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A path convergence theorem and construction of fixed points for nonexpansive mappings in certain Banach spaces

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ABSTRACT. In this paper, we introduce a new iterative process to approximate fixed points of nonexpansive maps in real Banach spaces having weakly continuous duality map and establish strong convergence theorems for the proposed iterative process. There is no compactness assumption on K or on T. Our results improve important recent results.

1. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0, +\infty) \to [0, \infty)$ be a stricly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge φ is a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, x \in E. \tag{1.1}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), x \neq 0.$$

Recall that a Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exist for each $x, y \in S_E$ (Here $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$. It is known that E is smooth if only if each duality map J_φ is single-valued, that E is Fréchet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E.

Following Browder [2], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is a single-valued and is weak-to-weak* sequentially continous, i.e., if $\{x_n\} \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. (see e.g., [8] fore more detais on duality maps). Finally recall that a Banach space E satisfies *Opial's*

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property (see, e.g., [18]) if $\limsup_{n\to+\infty}\|x_n - x\| < \limsup_{n\to+\infty}\|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property.

Let E be a real normed linear space and K be a nonempty subset of E. A map $T : K \to E$ is said to be *Lipschitz* if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in K; \tag{1.3}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.

For $T : K \to K$ nonexpansive with a fixed point, where K is a closed convex nonempty subset of a real Banach space E, unlike in the case of Banach contraction mapping principle, trivial examples show that the sequence $\{x_n\}$ generated by the *Picard iterates*, $x_{n+1} := Tx_n, n \geq 0$, may fail to converge to such a fixed point even when such a fixed point is unique. More precisely, let B be the closed unit ball of $\mathbb{R}^2$ and let T be the anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then, T is nonexpansive with the origin as the only fixed point. Moreover, the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n, n \geq 0$ with $x_0 = (0, 1) \in B$, does not converge to (0,0) (see, e.g., Chidume[5]).

Krasnoselskii [14], however, showed that in this example, one can obtain convergent sequence of successive approximations if $\frac{1}{2}(I + T)$ is used instead of T where I denotes the identity map on $\mathbb{R}^2$, that is, if the sequence of successive approximations is defined by $x_0 \in K$,

$$x_{n+1} = \frac{1}{2}\Big(x_n + Tx_n\Big), \quad n \geq 0, \tag{1.4}$$

instead of the usual Picard iterates, $x_{n+1} = Tx_n, x_0 \in K, n \geq 0$. Clearly, the fixed point sets of T and $\frac{1}{2}(I + T)$ are the same so that the limit of a convergent sequence defined by (1.4) is necessarily a fixed point T.

A generalization of equation (1.4) which has proved successful in the approximation of fixed points of nonexpansive maps $T : K \to K$ (when they exist), K is a closed convex subset of a normed linear space, is the following scheme: $x_0 \in K$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0;\ \lambda \in (0, 1), \tag{1.5}$$

(see, e.g., Schaefer [22]). However, the most general iterative scheme now studied for approximating fixed point of *nonexpansive* mappings is the following: $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.6}$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying appropriate conditions (see, e.g., Chidume [6], Eldestein and O'Brain [9], Ishikawa [13]). Under the conditions that; $\lim \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, the sequence $\{x_n\}$ generated by (1.6) is generally referred to as $Mann$ sequence in the light of Mann [17].

This Mann's method is remarkably useful for finding fixed points of a nonexpansive mapping and provides a unified framework for some kinds of algorithms from various different fields. In this respect, the following result is basic and important.

Theorem 1.1. *Let X be an Opial space and $T : K \to K$ be a nonexpansive self-mapping of a nonempty weakly compact convex subset K of X. For any $x_0 \in K$, let $\{x_n\} \subset K$ be the sequence given by* (1.6) *where $\{\alpha_n\} \subset (0, 1)$ is a non-increasing real sequence satisfying: $0 < a \leq \alpha_n < 1$ for all $n \geq 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

However, as in Theorem 1.1, Mann's method for nonexpansive mappings has only weak convergence. Thus a natural question rises: could we obtain a strong convergence theorem by using the well-known Mann's method for non-expansive mappings? In this connection, in 1975, Genel and Lindenstrass[10] gave a counter-example. Hence the modification is necessary in order to guarantee the strong convergence of Mann's method.

