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Some additive results for the Wg-Drazin inverse of Banach space operators

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ABSTRACT. We investigate additive properties of Wg–Drazin inverse of two Wg-Drazin invertible linear operators in Banach spaces under the commutative case. These results generalize recent works on the Drazin inverse of complex matrices.

1. INTRODUCTION

Let X and Y be arbitrary Banach spaces. Denote by $\mathbf{B}(X,Y)$ the set of all linear bounded operators from X to Y. Set $\mathbf{B}(X) = \mathbf{B}(X,X)$. For an operator $A \in \mathbf{B}(X,Y)$, the symbols $N(A)$, $R(A)$, $\sigma(A)$, $\rho(A)$ and $\operatorname{acc}\sigma(A)$ will denote the null space, the range, the spectrum, the spectral radius of A and the set of all accumulation points of $\sigma(A)$, respectively.

An operator $A \in \mathbf{B}(X)$ is called the *generalized Drazin invertible* (or Koliha-Drazin inverse or quasipolar), if there exists some $B \in \mathbf{B}(X)$ satisfying

$$BAB = B, \qquad AB = BA, \qquad A - A^2B \text{ is quasinilpotent.}$$

The generalized Drazin inverse B of A is unique, if it exists, and denoted by A^d (see [6, Theorem 7.5.3], [9]). The Drazin inverse is a special case of the generalized Drazin inverse for which $A - A^2B$ is nilpotent. It is easy to see that if A is a quasinilpotent operator, then A^d exists and $A^d = 0$. The generalized Drazin inverse of A is in the double commutant of A, that is, for $C \in \mathbf{B}(X)$, $AC = CA$ implies $A^dC = CA^d$ [9, Theorem 4.4].

We recall the reader that, for $A \in \mathbf{B}(X)$, A^d exists if and only if $0 \notin \operatorname{acc}\sigma(A)$. If $A \in \mathbf{B}(X)$ is the generalized Drazin invertible, then the spectral idempotent A^π of A corresponding to $\{0\}$ is given by $A^\pi = I - AA^d$.

Let $\mathbf{B}_W(X,Y)$ be the space $\mathbf{B}(X,Y)$ equipped with the multiplication $A * B = AWB$ and the norm $\|A\|_W = \|A\|\|W\|$, for some fixed $W \in \mathbf{B}(Y,X)$. Then $\mathbf{B}_W(X,Y)$ becomes a Banach algebra [2]. $\mathbf{B}_W(X,Y)$ has the unit if and only if W is invertible, in which case W^{-1} is that unit.

Let $W \in \mathbf{B}(Y,X)$ be a fixed nonzero operator. An operator $A \in \mathbf{B}(X,Y)$ is called W*g*–Drazin invertible if A is quasipolar in the Banach algebra $\mathbf{B}_W(X,Y)$. The W*g*–Drazin inverse $A^{d,W}$ of A is defined as the *g*–Drazin inverse of A in the Banach algebra $\mathbf{B}_W(X,Y)$ [2].

Let us recall that if $A \in \mathbf{B}(X,Y)$ and $W \in \mathbf{B}(Y,X)$ then the following conditions are equivalent [2]:

(1) A is Wg-Drazin invertible,
(2) AW is quasipolar in $\mathbf{B}(Y)$ with $(AW)^d = A^{d,W}W$,
(3) WA is quasipolar in $\mathbf{B}(X)$ with $(WA)^d = WA^{d,W}$.

