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Rings in which nilpotents form a subring

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ABSTRACT. Let R be a ring with the set of nilpotents $\mathrm{Nil}(R)$. We prove that the following are equivalent: (i) $\mathrm{Nil}(R)$ is additively closed, (ii) $\mathrm{Nil}(R)$ is multiplicatively closed and R satisfies Köthe's conjecture, (iii) $\mathrm{Nil}(R)$ is closed under the operation $x \circ y = x + y - xy$, (iv) $\mathrm{Nil}(R)$ is a subring of R. Some applications and examples of rings with this property are given, with an emphasis on certain classes of exchange and clean rings.

1. INTRODUCTION

Rings in which nilpotents form a subring (we will call these rings *NR rings* hereafter) are closely related to Armendariz rings and their variations. A ring R is called *Armendariz* if, given any two polynomials $f(x) = a_0 + a_1x + \ldots + a_mx^m$ and $g(x) = b_0 + b_1x + \ldots + b_nx^n$ over R, $f(x)g(x) = 0$ implies $a_ib_j = 0$ for all i and j. Antoine [1] studied the structure of the set of nilpotents in Armendariz rings and proved that in these rings nilpotents always form a subring. He also proved in [1] that the same is true under a slightly weaker condition that the ring is *nil-Armendariz*. Some other results relating the Armendariz and NR conditions can be also found in [5], and recently [4].

In this paper we prove some results which concern the question of when the set of nilpotents in a ring is a subring in general, not in connection with any of the above mentioned Armendariz conditions. Roughly speaking, our main theorem shows that the set of nilpotents $\mathrm{Nil}(R)$ of a ring R is a subring whenever R satisfies Köthe's conjecture (which is a weak assumption) and $\mathrm{Nil}(R)$ is closed under *any* of the most commonly used operations in rings, such as addition, multiplication, the "circle" operation $x \circ y = x + y - xy$, the commutator operation $(x, y) \mapsto xy - yx$, or even some more (see Theorem 2.1 and Remark 2.2). Therefore, very little needs to be assumed for the set of nilpotents to be a subring.

Let us provide here some notations and conventions used throughout the paper. All rings in the paper are associative and not necessarily unital, unless otherwise stated. For a ring R and for any two elements $x, y \in R$, we denote $x \circ y = x + y - xy$. Then $(R, \circ)$ is a monoid. We denote by $Q(R)$ its group of units, i.e.

$$Q(R) = \{q \in R \mid \text{there exists } r \in R \text{ such that } q \circ r = r \circ q = 0\}.$$

If R is unital, then the group of units of R, denoted $U(R)$, is precisely $U(R) = 1 - Q(R)$.

We denote by $J(R)$, $\mathrm{Nil}^*(R)$, and $\mathrm{Nil}(R)$ the Jacobson radical, the upper nilradical, and the set of nilpotents of a ring R, respectively. Note that $\mathrm{Nil}^*(R) \subseteq J(R) \subseteq Q(R)$ and $\mathrm{Nil}^*(R) \subseteq \mathrm{Nil}(R) \subseteq Q(R)$. The ring R is said to be of *bounded index* if there exists a positive integer n such that $x^n = 0$ for all $x \in \mathrm{Nil}(R)$, and R is of bounded index n if n is the least integer with this property. We call R a *NI* ring if $\mathrm{Nil}(R)$ is an ideal in R, and a *NR* ring if $\mathrm{Nil}(R)$ is a subring of R [5].

Our main result will be applied to certain classes of exchange rings. For a ring R, we denote by $\mathrm{Id}(R)$ its set of idempotents. We say that R is an *exchange* ring if for every $a \in R$ there exists $e \in \mathrm{Id}(R)$ such that $e = ra = s \circ a$ for some $r, s \in R$ [2]. If R has a unit, this

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is equivalent to saying that for every $a \in R$ there exists $e \in \mathrm{Id}(R)$ such that $e \in Ra$ and $1-e \in R(1-a)$ (see [2]).

