frac{1}{2}}, x \in X$. A Hilbert A-module is *full* if the pro-C^*-subalgebra of A generated by $\{\langle x,y\rangle_A; x,y \in X\}$ coincides with A. *A left Hilbert pro-C^*-module* X over a pro-C^*-algebra A is defined in the same way, where for instance the completeness is requested with respect to the family of seminorms $\{{}^Ap_\lambda\}_{\lambda\in\Lambda}$, where ${}^Ap_\lambda(x) = p_\lambda({}_A\langle x,x\rangle)^{\frac{1}{2}}, x \in X$.

In the case when X is a left Hilbert pro-C^* module over A and a right Hilbert pro-C^* module over B, the topology on B is given by the family of C^*-seminorms $\{q_\lambda\}_{\lambda\in\Lambda}$ such that the following relations hold:

- ${}_A\langle x,y\rangle z = x\langle y,z\rangle_B$ for all $x,y,z \in X$;
- $q^B_\lambda(ax) \leq p_\lambda(a)q^B_\lambda(x)$ and ${}^Ap_\lambda(xb) \leq q_\lambda(b)^Ap_\lambda(x)$ for all $x \in X$, $a \in A$, $b \in B$ and for all $\lambda \in \Lambda$,

then we say that X is *a Hilbert $A - B$ pro-C^* -bimodule*. A Hilbert $A - B$ pro-C^*-bimodule X is *full* if it is full as a right as well as a left Hilbert pro-C^*-module.

Clearly, any pro-C^*-algebra A has a canonical structure of full Hilbert $A - A$ bimodule under the inner products given by ${}_A\langle a,b\rangle = ab^*$, respectively $\langle a,b\rangle_A = a^*b$ for all $a,b \in A$.

A morphism of Hilbert pro-C^*-bimodules from a Hilbert $A - A$ pro-C^*-bimodule X to a Hilbert $B - B$ pro-C^*-bimodule Y is a pair (Φ,φ) consisting of a pro-C^*-morphism $\varphi : A \to B$ and a map $\Phi : X \to Y$ such that the following relations hold:

- ${}_B\langle\Phi(x),\Phi(y)\rangle = \varphi({}_A\langle x,y\rangle)$ and $\Phi(ax) = \varphi(a)\Phi(x)$ for all $x,y \in X$, $a \in A$,
- $\langle\Phi(x),\Phi(y)\rangle_B = \varphi(\langle x,y\rangle_A)$ and $\Phi(xa) = \Phi(x)\varphi(a)$ for all $x,y \in X$, $a \in A$.

A morphism of Hilbert pro-C^*-bimodules $(\Phi,\varphi) : (X,A) \to (Y,B)$ is *an isomorphism of Hilbert pro-C^*-bimodules* if Φ and φ are invertible and $\left(\Phi^{-1},\varphi^{-1}\right) : (Y,B) \to (X,A)$ is a morphism of Hilbert pro-C^*-bimodules.

Let X be a Hilbert $A - A$ pro-C^*-bimodule. Then, for each $\lambda \in \Lambda$, ${}^Ap_\lambda(x) = p^A_\lambda(x)$ for all $x \in X$, and the normed space $X_\lambda = X/\ker p^A_\lambda$, where $\ker p^A_\lambda = \{x \in X; p^A_\lambda(x) = 0\}$, is complete in the norm $||x + \ker p^A_\lambda||_{X_\lambda} = p^A_\lambda(x), x \in X$. Moreover, X_λ has a canonical structure of a Hilbert $A_\lambda-$ A_λ C^*-bimodule with $\left\langle x + \ker p^A_\lambda, y + \ker p^A_\lambda\right\rangle_{A_\lambda} = \langle x,y\rangle_A + \ker p_\lambda$ and ${}_{A_\lambda}\left\langle x + \ker p^A_\lambda, y + \ker p^A_\lambda\right\rangle = {}_A\langle x,y\rangle + \ker p_\lambda$ for all $x,y \in X$. The canonical surjection from X to X_λ is denoted by σ^X_λ. For $\lambda,\mu \in \Lambda$ with $\lambda \geq \mu$, there is a canonical surjective linear map $\sigma^X_{\lambda\mu} : X_\lambda \to X_\mu$ such that $\sigma^X_{\lambda\mu}\left(x + \ker p^A_\lambda\right) = x + \ker p^A_\mu$ for all $x \in X$. For $\lambda,\mu \in \Lambda$ with $\lambda \geq \mu$, $\left(\sigma^X_{\lambda\mu}, \pi^A_{\lambda\mu}\right)$ is a morphism of Hilbert C^*-bimodule and $\lim\limits_{\leftarrow\lambda} X_\lambda$ has a canonical structure of Hilbert $\lim\limits_{\leftarrow\lambda} A_\lambda - \lim\limits_{\leftarrow\lambda} A_\lambda$ bimodule. Moreover, $X = \lim\limits_{\leftarrow\lambda} X_\lambda$, up to an isomorphism of Hilbert pro-C^*- bimodules.