Some attempts to construct iteration algorithm so that strong convergence is guaranteed have been made.

Let E be a real Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a nonexpansive mapping. For fixed $t \in (0,1)$ and arbitrary $u \in K$, let $z_t \in K$ denote the unique fixed point of T_t defined by $T_t x := tu + (1-t)Tx, x \in K$. Assume that $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Browder [3] proved that if $E = H$, a Hilbert space, then $\lim_{t\to 0} z_t$ exists and is a fixed point of T. Reich [20] extended this result to uniformly smooth Banach spaces. Kirk [15] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. (see also, [7]).

For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_n u + (1-\alpha_n)Tx_n, n \geq 0. \tag{1.7}$$

Concerning this process, Reich [20] posed the following question.

Question. *Let E be a real Banach space. Is there a sequence $\{\alpha_n\}$ such that whenever a weakly compact convex subset K of E has the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (1.7) converges to a fixed point of T for arbitrary fixed $u \in K$ and all nonexpansive $T : K \to K$?*

Halpern [12] was the first to study the convergence of the algorithm (1.7) in the framework of Hilbert spaces. He proved the following theorem.

Theorem 1.2 (Halpern, [12]). *Let K be a nonempty bounded closed convex subset of a Hilbert space H and $T : K \to K$ be a nonexpansive mapping. Let $u \in K$ be arbitrary. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $\theta \in (0,1)$. Define a sequence $\{x_n\}$ in K by $x_1 \in K$, $x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n$, $n \geq 1$. Then, $\{x_n\}$ converges strongly to the element of $F(T) := \{x \in K : Tx = x\}$ nearest to u.*

An iteration method with recursion formula of the form (1.7) is referred to as a *Halpern-type iteration method*.

Lions [16] improved Theorem 1.2, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; and (iii) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0$.

Reich [20] gave an affirmative answer to the above question in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$. It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [25] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

$$(i) \lim_{n\to\infty} \alpha_n = 0;\ (ii) \sum_{n=1}^{\infty} \alpha_n = \infty;\ \text{and ;}\ (iii) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{1.8}$$

Reich [19] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps (e.g., $l_p (1 < p < \infty)$), where the sequence $\{\alpha_n\}$ is required to satisfy conditions (i) and (ii) of (1.8) and to be decreasing (and hence also satisfies (iii) of (1.8)). Shioji and Takahashi [23] extended

Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings (e.g., L_p spaces $(1 < p < \infty)$). They proved the followng theorem.

Theorem ST ([23]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let K be a nonempty closed convex subset of E. Let T be a nonexpansive mapping from K into K such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a sequence which satisfies the following conditions: $0 \leq \alpha_n \leq 1,\ \lim \alpha_n = 0,\ \sum \alpha_n = \infty,\ \sum |\alpha_{n+1} - \alpha_n| < \infty$. Let $u \in K$ and let $\{x_n\}$ be the sequence defined by $x_0 \in K,\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0$. Assume that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for $0 < t < 1, z_t$ is the unique element of K which satisfies $z_t = tu + (1 - t)Tz_t$. Then, $\{x_n\}$ converges strongly to z.*

A result of Reich [21] and that of Takahashi and Ueda [24] show that if K satisfies some additional assumption, then $\{z_t\}$ defined above converges strongly to a fixed point of T. In particular, the following is true.

Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let K be a weakly compact convex subset of E. Let T be a nonexpansive mapping from K into K. Let $u \in K$ and let z_t be the unique element of K which satisfies $z_t = tu + (1 - t)Tz_t$ for $0 < t < 1$. Assume that each nonempty $T-$invariant closed convex subset of K contains a fixed point of T. Then, $\{z_t\}$ converges strongly to a fixed point of T.

It is our purpose in this paper to construct a new iterative algorithm and prove strong convergence theorems for approximating fixed points of nonexpansive mappings in reflexive real Banach spaces having weakly continuous duality maps. No compactness assumption is made. Our theorems are important improvement of important recent results.

2. Preliminaries

We start with the following *demiclosedness principle* for nonexpansive mappings.