Throughout the text, $\mathbb{Z}$ will denote the set of integers and $\mathbb{N}$ the set of positive integers. $R[x]$ and $M_n(R)$ stand for the polynomial ring and the ring of $n \times n$ matrices over R, respectively.

2. The Main Result

In this section we prove our main theorem. For every nilpotent $x \in \mathrm{Nil}(R)$, we define the *index* of x as the smallest positive integer n such that $x^n = 0$.

Lemma 2.1. *In any ring R, if $x, y \in \mathrm{Nil}(R)$ are of index 2 and $x+y \in \mathrm{Nil}(R)$, then $xy \in \mathrm{Nil}(R)$.*

Proof. Let x, y be elements with the given properties. Since $x+y \in \mathrm{Nil}(R)$, $xy+yx = (x+y)^2 \in \mathrm{Nil}(R)$. We have $(xy+yx)^k = (xy)^k + (yx)^k$ for every k, hence $(xy)^k + (yx)^k = 0$ for some k, which gives $(xy)^{k+1} = 0$. □

Lemma 2.2. *In any ring R, if $x, y \in \mathrm{Nil}(R)$ are of index 2 and $x \circ y \in \mathrm{Nil}(R)$, then $xy \in \mathrm{Nil}(R)$.*

Proof. Let x, y be elements with the given properties. First note that we can embed R into a unital ring, so that we may assume that R is already unital.

Let $z = x \circ y \in \mathrm{Nil}(R)$ and $u = 1 - z \in U(R)$. A direct verification shows that $z^2 = (xy+yx)u$. Since z and u commute and z is a nilpotent, it follows that $xy+yx$ is a nilpotent. Since $(xy+yx)^k = (xy)^k + (yx)^k$ for every k, it follows that $(xy)^k + (yx)^k = 0$ for some k, which gives $(xy)^{k+1} = 0$. □

We say that a ring R satisfies *Köthe's conjecture* if every nil left ideal of R is contained in a nil two-sided ideal. Many other equivalent formulations exist; one of them, for example, is that the sum of two nil left ideals in R is a nil left ideal (see, for example, [12, p. 164]). Whether or not every ring satisfies Köthe's conjecture is a long-standing open question. An important fact that will be needed in our computations is that the rings we are dealing with all satisfy Köthe's conjecture.

Lemma 2.3. *Let R be a ring such that the set of nilpotents $\mathrm{Nil}(R)$ is closed under $+$ or $\circ$. Then R satisfies Köthe's conjecture.*

Proof. It suffices to prove that the sum of two nil left ideals in R is a nil left ideal. If $\mathrm{Nil}(R)$ is closed under $+$, this is trivial, so let us assume that $\mathrm{Nil}(R)$ is closed under $\circ$. Let I and J be nil left ideals of R, and take $x \in I$ and $y \in J$. We wish to prove that $x+y$ is nilpotent. Choose $n \in \mathbb{N}$ such that $y^n = 0$, and define $z = x + yx + y^2x + \ldots + y^{n-1}x$. Then $z - yz = x$, and hence $x + y = y \circ z$. Since I is a nil left ideal, $z \in I$ is nilpotent. Hence the conclusion follows from the fact that $\mathrm{Nil}(R)$ is closed under $\circ$. □

Recall that the upper radical of a ring R, $\mathrm{Nil}^*(R)$, is the sum of all nil (two-sided) ideals. If R satisfies Köthe's conjecture then every nil left (or right) ideal in R is contained in $\mathrm{Nil}^*(R)$. In particular, if $\mathrm{Nil}^*(R) = 0$ and R satisfies Köthe's conjecture, then R contains no nonzero nil left ideals.