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Then, the Wg-Drazin inverse $A^{d,W}$ of A satisfies

$$A^{d,W} = ((AW)^d)^2 A = A((WA)^d)^2.$$

Lemma 1.1. [2] *Let $A \in \mathbf{B}(X,Y)$ and $W \in \mathbf{B}(Y,X)\backslash\{0\}$. Then A is Wg-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \tag{1.1}$$

where $A_i \in \mathbf{B}(X_i, Y_i)$, $W_i \in \mathbf{B}(Y_i, X_i)$, with A_1, W_1 invertible, and W_2A_2 and A_2W_2 quasinilpotent in $\mathbf{B}(X_2)$ and $\mathbf{B}(Y_2)$, respectively. The Wg-Drazin inverse of A is given by

$$A^{d,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.2}$$

with $(W_1A_1W_1)^{-1} \in \mathbf{B}(X_1, Y_1)$ and $0 \in \mathbf{B}(X_2, Y_2)$.

Further results to the weighted Drazin inverse, and the particular case of group inverses, can be found in [8, 10, 11, 12, 13, 15].

The following auxiliary results give some properties of quasinilpotent elements.

Lemma 1.2. [6] *Let $\mathbf{R}$ be an associative ring and let $q \in \mathbf{R}$. Then q is quasinilpotent if and only if $1 + xq$ is invertible for all $x \in \mathbf{R}$ satisfying $qx = xq$.*

Lemma 1.3. [5, Lemma 2.1] *If $P, Q \in \mathbf{B}(X)$ are quasinilpotent, and $PQ = 0$ or $PQ = QP$, then $P + Q$ is quasinilpotent and $(P + Q)^d = 0$.*

Lemma 1.4. [3, Corollary 2(3)] *Let $P, Q \in \mathbf{B}(X)$ be generalized Drazin invertible such that $PQ = QP$ and $I + P^dQ$ is generalized Drazin invertible. If Q is quasinilpotent, then $(P+Q)^d = (I + P^dQ)^{-1}P^d$.*

The next lemma deals with the generalized Drazin inverse of a product.

Lemma 1.5. [9, Theorem 5.5] *Let $\mathbf{A}$ be a complex unital Banach algebra, $a, b \in \mathbf{A}$ be generalized Drazin invertible and let $ab = ba$. Then ab is generalized Drazin invertible and $(ab)^d = b^da^d = a^db^d$.*

We state the form for the generalized Drazin inverse of an upper triangular operator matrix.

Lemma 1.6. [4, Theorem 5.1] *If $A \in \mathbf{B}(X)$ and $B \in \mathbf{B}(Y)$ are generalized Drazin invertible, $C \in \mathbf{B}(Y,X)$, then $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is generalized Drazin invertible and $M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}$, where $S = \sum\limits_{n=0}^{\infty} (A^d)^{n+2}CB^nB^\pi + A^\pi \sum\limits_{n=0}^{\infty} A^nC(B^d)^{n+2} - A^dCB^d$.*

Many authors have studied additive properties and the explicit expressions for the generalized Drazin inverse of matrices or operators on an infinite dimensional space [1, 5, 7]. Deng and Wei [3] have obtained several representations for the generalized Drazin inverse of the sum of two bounded linear operators on Banach space under the commutative condition, using the technique of block operator matrices.

The motivation for this paper was the paper of Wei and Deng [14]. In this paper, for two square matrices A and E such that $AE = EA$, explicit representations for the Drazin inverse $(A + E)^D$ in terms of A, E, A^D, E^D and $(I + A^DE)^D$ were considered, using the technique of Jordan canonical decomposition.

In the present paper we investigate additive results for the Wg-Drazin inverse of Banach space operators. Using the operator matrices, we extend some results from [14] to more general settings getting formulae with different expressions that in [3].

2. Additive results

First, we introduce an explicit representation of the Wg–Drazin inverse $(A+E)^{d,W}$ in terms of A, E, $A^{d,W}$ and $E^{d,W}$ under the condition $AWE = EWA$.