Lemma 2.1 (demiclosedness principle, Browder [11]). *Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed; that is,*

$$\{x_n\} \subset K,\ x_n \rightharpoonup x \in K \ \textit{and} \ (I - T)x_n \to y \ \textit{implies that} \ (I - T)x = y.$$

Lemma 2.2 (Xu, [26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

$$(a) \sum_{n=0}^{\infty} \alpha_n = \infty,\ (b) \limsup_{n\to\infty} \sigma_n \leq 0 \text{ or } \sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty. \text{ Then } \lim_{n\to\infty} a_n = 0.$$

3. Main results

Lemma 3.3. *Let E be a real Banach space and K a nonempty,closed convex cone of E. Let $T : K \to K$ be a nonexpansive mapping, and λ be a constant in $(0,1)$. Then, for each $t \in (0,1)$, there exists a unique $x_t \in K$ such that*

$$x_t = t(\lambda x_t) + (1 - t)Tx_t.$$

Proof. For each $t \in (0,1)$, define the mapping $T_t : K \to K$ by:

$$T_t x = t(\lambda x) + (1 - t)Tx,\ \forall x \in K.$$

We show that T_t is a contraction. For this, let $x, y \in K$. We have

$$\|T_t x - T_t y\| = \|[t(\lambda x_t) + (1-t)Tx] - [t(\lambda y) + (1-t)Ty]\| \le [1-(1-\lambda)t]\|x-y\|.$$

Therefore, T_t is a contraction. Using Banach's contraction principle, there exists a unique fixed point x_t of T_t in K, i.e,

$$x_t = t(\lambda x_t) + (1-t)Tx_t. \tag{3.9}$$

□

We now prove the following theorem.

Theorem 3.3. *Let E be a reflexive real Banach space having a weakly continuous duality map and K a nonempty, closed convex cone of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then as $t \to 0$, the net $\{x_t\}$ defined by* (3.9) *converges strongly to a fixed point of T.*

Proof.

Step 1. We prove that $\{x_t\}$ is bounded. Let $u \in F(T)$. From (3.9), we have

$$\begin{aligned} \|x_t - u\| &= \|t(\lambda x_t) + (1-t)Tx_t - u\| \\ &\le t\lambda\|x_t - u\| + (1-t)\|Tx_t - u\| + (1-\lambda)t\|u\| \\ &\le [1-(1-\lambda)t]\|x_t - u\| + (1-\lambda)t\|u\|, \end{aligned}$$

which implies that

$$\|x_t - u\| \le \|u\|.$$

Hence, $\{x_t\}$ is bounded.

Step 2. We show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Using (3.9), we have

$$\|x_t - Tx_t\| = t\|\lambda x_t - Tx_t\| \to 0, \text{ as } t \to 0. \tag{3.10}$$

Now, let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to +\infty$. Set $x_n := x_{t_n}$. We show that $\{x_n\}$ has a convergence subsequence. To this end, from (3.10), we have

$$\|x_n - Tx_n\| \to 0. \tag{3.11}$$

Let φ be a gauge such that the corresponding duality map J_φ is single valued and weak-to-weak* seqentially continuous from E to E^*. Let $u \in F(T)$. From (1.1) and (3.9), we have

$$\begin{aligned} \|x_t - u\|\varphi(\|x_t - u\|) &= \langle t(\lambda x_t) + (1-t)Tx_t - u, J_\varphi(x_t - u)\rangle \\ &= t\lambda\langle x_t - u, J_\varphi(x_t - u)\rangle + (1-t)\langle Tx_t - u, J_\varphi(x_t - u)\rangle \\ &\quad -(1-\lambda)t\langle u, J_\varphi(x_t - u)\rangle \\ &\le [1-(1-\lambda)t]\|x_t - u\|\varphi(\|x_t - u\|) - (1-\lambda)t\langle u, J_\varphi(x_t - u)\rangle. \end{aligned}$$

So,

$$\|x_t - u\|\varphi(\|x_t - u\|) \le \langle u, J_\varphi(u - x_t)\rangle.$$

In particular,

$$\|x_n - u\|\varphi(\|x_n - u\|) \le \langle u, J_\varphi(u - x_n)\rangle \ \forall u \in F(T), \tag{3.12}$$

which implies that

$$\|x_n - u\| \le \|u\|.$$

Therefore, $\{x_n\}$ is bounded.