Lemma 2.4. *Let R be a ring such that $\mathrm{Nil}^*(R) = 0$ and R satisfies Köthe's conjecture, and suppose that $xy \in \mathrm{Nil}(R)$ for all $x, y \in \mathrm{Nil}(R)$ with $x^2 = y^2 = 0$. Then R satisfies the following condition:*

$$axb = 0 \quad \text{whenever} \quad x \in \mathrm{Nil}(R) \text{ and } a, b \in R \text{ with } ab = 0. \tag{2.1}$$

Proof. Take $x \in \mathrm{Nil}(R)$, with $x^n = 0$, and $a, b \in R$ with $ab = 0$. We will prove the needed equality $axb = 0$ by induction on n.

If $n = 1$ there is nothing to prove. Thus suppose that $n \geq 2$, and take any $t \in R$. From $ab = 0$ we have $(bta)^2 = 0$. Moreover, since $(x^2)^{n-1} = 0$, by the inductive hypothesis we have $ax^2b = 0$, so that $(xbtax)^2 = 0$. Therefore, the assumption of the lemma yields $bta \cdot xbtax \in \mathrm{Nil}(R)$ and hence $btax \in \mathrm{Nil}(R)$, i.e. $taxb \in \mathrm{Nil}(R)$. As t was arbitrary, this means that axb generates a nil left ideal in R. (Note that in any ring R, if $x \in R$ satisfies $Rx \subseteq \mathrm{Nil}(R)$, then the left ideal generated by x, which is $\mathbb{Z}x + Rx$, is also nil.) Since R contains no nonzero nil left ideals, it follows that $axb = 0$, as desired. □

Rings satisfying the condition (2.1) above were studied in [10] and called there *NZI* rings, and also in [8] as *INFP* rings. In [8, Proposition 2.1] it is proven that if R is any ring satisfying (2.1) then $\mathrm{Nil}(R)$ is a subring of R. With this result and our lemmas at hand, we are ready to give the main theorem:

Theorem 2.1. *Let R be a ring with the set of nilpotents $\mathrm{Nil}(R)$. The following are equivalent:*

(i) $\mathrm{Nil}(R)$ *is additively closed.*
(ii) $\mathrm{Nil}(R)$ *is multiplicatively closed and R satisfies Köthe's conjecture.*
(iii) $\mathrm{Nil}(R)$ *is closed under $\circ$.*
(iv) $\mathrm{Nil}(R)$ *is a subring of R.*

Proof. (iv) ⇒ (i), (ii), (iii) is trivial.

(i) ⇒ (iv): Let R satisfy (i). Denote the factor ring $R' = R/\mathrm{Nil}^*(R)$. Then R' also satisfies (i) (indeed, $\mathrm{Nil}(R')$ is nothing but the set of all $x + \mathrm{Nil}^*(R)$ with $x \in \mathrm{Nil}(R)$). By Lemmas 2.1 and 2.3, R' satisfies the hypotheses of Lemma 2.4, so that $\mathrm{Nil}(R')$ is a subring of R' by [8, Proposition 2.1]. It follows that $\mathrm{Nil}(R)$ is a subring of R as well.

(ii) ⇒ (iv): Suppose that R satisfies (ii). Similarly as above, we see that $R' = R/\mathrm{Nil}^*(R)$ also satisfies (ii) (note that R' satisfies Köthe's conjecture since R does). Hence R' satisfies the hypotheses of Lemma 2.4, so that [8, Proposition 2.1] again gives that $\mathrm{Nil}(R')$ is a subring of R' and hence $\mathrm{Nil}(R)$ is a subring of R.

(iii) ⇒ (iv): Similarly as before, denote again $R' = R/\mathrm{Nil}^*(R)$, and observe that R' also satisfies (iii). Hence Lemmas 2.2 and 2.3 give that R' satisfies the hypotheses of Lemma 2.4 and the conclusion follows as above. □

It is a natural question if the assumption 'R satisfies Köthe's conjecture' in (ii) of the above theorem is actually superfluous. Although the assumption that $\mathrm{Nil}(R)$ is multiplicatively closed seems to be quite restrictive, we have not been able to prove that in that case, R actually satisfies Köthe's conjecture. Providing a counterexample to this problem might even be more difficult since such an example would certainly settle Köthe's conjecture in the negative.