Theorem 2.1. *Let $W \in \mathbf{B}(Y,X)$, and let $A, E \in \mathbf{B}(X,Y)$ be Wg–Drazin invertible. If $AWE = EWA$, then the following statements are equivalent:*

(i) *$A+E$ is Wg–Drazin invertible,*

(ii) *$I + A^{d,W}WEW$ is generalized Drazin invertible,*

(iii) *$AWA^{d,W}W(I + A^{d,W}WEW)$ is generalized Drazin invertible.*

If one of conditions (i)-(iii) is satisfied, then

$$\begin{aligned}(A+E)^{d,W} &= \Big\{\big[(I + A^{d,W}WEW)^d A^{d,W}W\big]^2 \\ &+ \big[(I - AWA^{d,W}W)E^{d,W}WR^{-1}\big]^2\Big\}(A+E) \\ &= \Big\{\big[(I + A^{d,W}WEW)^d A^{d,W}W\big]^2 \\ &+ \left[(I - AWA^{d,W}W)\sum_{n=0}^{\infty}(-AW)^n(E^{d,W}W)^{n+1}\right]^2\Bigg\}(A+E),\end{aligned}$$

where $R = I + AW(I - AWA^{d,W}W)E^{d,W}W$.

Proof. Let A, W and $A^{d,W}$ have the matrix forms as in (1.1) and (1.2) with respect to topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. The equality $AWE = EWA$ gives $A^{d,W}WE = EWA^{d,W}$ and

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $A_1W_1E_1 = E_1W_1A_1$ and $A_2W_2E_2 = E_2W_2A_2$. Since E is Wg–Drazin invertible, we deduce that E_1 is W_1g–Drazin invertible, E_2 is W_2g–Drazin invertible and

$$E^{d,W} = \begin{bmatrix} E_1^{d,W_1} & 0 \\ 0 & E_2^{d,W_2} \end{bmatrix}.$$

From $A_2W_2E_2 = E_2W_2A_2$, we have $A_2W_2E_2^{d,W_2} = E_2^{d,W_2}W_2A_2$. Because A_2W_2 is quasi–nilpotent, by Lemma 1.2, $I + E_2^{d,W_2}W_2A_2W_2$ is invertible. Using Lemma 1.4, we get $(A_2W_2+E_2W_2)^d = (I+E_2^{d,W_2}W_2A_2W_2)^{-1}E_2^{d,W_2}W_2$. So, A_2+E_2 is W_2g–Drazin invertible.

(i) $\Leftrightarrow$ (ii): Notice that, $A+E = \begin{bmatrix} A_1+E_1 & 0 \\ 0 & A_2+E_2 \end{bmatrix}$ is Wg–Drazin invertible if and only if A_1+E_1 is W_1g–Drazin invertible, i.e. $A_1W_1+E_1W_1$ is generalized Drazin invertible. By

$$\begin{aligned} I + A^{d,W}WEW &= \begin{bmatrix} I + (A_1W_1)^{-1}E_1W_1 & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} (A_1W_1)^{-1}(A_1W_1 + E_1W_1) & 0 \\ 0 & I \end{bmatrix},\end{aligned}$$

we conclude that $I + A^{d,W}WEW$ is generalized Drazin invertible if and only if $A_1W_1 + E_1W_1$ is generalized Drazin invertible. Applying Lemma 1.5,

$$(I + A^{d,W}WEW)^d A^{d,W}W = \begin{bmatrix} (A_1W_1 + E_1W_1)^d & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the quasinilpotent element $AW(I - AWA^{d,W}W)$ commutes with $E^{d,W}W$, $R = I + AW(I - AWA^{d,W}W)E^{d,W}W$ is invertible and

$$\begin{aligned}(I - AWA^{d,W}W)E^{d,W}WR^{-1} &= \begin{bmatrix} 0 & 0 \\ 0 & E_2^{d,W_2}W_2 \end{bmatrix} \\ &\times \begin{bmatrix} I & 0 \\ 0 & (I + E_2^{d,W_2}W_2A_2W_2)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (A_2W_2 + E_2W_2)^d \end{bmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(A+E)^{d,W} &= \left[(AW + EW)^d\right]^2 (A + E) \\ &= \left[(I + A^{d,W}WEW)^d A^{d,W}W\right. \\ &+ \left.(I - AWA^{d,W}W)E^{d,W}WR^{-1}\right]^2 (A + E) \\ &= \left\{\left[(I + A^{d,W}WEW)^d A^{d,W}W\right]^2\right. \\ &+ \left.\left[(I - AWA^{d,W}W)E^{d,W}WR^{-1}\right]^2\right\}(A + E).\end{aligned}$$