Since E is reflexive and K is closed and convex, there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ that converges weakly to $x^* \in K$. Using Lemma 2.1, it follows that $x^* \in F(T)$. Replacing u by x^* in (3.12), we have:

$$\|x_{n_k} - x^*\|\varphi(\|x_{n_k} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x_{n_k})\rangle \ \forall k \geq 1. \tag{3.13}$$

Using (3.13), the fact that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$ and J_φ is weakly continuous, it follows that

$$\|x_{n_k} - x^*\|\varphi(\|x_{n_k} - x^*\|) \to 0 \ \text{ as } \ k \to \infty. \tag{3.14}$$

Using (3.14), the fact that $\{x_n\}$ is bounded, and φ is continuous and satisfies $\varphi(t) = 0$ if and only if $t = 0$, we deduce that $\|x_{n_k} - x^*\| \to 0$ as $k \to \infty$. Hence $x_{n_k} \to x^*$. This proves the relatively compactness of the net $\{x_t\}$.

Step 3. We show that the entire net $\{x_t\}$ converge to a fixed point of T. We claim that the net $\{x_t\}$ has a unique cluster point. From step2, the net $\{x_t\}$ has a cluster point. Now suppose that $x^* \in E$ and $x^{**} \in E$ are two cluster points of $\{x_t\}$. Let $\{t_n\} \subset (0,1)$ such that $t_n \to 0$ and $x_{t_n} \to x^*$, as $n \to \infty$ and $\{s_n\} \subset (0,1)$ such that $s_n \to 0$ and $x_{s_n} \to x^{**}$, as $n \to \infty$. Set $x_n = x_{t_n}$ and $z_n = x_{s_n}$.

Following the same arguments as in step2, it follows that $x^*, x^{**} \in F(T)$, and the following estimates hold:

$$\|x_n - x^{**}\|\varphi(\|x_n - x^{**}\|) \leq \langle x^{**}, J_\varphi(x^{**} - x_n)\rangle, \tag{3.15}$$

and

$$\|z_n - x^*\|\varphi(\|z_n - x^*\|) \leq \langle x^*, J_\varphi(x^* - z_n)\rangle. \tag{3.16}$$

Letting $n \to \infty$ in (3.15) and (3.16) gives

$$\|x^* - x^{**}\|\varphi(\|x^* - x^{**}\|) \leq \langle x^{**}, J_\varphi(x^{**} - x^*)\rangle. \tag{3.17}$$

and

$$\|x^{**} - x^*\|\varphi(\|x^{**} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x^{**})\rangle. \tag{3.18}$$

Adding up (3.17) and (3.18) yields

$$2\|x^* - x^{**}\|\varphi(\|x^* - x^{**}\|) \leq \|x^* - x^{**}\|\varphi(\|x^* - x^{**}\|),$$

which implies that $x^* = x^{**}$.This complete the proof. □

We now apply Theorem 3.3 to approximate fixed points of nonexpansive mappings.

Theorem 3.4. *Let E be a uniformly smooth real Banach space having a weakly continuous duality map and K a nonempty, closed and convex cone of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let λ_n and α_n be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)Tx_n \ n \geq 0. \tag{3.19}$$

Suppose the following conditions hold :

(a) $\lim_{n\to\infty} \alpha_n = 0$; (b) $\lim_{n\to\infty} \lambda_n = 1$;
(c) $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.
Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $u \in F(T)$. From (3.19), we have

$$\begin{aligned}\|x_{n+1}-u\| &= \|\alpha_n(\lambda_n x_n)+(1-\alpha_n)Tx_n-u\| \\ &\leq \alpha_n\lambda_n\|x_n-u\|+(1-\lambda_n)\alpha_n\|u\|+(1-\alpha_n)\|Tx_n-u\| \\ &= [1-(1-\lambda_n)\alpha_n]\|x_n-u\|+(1-\lambda_n)\alpha_n\|u\| \leq \max\{\|x_n-u\|,\|u\|\}.\end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