Question 1. Let R be a ring such that $\mathrm{Nil}(R)$ is multiplicatively closed. Does R satisfy Köthe's conjecture?

Remark 2.1. To answer Question 1, one may assume that R is a radical ring (i.e. $R = J(R)$). Indeed, if R is a ring with $\mathrm{Nil}(R)$ multiplicatively closed, and I, J are two nil left ideals in R such that $I + J$ is not nil, then I and J are nil left ideals in $J(R)$, which is a radical ring with $\mathrm{Nil}(J(R))$ multiplicatively closed.

Remark 2.2. In Theorem 2.1, we could add some more equivalent statements. For example, one such statement, equivalent to (i)–(iv) of Theorem 2.1, would be that $\mathrm{Nil}(R)$ is closed under the operation $x * y = x + y + xy$. (In fact, this can be easily seen to be equivalent to (iii) of Theorem 2.1.) Moreover, another statement would be that $\mathrm{Nil}(R)$ is closed under the operation $(x, y) \mapsto xy + yx$, or under the operation $(x, y) \mapsto [x, y] = xy - yx$, and R satisfies Köthe's conjecture. (In order to prove this, note that if $x, y \in \mathrm{Nil}(R)$ are of index 2 and $xy + yx \in \mathrm{Nil}(R)$ (or $xy - yx \in \mathrm{Nil}(R)$), then $xy \in \mathrm{Nil}(R)$. For the rest of the proof, use Lemma 2.4.) Similarly as above, we do not know if the 'Köthe conjecture' assumption in these cases is superfluous or not.

The most interesting part of Theorem 2.1 for us will be the equivalence (iii) $\Leftrightarrow$ (iv). In the following we provide a few corollaries of that equivalence. Following [5], we call a ring a *NR ring* if its set of nilpotents forms a subring.

Note that in any ring R, $\mathrm{Nil}(R)$ is a subset of the group $(Q(R), \circ)$ which is closed with respect to taking inverses. Therefore, $\mathrm{Nil}(R)$ is a subgroup of $Q(R)$ if and only if it is closed under $\circ$.

Corollary 2.1. *Let R be a ring. Then R is NR if and only if $\mathrm{Nil}(R)$ is a subgroup of $Q(R)$.* □

In particular, when $\mathrm{Nil}(R)$ is the whole group $Q(R)$, we have:

Corollary 2.2. *Let R be a ring such that $\mathrm{Nil}(R) = Q(R)$. Then R is NR.* □

If R is a unital ring, then saying that $\mathrm{Nil}(R)$ is closed under $\circ$ is the same as saying that the set $1 + \mathrm{Nil}(R) = \{1 + x \mid x \in \mathrm{Nil}(R)\}$ is closed under multiplication (and thus forms a multiplicative subgroup of $U(R)$). Thus, for unital rings, an equivalent form of (iii) $\Leftrightarrow$ (iv) of Theorem 2.1 might be:

Corollary 2.3. *Let R be a unital ring. Then R is NR if and only if $1 + \mathrm{Nil}(R)$ is a multiplicative subgroup of $U(R)$.* □

A unital ring R is called a *UU* ring if all units are unipotent, i.e. $U(R) = 1 + \mathrm{Nil}(R)$. These rings were studied in [3] and [6].

Corollary 2.4. *Every UU unital ring is NR.* □

In [4], Chen called R a *NDG* ring if $\mathrm{Nil}(R)$ is additively closed, and proved that the polynomial ring $R[x]$ is NDG if and only if $\mathrm{Nil}(R)[x] = \mathrm{Nil}(R[x])$. (While this result is stated only for unital rings in [4], the proof also works for general rings.) By Theorem 2.1, NDG rings are just NR rings. Thus, we obtain the following interesting criterion when the polynomial ring is NR:

Corollary 2.5. *Let R be a ring. Then $R[x]$ is NR if and only if $\mathrm{Nil}(R)[x] = \mathrm{Nil}(R[x])$.* □

In particular, if R is a nil ring, then the above proposition says that $R[x]$ is NR if and only if it is nil. It is known that the polynomial ring over a nil ring need not be nil [14]. Thus, we recover the observation noted in [5, Example 2.6] that the polynomial ring over a NR ring need not be NR.