Further, by

$$\sigma(AW(I - AWA^{d,W}W)E^{d,W}W) \subset \sigma(AW(I - AWA^{d,W}W))\sigma(E^{d,W}W) = \{0\},$$

we deduce that $\rho(AW(I-AWA^{d,W}W)E^{d,W}W) = 0 < 1$ and geometric series $\sum\limits_{n=0}^{\infty}(-AW(I-AWA^{d,W}W)E^{d,W}W)^n$ converges. Thus,

$$\begin{aligned}(A+E)^{d,W} &= \left\{\left[(I + A^{d,W}WEW)^d A^{d,W}W\right]^2\right. \\ &+ \left.\left[(I - AWA^{d,W}W)\sum_{n=0}^{\infty}(-AW)^n(E^{d,W}W)^{n+1}\right]^2\right\}(A + E).\end{aligned}$$

(i) $\Leftrightarrow$ (ii): Observe that

$$AWA^{d,W}W(I + A^{d,W}WEW) = \begin{bmatrix} (A_1W_1)^{-1}(A_1W_1 + E_1W_1) & 0 \\ 0 & 0 \end{bmatrix}$$

is generalized Drazin invertible if and only if $A_1W_1 + E_1W_1$ is generalized Drazin invertible if and only if $I + A^{d,W}WEW$ is generalized Drazin invertible. $\square$

Remark 2.1. Let $W \in \mathbf{B}(Y, X)$, and let $A, E \in \mathbf{B}(X, Y)$ be Wg–Drazin invertible. Suppose that $AWE = EWA$ and $I + A^{d,W}WEW$ (or $AWA^{d,W}W(I + A^{d,W}WEW)$) is generalized Drazin invertible. Applying Theorem 2.1, we obtain some special expressions for $(A + E)^{d,W}$.

(1) If $AWE = 0$, then $(A + E)^{d,W} = A^{d,W} + E^{d,W}$.

(2) If EW is quasinilpotent, then

$$\begin{aligned}(A+E)^{d,W} &= \left[(I + A^{d,W}WEW)^{-1}A^{d,W}W\right]^2 (A + E) \\ &= \left[\sum_{n=0}^{\infty}(A^{d,W}W)^{n+1}(-EW)^n\right]^2 (A + E).\end{aligned}$$

Using Theorem 2.1, we can obtain the following result for the generalized Drazin inverse $(A + E)^d$ which recovers [14, Theorem 2] for Drazin inverse of complex matrices.

Corollary 2.1. *Let $A, E \in \mathbf{B}(X)$ be generalized Drazin invertible and $AE = EA$. Then the following statements are equivalent:*

(i) *$A + E$ is generalized Drazin invertible,*
(ii) *$I + A^d E$ is generalized Drazin invertible,*
(iii) *$AA^d(I + A^d E)$ is generalized Drazin invertible.*

If one of conditions (i)-(iii) is satisfied, then

$$\begin{aligned}(A+E)^d &= (I + A^d E)^d A^d + A^\pi E^d (I + AA^\pi E^d)^{-1} \\ &= (I + A^d E)^d A^d + A^\pi \sum_{n=0}^{\infty} (-A)^n (E^d)^{n+1}\end{aligned}$$

and

$$(A+E)^d(A+E) = (I + A^d E)^d (A+E) A^d + A^\pi E E^d.$$

Observe that, the representation of $(A+E)^d$ in the previous corollary is quite different from the expressions for the generalized Drazin inverse in [3, 5].