From (3.19), it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n(\lambda_n x_n)+(1-\alpha_n)Tx_n-\alpha_{n-1}(\lambda_{n-1}x_{n-1})+(1-\alpha_{n-1})Tx_{n-1}\| \\ &= \|\alpha_n\lambda_n(x_n-x_{n-1})+\alpha_n(\lambda_n-\lambda_{n-1})x_{n-1}+(\alpha_n-\alpha_{n-1})(\lambda_{n-1}x_{n-1}) \\ &\quad +(1-\alpha_n)(Tx_n-Tx_{n-1})+(\alpha_{n-1}-\alpha_n)Tx_{n-1}\| \\ &\leq \alpha_n\lambda_n\|x_n-x_{n-1}\|+(1-\alpha_n)\|Tx_n-Tx_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\| \\ &\quad +\|Tx_{n-1}\|)+\ \alpha_n|\lambda_n-\lambda_{n-1}|\|x_{n-1}\| \\ &\leq [1-(1-\lambda_n)\alpha_n]\|x_n-x_{n-1}\|+(|\alpha_n-\alpha_{n-1}|+\alpha_n|\lambda_n-\lambda_{n-1}|)M_1;\end{aligned}$$

Hence,

$$\|x_{n+1}-x_n\| \leq [1-(1-\lambda_n)]\|x_n-x_{n-1}\|+(|\alpha_n-\alpha_{n-1}|+\alpha_n|\lambda_n-\lambda_{n-1}|)M_1, \tag{3.20}$$

where $M_1>0$ is such that $\sup_n\{\|x_{n-1}\|+\|Tx_{n-1}\|\} \leq M_1$. Hence, from (3.24) and Lemma 2.2, we deduce

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|=0.$$

At the same time, we note that

$$\|x_{n+1}-Tx_n\|=\alpha_n\|(\lambda_n x_n)-Tx_n\| \to 0.$$

Therefore, we have

$$\lim_{n\to+\infty}\|x_n-Tx_n\|=0. \tag{3.21}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J(x^*-x_n)\rangle \leq 0$, where $x^*=\lim_{t\to 0}x_t$ and $\{x_t\}$ is the net defined by (3.9). From (1.1), (3.9), the fact that T is nonexpansive and the $\{x_t\}$ and $\{x_n\}$ are bounded, we have the following estimates

$$\begin{aligned}\|x_t-x_n\|^2 &= \langle x_t-x_n, J(x_t-x_n)\rangle = t\langle x_t-x_n, J(x_t-x_n)\rangle-(1-\lambda)t\langle x_t, J(x_t-x_n)\rangle \\ &+ (1-t)\langle Tx_t-Tx_n, J(x_t-x_n)\rangle+(1-t)\langle Tx_n-x_n, J(x_t-x_n)\rangle \\ &\leq \|x_t-x_n\|^2-(1-\lambda)t\langle x_t, x_t-x_n\rangle+(1-t)\langle Tx_n-x_n, x_t-x_n\rangle \\ &\leq \|x_t-x_n\|^2-(1-\lambda)t\langle x_t, J(x_t-x_n)\rangle+M_2\|Tx_n-x_n\|,\end{aligned}$$

where $M_2>0$ such that $\sup\{\|x_t-x_n\|, t\in(0,1), n\geq 0\} \leq M_2$. Therefore, we have

$$\langle x_t, J(x_t-x_n)\rangle \leq \frac{M_2}{(1-\lambda)t}\|Tx_n-x_n\|. \tag{3.22}$$

From (3.22) and (3.26), we obtain

$$\limsup_{n\to+\infty}\langle x_t, J(x_t-x_n)\rangle \leq 0. \tag{3.23}$$

Letting $t\to 0$, noting the fact that $x_t\to x^*$ in norm and the fact that the duality map J is norm-to-norm uniformly continuous on bounded subsets of E, we get

$$\limsup_{n\to+\infty}\langle x^*, J(x^*-x_n)\rangle \leq 0.$$

Finally, we show that $x_n \to x^*$. From (3.19), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, J(x_{n+1}-x^*)\rangle = \alpha_n\lambda_n\langle x_n-x^*, J(x_{n+1}-x^*)\rangle \\ &+ (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle + (1-\alpha_n)\langle Tx_n-x^*, J(x_{n+1}-x^*)\rangle \\ &\leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle \\ &\leq \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2) + (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|^2 \leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle.$$