A covariant representation of a Hilbert pro-C^*-bimodule (X, A) on a pro-C^*-algebra B is a morphism of Hilbert pro-C^*-bimodules from (X, A) to the Hilbert pro-C^*-bimodule (B, B).

The crossed product of A by a Hilbert pro-C^*-bimodule (X, A) is a pro-C^*-algebra, denoted by $A \times_X \mathbb{Z}$, and a covariant representation (i_X, i_A) of (X, A) on $A \times_X \mathbb{Z}$ with the property that for any covariant representation (φ_X, φ_A) of (X, A) on a pro-C^*-algebra B, there is a unique pro-C^*-morphism $\varphi_X \times \varphi_A : A \times_X \mathbb{Z} \to B$ such that $\varphi_X \times \varphi_A \circ i_X = \varphi_X$ and $\varphi_X \times \varphi_A \circ i_A = \varphi_A$ [11, Definition 3.3].

If (Φ, φ) is a morphism of Hilbert pro-C^*-bimodules from (X, A) to (Y, B), then $(i_Y \circ \Phi,$ $i_B \circ \varphi)$ is a covariant representation of (X, A) on $B \times_Y \mathbb{Z}$ and by the universal property of $A \times_X \mathbb{Z}$ there is a unique pro-C^* -morphism $\Phi \times \varphi$ from $A \times_X \mathbb{Z}$ to $B \times_Y \mathbb{Z}$ such that $(\Phi \times \varphi) \circ i_A = i_B \circ \varphi$ and $(\Phi \times \varphi) \circ i_X = i_Y \circ \Phi$.

2. Group actions on Hilbert pro-C^*-bimodules

An action of a locally compact group G on a pro-C^*-algebra A is a group morphism α from G to the group $\text{Aut}(A)$ of all automorphisms of A such that the map $g \mapsto \alpha_g(a)$ from G to A is continuous for all $a \in A$.

The action α is *an inverse limit action*, if $p_\lambda(\alpha_g(a)) = p_\lambda(a)$ for all $a \in A$, for all $g \in G$ and for all $\lambda \in \Lambda$. If α is an inverse limit action, then, for each $\lambda \in \Lambda$, there is an action α^λ of G on A_λ such that $\alpha_g^\lambda \circ \pi_\lambda^A = \pi_\lambda^A \circ \alpha_g$ for all $g \in G$, and $\alpha_g = \lim\limits_{\leftarrow \lambda} \alpha_g^\lambda$ for all $g \in G$. *The full crossed product* of A by α is a pro-C^*-algebra, denoted by $A \times_\alpha G$, which is isomorphic to $\lim\limits_{\leftarrow \lambda} A_\lambda \times_{\alpha^\lambda} G$ [6, Corollary 1.3.7], and *the reduced crossed product* of A by α is a pro-C^*-algebra, denoted by $A \times_{\alpha,r} G$, which is isomorphic to $\lim\limits_{\leftarrow \lambda} A_\lambda \times_{\alpha^\lambda,r} G$ [6, Corollary 1.4.4]. The canonical pro-C^*-morphism from (A, G, α) to the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$ is denoted by (ι_A, ι_G) [8, Proposition 3.1].

An action of a locally compact group G on a Hilbert pro-C^*-bimodule (X, A) is a pair (γ, α), where α is an action of G on A, γ is a group morphism from G to the group $\text{Aut}(X)$ of all automorphisms of (X, A) such that for each $g \in G, (\gamma_g, \alpha_g)$ is an isomorphism of Hilbert pro-C^*-bimodules from (X, A) to (X, A). If α is an inverse limit action, then for each $\lambda \in \Lambda$, there is an action $\left(\gamma^\lambda, \alpha^\lambda\right)$ of G on (X_λ, A_λ) such that $\gamma_g = \lim\limits_{\leftarrow \lambda} \gamma_g^\lambda$ for all $g \in G$.