We prove the following result.

Theorem 2.2. *Let $W \in \mathbf{B}(Y, X)$, $A \in \mathbf{B}(X, Y)$ be Wg–Drazin invertible, and let $E \in \mathbf{B}(X, Y)$ such that EW is quasinilpotent. If $EWA^{d,W} = 0$ and $(I - AWA^{d,W}W)AWE = (I - AWA^{d,W}W)EWA$, then $A + E$ is Wg–Drazin invertible and*

$$(A+E)^{d,W} = \left[A^{d,W}W + \sum_{n=0}^{\infty} (A^{d,W}W)^{n+2} EWT(n) \right]^2 (A+E),$$

where $T(n) = (I - AWA^{d,W}W) \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) (AW)^i (EW)^{n-i}$.

Proof. If A, W and $A^{d,W}$ have the matrix forms as in (1.1) and (1.2), by $EWA^{d,W} = 0$, we obtain that E has the matrix form

$$E = \begin{bmatrix} 0 & E_1 \\ 0 & E_2 \end{bmatrix}.$$

By $\sigma(E_2W_2) \subset \sigma(0) \cup \sigma(EW) = \{0\}$, E_2W_2 is quasinilpotent. Since $(I - AWA^{d,W}W)AWE = (I - AWA^{d,W}W)EWA$, we get that $A_2W_2E_2 = E_2W_2A_2$ and, by Lemma 1.3, $A_2W_2 + E_2W_2$ is quasinilpotent and $(A_2W_2 + E_2W_2)^d = 0$.

Using Lemma 1.6, $AW + EW$ is generalized Drazin invertible and

$$(AW + EW)^d = \begin{bmatrix} A_1W_1 & E_1W_2 \\ 0 & A_2W_2 + E_2W_2 \end{bmatrix}^d = \begin{bmatrix} (A_1W_1)^{-1} & S \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} S &= \sum_{n=0}^{\infty} (A_1W_1)^{-(n+2)} E_1W_2 (A_2W_2 + E_2W_2)^n \\ &= \sum_{n=0}^{\infty} (A_1W_1)^{-(n+2)} E_1W_2 \left(\sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) (A_2W_2)^i (E_2W_2)^{n-i} \right). \end{aligned}$$

Now, from

$$\sum_{n=0}^{\infty} (A^{d,W}W)^{n+2} EWT(n) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

we deduce that

$$\begin{aligned}(A+E)^{d,W} &= \left[(AW+EW)^d\right]^2(A+E)\\ &= \left[A^{d,W}W+\sum_{n=0}^{\infty}(A^{d,W}W)^{n+2}EWT(n)\right]^2(A+E).\end{aligned}$$

□

If we suppose that $W = I$ in Theorem 2.2, we get the next corollary which covers the result for Drazin inverse of complex matices [14, Theorem 4].

Corollary 2.2. *Let $A \in \mathbf{B}(X)$ be generalized Drazin invertible and $E \in \mathbf{B}(X)$ be quasinilpotent. If $EA^d = 0$ and $A^\pi AE = A^\pi EA$, then $A+E$ is generalized Drazin invertible and*

$$(A+E)^d = A^d + \sum_{n=0}^{\infty}(A^d)^{n+2}ET(n),$$

where $T(n) = A^\pi \sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix} A^i E^{n-i}$.

Also, we obtain the next theorem which includes the other representation of $(A+E)^{d,W}$.