Another consequence of our result is the following corollary which relates INFP rings studied in [8] and NR rings. Recall that a ring R is INFP if it satisfies the condition (2.1). By Lemma 2.4, $R/\mathrm{Nil}^*(R)$ is INFP for every NR ring R. Conversely, if $R' = R/\mathrm{Nil}^*(R)$ is INFP then it is NR by [8, Proposition 2.1], so that R is also NR. Thus we have:

Corollary 2.6. *A ring R is NR if and only if $R/\mathrm{Nil}^*(R)$ is INFP.* □

We close this section with a few limiting examples which show that the statements of the lemmas applied in our main theorem cannot be much generalized.

In view of Lemmas 2.1 and 2.2, one might wonder if some kind of Theorem 2.1 would also hold elementwise. For example, one might ask, if x, y are nilpotents and $x + y$ (or xy or $x \circ y$) is a nilpotent, is then any other among $x + y, xy, x \circ y$ nilpotent? While Lemmas 2.1 and 2.2 say that this is actually the case if the indices of the nilpotents are sufficiently low, it is not true in general.

Example 2.1. Let F be a field and $R = M_3(F)$. Define the following elements in R:

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad w = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $x, y, x + y \in \mathrm{Nil}(R)$ but $xy, x \circ y \notin \mathrm{Nil}(R)$, $x, z, xz \in \mathrm{Nil}(R)$ but $x + z, x \circ z \notin \mathrm{Nil}(R)$, and $x, w, x \circ w \in \mathrm{Nil}(R)$ but $x + w, xw \notin \mathrm{Nil}(R)$.

Lemma 2.4 raises the question if every ring R in which $\mathrm{Nil}(R)$ is a subring satisfies that, whenever $x, y \in R$ are nilpotents and x is of index 2, xy is also of index at most 2. Note that by Lemma 2.4, this is the case if $\mathrm{Nil}^*(R) = 0$. The following example shows that in general it is not true.

Example 2.2. Let $n \geq 3$ and let T be the ring of strictly upper triangular $2n \times 2n$ matrices over a field F. Set $R = M_2(T)$. Clearly, R is a nil ring, i.e. $\mathrm{Nil}(R) = R$. (In fact, R is even nilpotent of index $2n$.) Let $A \in T$ be the matrix with ones above the main diagonal and zeros elsewhere. Then A is of index $2n$. Letting $x = \left[\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right] \in R$ and $y = \left[\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right] \in R$, we see that x and y are of index 2, but $xy = \left[\begin{smallmatrix} A^2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is of index n.

3. Exchange NR Rings

In this section we study the NR condition for the class of exchange rings. We begin with a proposition which relates the Abelian and NR conditions, and holds for any ring, regardless of the exchange property. Recall that a ring R is *Abelian* if idempotents in R are central.

Proposition 3.1. *Let R be a NR ring. Then $R/\mathrm{Nil}^*(R)$ is Abelian.*

In the unital case this proposition follows from Corollary 2.6 and [8, Proposition 1.5] where it is shown that INFP unital rings are Abelian. However, to cover also the nonunital case, we include the short proof here.