If (γ, α) is an inverse limit action, then $C_c(G, X)$, the vector space of all continuous functions from G to X with compact support, has a structure of pre-Hilbert bimodule over $C_c(G, A)$ with the bimodule operations given by

$$(\xi f)(g) = \int_G \xi(s) \alpha_s\left(f\left(s^{-1}g\right)\right) ds$$

and

$$(f\xi)(g) = \int_G f(s) \gamma_s\left(s^{-1}g\right) dg$$

for all $\xi \in C_c(G, X)$, for all $f \in C_c(G, A)$, and the inner products given by

$${}_{C_c(G,A)}\langle \xi, \eta \rangle (g) = \int_G \Delta\left(g^{-1}s\right)_A \left\langle \xi(s), \gamma_g\left(\eta\left(g^{-1}s\right)\right)\right\rangle,$$

where Δ denotes the modular function on G, and

$$\langle \xi, \eta \rangle_{C_c(G,A)}(g) = \int_G \alpha_{s^{-1}}\left(\langle \xi(s), \eta(sg) \rangle_A\right) ds$$

for all $\xi, \eta \in C_c(G, X)$. *The full crossed product* is the completion $(X \times_\gamma G, A \times_\alpha G)$ of the pre-Hilbert bimodule $(C_c(G, X), C_c(G, A))$ [6, pp.64-69]. *The reduced crossed product* $(X \times_{\gamma,r} G, A \times_{\alpha,r} G)$ is defined in the same way. The canonical homomorphism from (X, A) to $(M(X \times_\gamma G), M(A \times_\alpha G))$, the multiplier module of $(X \times_\gamma G, A \times_\alpha G)$, is denoted by (ι_X, ι_A) [8, Proposition 4.1].

Let (γ, α) be an action of G on (X, A). Then, for each $g \in G$, there is a pro-C^*-morphism $\gamma_g \times \alpha_g : A \times_X \mathbb{Z} \to A \times_X \mathbb{Z}$ such that $(\gamma_g \times \alpha_g) \circ i_X = i_X \circ \gamma_g$ and $(\gamma_g \times \alpha_g) \circ i_A = i_A \circ \alpha_g$. It is easy to check that $g \mapsto (\gamma \times \alpha)_g = \gamma_g \times \alpha_g$ is an action of G on $A \times_X \mathbb{Z}$. Moreover, if (γ, α) is an inverse limit action, then $\gamma \times \alpha$ is an inverse limit action, and $(\gamma \times \alpha)_g = \lim\limits_{\leftarrow \lambda} \left(\gamma^\lambda \times \alpha^\lambda\right)_g$ for all $g \in G$.

Given an action (γ, α) of a locally compact group G on a Hilbert pro-C^*-bimodule (X, A), it is natural to ask if $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G$ is isomorphic to the crossed product of a pro-C^*-algebra by a Hilbert pro-C^*-bimodule. In [1, Proposition 4.5], Abadie proved that if G is amenable and (γ, α) is an action of G on the Hilbert C^* -bimodule (X, A), then the C^*-algebras $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G$ and $(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$ are isomorphic.

It is known that every Hilbert C^*-bimodule (X, A) can be regarded as a C^*-correspondence (X, A, φ), where the C^*-morphism $\varphi : A \to L(X)$ is given by $\varphi(a)x = ax$. Moreover, the crossed product of A by X is isomorphic to the C^* -algebra $\mathcal{O}_X$ associated to (X, A, φ) [12, Proposition 3.7]. If (X, A) is full, then the Katsura ideal J_X coincides with A [12, Lemma 3.3]. If (γ, α) is an action of a locally compact group G on a full Hilbert C^*-bimodule (X, A), then $(X \times_\gamma G, A \times_\alpha G)$ and $(X \times_{\gamma,r} G, A \times_{\alpha,r} G)$ are full Hilbert C^*-bimodules, and so $J_{X \times_\gamma G} = A \times_\alpha G$ and $J_{X \times_{\gamma,r} G} = A \times_{\alpha,r} G$. Using these facts and [3, Theorems 4.1 and 5.3], we obtain the following result.