Theorem 2.3. *Let $W \in \mathbf{B}(Y,X)$, and let $A, E \in \mathbf{B}(X,Y)$ be Wg–Drazin invertible. Assume that $Q \in \mathbf{B}(X,Y)$ satisfies QW is idempotent, $AWQ = QWA$, $EWQ = 0$ and $(I - QW)AWE = (I - QW)EWA$. If one of the following equivalent conditions holds:*

(i) *$I + A^{d,W}WEW$ is generalized Drazin invertible,*
(ii) *$(I - A^{d,W}WAW)(I + A^{d,W}WEW)$ is generalized Drazin invertible,*

then $A+E$ is Wg–Drazin invertible and

$$\begin{aligned}(A+E)^{d,W} &= \{QWA^{d,W}W + (I-QW)R - QWA^{d,W}WEWR\\ &+ QW(I - A^{d,W}WAW)\sum_{n=0}^{\infty}(AW)^nEWR^{n+2}\\ &+ QW\sum_{n=0}^{\infty}(A^{d,W}W)^{n+2}EW(AW+EW)^n(I-QW)\\ &\times [I-(AW+EW)R]\}^2(A+E), \qquad (2.3)\end{aligned}$$

where

$$R = (I + A^{d,W}WEW)^dA^{d,W}W + (I - A^{d,W}WAW)\sum_{n=0}^{\infty}(E^{d,W}W)^{n+1}(-AW)^n.$$

Proof. Since $(QW)^2 = QW$, we obtain $X = R(QW) \oplus N(QW)$ and $QW = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ with respect to this decomposition. By $AWQW = QWAW$ and $EWQW = 0$, we obtain $AW = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ and $EW = \begin{bmatrix} 0 & F_1 \\ 0 & F_2 \end{bmatrix}$, where B_1, B_2 and F_2 are generalized Drazin invertible.

The equality $(I - QW)AWEW = (I - QW)EWAW$ implies $B_2F_2 = F_2B_2$. Hence, $I + A^{d,W}WEW = \begin{bmatrix} I & B_1^dF_1 \\ 0 & I + B_2^dF_2 \end{bmatrix}$ is generalized Drazin invertible if and only if $I + B_2^dF_2$ is generalized Drazin invertible if and only if $(I - A^{d,W}WAW)(I + A^{d,W}WEW)$ is generalized Drazin invertible. By Theorem 2.1, we conclude B_2+F_2 is generalized Drazin invertible and

$$(B_2+F_2)^d = (I+B_2^dF_2)^dB_2^d + B_2^\pi\sum_{n=0}^{\infty}(-B_2)^n(F_2^d)^{n+1}.$$

Observe that, by Lemma 1.6, $AW+EW$ is generalized Drazin invertible and

$$(AW+EW)^d = \begin{bmatrix} B_1^d & S \\ 0 & (I+B_2^dF_2)^dB_2^d + B_2^\pi\sum\limits_{n=0}^{\infty}(-B_2)^n(F_2^d)^{n+1} \end{bmatrix},$$

where

$$\begin{aligned} S &= \sum_{n=0}^{\infty}(B_1^d)^{n+2}F_1(B_2+F_2)^n(B_2+F_2)^\pi + B_1^\pi\sum_{n=0}^{\infty}B_1^nF_1[(B_2+F_2)^d]^{n+2} \\ &- B_1^dF_1(B_2+F_2)^d. \end{aligned}$$

Using the following equalities

$$QWA^{d,W}W = \begin{bmatrix} B_1^d & 0 \\ 0 & 0 \end{bmatrix}, \qquad (I-QW)R = \begin{bmatrix} 0 & 0 \\ 0 & (B_2+F_2)^d \end{bmatrix},$$

$$QWA^{d,W}WEWR = \begin{bmatrix} 0 & B_1^dF_1(B_2+F_2)^d \\ 0 & 0 \end{bmatrix},$$

$$QW(I-A^{d,W}WAW)\sum_{n=0}^{\infty}(AW)^nEWR^{n+2} = \begin{bmatrix} 0 & B_1^\pi\sum\limits_{n=0}^{\infty}B_1^nF_1[(B_2+F_2)^d]^{n+2} \\ 0 & 0 \end{bmatrix},$$

$$QW\sum_{n=0}^{\infty}(A^{d,W}W)^{n+2}EW(AW+EW)^n(I-QW)[I-(AW+EW)R] =$$

$$= \begin{bmatrix} 0 & \sum\limits_{n=0}^{\infty}(B_1^d)^{n+2}F_1(B_2+F_2)^n(B_2+F_2)^\pi \\ 0 & 0 \end{bmatrix},$$

we get the formula (2.3). □

Remark 2.2. If $Q = A^{d,W}WA$ in Theorem 2.3, we can obtain the next formula.