We can check that all the assumptions of Lemma 2.2 are satisfied. Therefore, we deduce $x_n \to x^*$. This complete the proof. □

Corollary 3.1. *Assume that $E = l_p,\ 1 < p < \infty$. Let K be a nonempty, closed and convex cone of $E$and $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let λ_n and α_n be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

(3.24) $$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n\ n \geq 0.$$

Suppose the following conditions hold :

$(a)\ \lim_{n\to\infty}\alpha_n = 0;\quad (b)\ \lim_{n\to\infty}\lambda_n = 1;$

$(c)\ \sum_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}| < \infty,\ \sum_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since l_p spaces, $1 < p < \infty$ have weakly continuous duality map (see, e.g., [8]), the proof follows from Theorem 3.4. □

Corollary 3.2. *Let H be a real Hilbert space and K a nonempty, closed and convex cone of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let λ_n and α_n be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

(3.25) $$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n\ n \geq 0.$$

Suppose the following conditions hold :

$(a)\ \lim_{n\to\infty}\alpha_n = 0;\quad (b)\ \lim_{n\to\infty}\lambda_n = 1;$

$(c)\ \sum_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}| < \infty,\ \sum_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.1. The Mann algorithm (see, [25]) for nonexpansive mappings, without any compactness assumptions on the set K or on the mapping T, gives only weak convergence of the associated sequence. Here, we prove strong convergence thereom without any compactness assumptions on the set K or on the map T. Therefore, our results improve many recent results using Mann's method to approximate fixed points of nonexpansive mappings.

We know give example of space E, set K and mapping T satisfying the assumptions of Theorem 3.4 and Corollary 3.1.

Let $E = l_p$ and K be the subset of E defined by:

$$K = \{x = (x_n) \in E : x_n \geq 0,\ \forall\, n \geq 1\}.$$

Finally, let $T : K \to K$ be the mapping defined by:

$$Tx = (x_2, x_3, \cdots, x_n, \cdots), \;\; x = (x_n)_{n\geq 1} \in K.$$

It is well known (see, e.g., [8]) that l_p, $1 < p < \infty$ has weakly continuous duality map. The set K is a nonempty, closed, convex cone in l_p and the map T is nonexpansive. Therefore, the spaces E, the set K and te map T satisfies all the assumptions of Theorem 3.4.

Remark 3.2. In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0, 1)$ and $x \in K$. Therefore, in the case where E is a real Hilbert space or $E = l_p,\; 1 < p < \infty$, our results can be used to approximated fixed ponts of nonexpansive mappings from the closed unit ball to itself.

In fact, we have the following.

Corollary 3.3. *Assume that* $E = l_p,\; 1 < p < \infty$ *or* E *is a real Hilbert space. Let* B *be the closed unit ball of* E *and* $T : B \to B$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Let* λ_n *and* α_n *be two sequences in* $(0, 1)$. *Let* $\{x_n\}$ *be a sequence generated iteratively from arbitrary* $x_0 \in B$ *by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)Tx_n \;\; n \geq 0. \tag{3.26}$$

Suppose the following conditions hold :

(a) $\lim_{n\to\infty} \alpha_n = 0$; $\;(b)$ $\lim_{n\to\infty} \lambda_n = 1$;

(c) $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 3.3. Note that Corollary 3.3 is not valid if the Mann iteration is used instead of (3.26) (see, e.g., Chidume [5] or Genel and Lindenstrass [10]).

Remark 3.4. For numerous applications to approximate fixed points of nonexpansive mappings, see the celebrated monograph of Berinde [1]. As remarked by Charles Byrne [4], most of the maps that arise in image reconstruction and signal processing are nonexpansive in nature.

Remark 3.5. Real sequences that satisfy conditions (i), (ii) and (iii) are given by: $\alpha_n = \frac{1}{\sqrt{n}}$ and $\lambda_n = 1 - \frac{1}{\sqrt{n}}$.

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