Theorem 2.1. *Let (X, A) be a full Hilbert C^*-bimodule and let (γ, α) be an action of a locally compact group G on (X, A). Then the C^*-algebras $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G$ and $(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$ are isomorphic as well as the C^*-algebras $(A \times_{\alpha,r} G) \times_{X \times_{\gamma,r} G} \mathbb{Z}$ and $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$.*

Proof. Since $\left(i_{X \times_\gamma G}, i_{A \times_\alpha G}\right)$ is a non-degenerate covariant representation of $(X \times_\gamma G, A \times_\alpha G)$ on $(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$ [11, Proposition 3.4], it extends to a unique covariant representation $\left(\overline{i_{X \times_\gamma G}}, \overline{i_{A \times_\alpha G}}\right)$ of $(M(X \times_\gamma G), M(A \times_\alpha G))$ on $M((A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$ [4, Theorem 1.30]. It is easy to check that $\left(\overline{i_{X \times_\gamma G}} \circ \iota_X, \overline{i_{A \times_\alpha G}} \circ \iota_A\right)$ is a covariant representation of (X, A) on $M((A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z})$, and then there is a C^*-morphism $\left(\overline{i_{X \times_\gamma G}} \circ \iota_X\right) \times \left(\overline{i_{A \times_\alpha G}} \circ \iota_A\right)$ from $A \times_X \mathbb{Z}$ to $M\left((A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}\right)$.

Moreover, $\left(\left(\overline{i_{X \times_\gamma G}} \circ \iota_X\right) \times \left(\overline{i_{A \times_\alpha G}} \circ \iota_A\right), \overline{i_{A \times_\alpha G}} \circ \iota_G\right)$ is a covariant morphism from $(A \times_X \mathbb{Z}, G, \gamma \times \alpha)$ to $M\left((A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}\right)$, and so there is a C^*-morphism

$$\Psi = \left(\left(\overline{i_{X \times_\gamma G}} \circ \iota_X\right) \times \left(\overline{i_{A \times_\alpha G}} \circ \iota_A\right)\right) \times \left(\overline{i_{A \times_\alpha G}} \circ \iota_G\right)$$

from $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G$ to $M\left((A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}\right)$. According to [3, Theorem 4.1], Ψ is a C^*-isomorphism onto $(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$, since $J_{X \times_\gamma G} = A \times_\alpha G = J_X \times_\alpha G$.

Since $i_A\left(\alpha_g(a)\right) = (\gamma \times \alpha)_g\left(i_A(a)\right)$ for all $a \in A$ and for all $g \in G$, there is a C^*-morphism $i_A \times_\alpha \mathrm{id}$ from $A \times_{\alpha,r} G$ to $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$ such that $(i_A \times_\alpha \mathrm{id})(f)(g) = i_A(f(g))$ for all $g \in G$ and $f \in C_c(G, A)$. Consider the map $i_X \times_\gamma \mathrm{id}: C_c(G, X) \to C_c(G, A \times_X \mathbb{Z})$ given by $(i_X \times_\gamma \mathrm{id})(\xi)(g) = i_X(\xi(g))$. Taking into account that

$$i_X\left(\gamma_g(x)\right) = (\gamma \times \alpha)_g\left(i_X(x)\right)$$

for all $x \in X$ and for all $g \in G$, it is easy to check that

$${}_{C_c(G, A \times_X \mathbb{Z})}\left\langle (i_X \times_\gamma \mathrm{id})(\xi), (i_X \times_\gamma \mathrm{id})(\eta)\right\rangle = (i_A \times_\alpha \mathrm{id})\left({}_{C_c(G,A)}\langle \xi, \eta\rangle\right)$$

and

$$\langle (i_X \times_\gamma \mathrm{id})(\xi), (i_X \times_\gamma \mathrm{id})(\eta) \rangle_{C_c(G, A \times_X \mathbb{Z})} = (i_A \times_\alpha \mathrm{id}) \left(\langle \xi, \eta \rangle_{C_c(G,A)} \right)$$

for all $\xi, \eta \in C_c(G, X)$. From these facts we deduce that $i_X \times_\gamma$ id extends to a map $i_X \times_\gamma \mathrm{id}: X \times_{\gamma,r} G \to (A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$, and moreover, $(i_X \times_\gamma \mathrm{id}, i_A \times_\alpha \mathrm{id})$ is a covariant representation of $(X \times_{\gamma,r} G, A \times_{\alpha,r} G)$ on $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$. Therefore, there is a C^*-morphism $\Psi_r = (i_X \times_\gamma \mathrm{id}) \times (i_A \times_\alpha \mathrm{id})$ from $(A \times_{\alpha,r} G) \times_{X \times_{\gamma,r} G} \mathbb{Z}$ to $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$, and since $J_{X \times_{\gamma,r} G} = A \times_{\alpha,r} G = J_X \times_{\alpha,r} G$, Ψ_r is a C^* -isomorphism [3, Theorem 5.3]. □

The following theorem extends in the context of pro-C^*-algebras the above results.