Let $W \in \mathbf{B}(Y,X)$, and let $A,E \in \mathbf{B}(X,Y)$ be Wg–Drazin invertible, and let $I + A^{d,W}WEW$ be generalized Drazin invertible. If EW is quasinilpotent, $EWA^{d,W}WA = 0$ and $(I-A^{d,W}WAW)AWE = (I-A^{d,W}WAW)EWA$, then $A+E$ is Wg–Drazin invertible and

$$(A+E)^{d,W} = \left[A^{d,W}W + \sum_{n=0}^{\infty}(A^{d,W}W)^{n+2}EW(AW+EW)^n\right]^2(A+E).$$

If there exists an idempotent $P^2 = P \in \mathbf{B}(X)$ such that $AEP = EAP$ (or $PAE = PEA$), then A and E partially commutative. As a consequence of Theorem 2.3, a representation of $(A+E)^d$ under partially commutative condition is presented in the following corollary.

Corollary 2.3. *Let $A,E \in \mathbf{B}(X)$ be generalized Drazin invertible. Assume that $Q \in \mathbf{B}(X)$ is idempotent such that $AQ = QA$, $EQ = 0$ and $(I-Q)AE = (I-Q)EA$. If one of the following equivalent conditions holds:*

(i) *$I + A^dE$ is generalized Drazin invertible,*

(ii) *$A^\pi(I + A^dE)$ is generalized Drazin invertible,* *then $A+E$ is generalized Drazin invertible and*

$$\begin{aligned}(A+E)^d &= QA^d+(I-Q)R-QA^dER+QA^\pi\sum_{n=0}^{\infty}A^nER^{n+2}\\ &+ Q\sum_{n=0}^{\infty}(A^d)^{n+2}E(A+E)^n(I-Q)[I-(A+E)R],\end{aligned}$$

where $R=(I+A^dE)^dA^d+A^\pi\sum\limits_{n=0}^{\infty}(E^d)^{n+1}(-A)^n$.

Observe that [14, Theorem 5] is a particular case of Corollary 2.3.

Applying [3, Theorem 1] instead of Theorem 2.1 in the proof of Theorem 2.3, it is interesting to note that we can obtain the following result.

Theorem 2.4. *Let $A, E\in\mathbf{B}(X)$ be generalized Drazin invertible. Assume that $Q\in\mathbf{B}(X)$ is idempotent such that $AQ=QA$, $EQ=0$ and $(I-Q)AE=(I-Q)EA$. If one of the following equivalent conditions holds:*

(i) *$I+A^dE$ is generalized Drazin invertible,*
(ii) *$A^\pi(I+A^dE)$ is generalized Drazin invertible,*

then $A+E$ is generalized Drazin invertible and

$$\begin{aligned}(A+E)^d &= QA^d+(I-Q)R_1-QA^dER_1+QA^\pi\sum_{n=0}^{\infty}A^nER_1^{n+2}\\ &+ Q\sum_{n=0}^{\infty}(A^d)^{n+2}E(A+E)^n(I-Q)[I-(A+E)R_1],\end{aligned}$$

where $R_1=A^d(I+A^dE)^dEE^d+E^\pi\sum\limits_{n=0}^{\infty}(-E)^n(A^d)^{n+1}+\sum\limits_{n=0}^{\infty}(E^d)^{n+1}(-A)^nA^\pi$.

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