Theorem 2.2. *Let (X, A) be a full Hilbert pro-C^*-bimodule and let (γ, α) be an inverse limit action of a locally compact group G on (X, A). Then the pro-C^*-algebras $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G$ and $(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z}$ are isomorphic as well as the pro-C^*-algebras $(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G$ and $(A \times_{\alpha,r} G) \times_{X \times_{\gamma,r} G} \mathbb{Z}$.*

Proof. By [11, Proposition 3.8] and [6, Corollaries 1.3.7 and 1.4.4]

$$(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha} G = \lim_{\leftarrow \lambda} (A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda \times \alpha_\lambda} G$$

and

$$(A \times_X \mathbb{Z}) \times_{\gamma \times \alpha, r} G = \lim_{\leftarrow \lambda} (A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda \times \alpha_\lambda, r} G$$

up to a pro-C^*-isomorphism as well as

$$(A \times_\alpha G) \times_{X \times_\gamma G} \mathbb{Z} = \lim_{\leftarrow \lambda} (A_\lambda \times_{\alpha^\lambda} G) \times_{X_\lambda \times_{\gamma_\lambda} G} \mathbb{Z}$$

and

$$(A \times_{\alpha,r} G) \times_{X \times_{\gamma,r} G} \mathbb{Z} = \lim_{\leftarrow \lambda} \left(A_\lambda \times_{\alpha^\lambda, r} G \right) \times_{X_\lambda \times_{\gamma_\lambda, r} G} \mathbb{Z}.$$

For each $\lambda \in \Lambda$, by Theorem 2.1 there are a C^*-isomorphism

$$\Psi_\lambda : (A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda \times \alpha_\lambda} G \to (A_\lambda \times_{\alpha_\lambda} G) \times_{X_\lambda \times_{\gamma_\lambda} G} \mathbb{Z}$$

and a C^*-isomorphism

$$\Psi_{r,\lambda} : (A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda \times \alpha_\lambda, r} G \to (A_\lambda \times_{\alpha_\lambda, r} G) \times_{X_\lambda \times_{\gamma_\lambda, r} G} \mathbb{Z}.$$

It is easy to check that $(\Psi_\lambda)_\lambda$ and $(\Psi_{r,\lambda})_\lambda$ are inverse systems of C^*-isomorphisms and the theorem is proved. □

It is known that $A \times_X \mathbb{Z}$ carries a natural action δ of the circle group $\mathbb{T}$, called the dual action, such that for each $z \in \mathbb{T}$, $\delta_z(i_A(a)) = i_A(a)$ for all $a \in A$ and $\delta_z(i_X(x)) = z\, i_X(x)$ for all $x \in X$. Moreover, $\delta = \delta_X \times \delta_A$, where (δ_X, δ_A) is the action of $\mathbb{T}$ on (X, A) given by $(\delta_X)_z(x) = zx$ and $(\delta_A)_z(a) = a$ fo all $z \in \mathbb{T}, x \in X, a \in A$. Clearly, (δ_X, δ_A) is an inverse limit action.

Corollary 2.1. *Let (X, A) be a full Hilbert pro-C^*-bimodule. Then $(A \times_X \mathbb{Z}) \times_{\delta_X \times \delta_A} \mathbb{T}$ and $(A \times_X \mathbb{Z}) \otimes C_0(\mathbb{Z})$ are isomorphic as pro-C^*-algebras.*

Proof. By Theorem 2.2, $(A \times_X \mathbb{Z}) \times_{\delta_X \times \delta_A} \mathbb{T}$ and $(A \times_{\delta_A} \mathbb{T}) \times_{X \times_{\delta_X} \mathbb{T}} \mathbb{Z}$ are isomorphic as pro-C^*-algebras. Since the Hilbert pro-C^* -bimodules $(X \times_{\delta_X} \mathbb{T}, A \times_{\delta_A} \mathbb{T})$ and $(X \otimes C_0(\mathbb{Z}), A \otimes C_0(\mathbb{Z}))$ are isomorphic, the pro-C^*-algebras $(A \times_{\delta_A} \mathbb{T}) \times_{X \times_{\delta_X} \mathbb{T}} \mathbb{Z}$ and $(A \otimes C_0(\mathbb{Z})) \times_{X \otimes C_0(\mathbb{Z})} \mathbb{Z}$ are isomorphic [7, Proposition 3.3]. On the other hand, $(A \otimes C_0(\mathbb{Z})) \times_{X \otimes C_0(\mathbb{Z})} \mathbb{Z}$ is isomorphic to $(A \times_X \mathbb{Z}) \otimes C_0(\mathbb{Z})$ [7, Theorem 3.4], and the corollary is proved. □

Let (γ, α) be an inverse limit action of a locally compact group G on the full Hilbert pro-C^*-bimodule (X, A). In the following corollary, we show that the maximal tensor product (the minimal tensor product) of the crossed product (the reduced crossed product) of $A \times_X \mathbb{Z}$ by the action induced by (γ, α) and a pro-C^*-algebra B is isomorphic to the crossed product of a pro-C^*-algebra by a Hilbert pro-C^*-module. For more details regarding tensor products of pro-C^*-algebras we refer the reader to [5].

Corollary 2.2. *Let (X, A) be a full Hilbert pro-C^*-bimodule, let (γ, α) be an inverse limit action of a locally compact group G on (X, A) and let B be a pro-C^*-algebra. Then the pro-C^* -algebras $((A \times_X \mathbb{Z}) \times_{\gamma\times\alpha} G) \otimes_{\max} B$ and $((A \times_\alpha G) \otimes_{\max} B) \times_{(X\times_\gamma G)\otimes_{\max} B} \mathbb{Z}$ are isomorphic as well as the pro-C^*-algebras $((A \times_X \mathbb{Z}) \times_{\gamma\times\alpha,r} G) \otimes_{\min} B$ and $((A \times_{\alpha,r} G) \otimes_{\min} B) \times_{(X\times_\gamma G)\otimes_{\min} B} \mathbb{Z}$.*

Proof. The result follows from [10, Theorems 7.2 and 7.3], Theorem 2.2 and [7, Theorems 3.4 and 3.6]. □

It is known that a pro-C^*-algebra A is nuclear if and only if the C^*-algebras $A_\lambda, \lambda \in \Lambda$ are nuclear (see, for example, [13, Section 3]).

Proposition 2.1. *Let (X, A) be a full Hilbert pro-C^*-bimodule, let (γ, α) be an inverse limit action of a locally compact group G on (X, A). If A is nuclear and G is amenable, then $(A \times_X \mathbb{Z}) \times_{\gamma\times\alpha} G$ is nuclear.*

Proof. Since for each $\lambda \in \Lambda$, the C^*-algebras $((A \times_X \mathbb{Z}) \times_{\gamma\times\alpha} G)_\lambda$ and $(A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda\times\alpha_\lambda} G$ are isomorphic, to show that $(A \times_X \mathbb{Z}) \times_{\gamma\times\alpha} G$ is nuclear, it is sufficient to show that for each $\lambda \in \Lambda$, the C^*-algebras $(A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda\times\alpha_\lambda} G$, $\lambda \in \Lambda$ are nuclear.

If A is nuclear, then, by [7, Proposition 4.3], $A \times_X \mathbb{Z}$ is nuclear. On the other hand, since for each $\lambda \in \Lambda$, the C^*-algebras $(A \times_X \mathbb{Z})_\lambda$ and $A_\lambda \times_{X_\lambda} \mathbb{Z}$ are isomorphic [11, Proposition 3.8], $A \times_X \mathbb{Z}$ is nuclear if and only if the C^*-algebras $A_\lambda \times_{X_\lambda} \mathbb{Z}, \lambda \in \Lambda$ are nuclear. Since G is amenable and $A_\lambda \times_{X_\lambda} \mathbb{Z}, \lambda \in \Lambda$ are nuclear, $(A_\lambda \times_{X_\lambda} \mathbb{Z}) \times_{\gamma_\lambda\times\alpha_\lambda} G, \lambda \in \Lambda$ are nuclear, and the proposition is proved. □

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