Proof. Let $R' = R/\mathrm{Nil}^*(R)$ and $e \in \mathrm{Id}(R')$, and take any $x, y \in R'$. Since R' is a NR ring and $(x - ex)e, e(y - ye) \in \mathrm{Nil}(R')$, we have $e(y - ye)(x - ex)e \in \mathrm{Nil}(R')$, hence $y(xe - exe) = (y - ye)(x - ex)e \in \mathrm{Nil}(R')$. Since this holds for every $y \in R'$, it follows that $xe - exe$ generates a nil left ideal in R'. Since R' contains no nonzero nil left ideals, it follows that $xe - exe = 0$. Similarly we prove $ex - exe = 0$, and hence $ex = xe$. □

Remark 3.3. In the above proposition, the ring R in general is not Abelian. For example, the ring of upper triangular matrices over a field is NI and not Abelian.

As noted in [5], NR rings in general are not NI. For example, if $R = F\langle x, y\rangle/(x^2)$ (i.e., R is the ring of formal polynomials in noncommuting variables x and y over a field F, modulo the ideal generated by x^2), then $\mathrm{Nil}(R) = Fx + xRx$. So $\mathrm{Nil}(R)$ is a subring of R with the trivial multiplication, but $yx \notin \mathrm{Nil}(R)$, and hence $\mathrm{Nil}(R)$ is not an ideal (see [5]).

Observe that, in this example, $J(R) = 0$, hence $\mathrm{Nil}(R)$ is not even contained in $J(R)$. However, if R is a NR exchange ring then always $\mathrm{Nil}(R) \subseteq J(R)$. This follows from [4, Corollary 2.17] (note that NR rings are linearly weak Armendariz, see [4] for details). Alternatively, the proof could be obtained by using Proposition 3.1.

It would be interesting to know if NR exchange rings are actually NI.

Question 2. Let R be an exchange ring. If R is NR, is then R NI?

Remark 3.4. To answer Question 2, we may assume that R is a radical ring (note that radical rings are exchange [2]). Indeed, if R is a NR exchange ring, then we know that $\mathrm{Nil}(R) \subseteq J(R)$, so that $J(R)$ is a NR radical ring. Assume that $\mathrm{Nil}(R)$ is an ideal in $J(R)$.

Then, for any $a \in R$ and $x \in \mathrm{Nil}(R)$, $axa \in J(R)$, hence $(ax)^2 = (axa)x \in \mathrm{Nil}(R)$ and therefore $ax \in \mathrm{Nil}(R)$, which proves that $\mathrm{Nil}(R)$ is also an ideal in R, as desired.

In the following we provide a partial answer to Question 2 for the case when R is of bounded index.

Lemma 3.5. *Let R be a NR ring of bounded index n with $\mathrm{Nil}^*(R) = 0$. If $x_1, \ldots, x_n \in \mathrm{Nil}(R)$ are of index 2, then $x_1 \ldots x_n = 0$.*

Proof. By assumption we have $(x_1 + \ldots + x_n)^n = 0$, i.e.

$$\sum_{\sigma \in X_n} x_{\sigma(1)} \ldots x_{\sigma(n)} = 0, \tag{3.2}$$

where X_n denotes the set of all functions $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. By Lemma 2.4 we have $x_i y x_i = 0$ for each i and $y \in \mathrm{Nil}(R)$, thus $x_{\sigma(1)} \ldots x_{\sigma(n)} = 0$ whenever $\sigma(i) = \sigma(j)$ for some $i \neq j$. Hence (3.2) becomes

$$\sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)} = 0,$$

where $S_n \subseteq X_n$ denotes the permutation group of $\{1, \ldots, n\}$.

Let $t \in R$. For every $\sigma \in S_n \setminus \{\mathrm{id}\}$ there exists $i = i(\sigma)$ such that $\sigma(i) > \sigma(i+1)$. Denote

$$y = y(\sigma) = x_{\sigma(i)} x_{\sigma(i)+1} \ldots x_n t x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(i)}.$$

Since $x_{\sigma(i)}^2 = 0$, $y \in \mathrm{Nil}(R)$, and hence $z = z(\sigma) = x_{\sigma(i+1)+1} \ldots x_{\sigma(i)-1} y \in \mathrm{Nil}(R)$. By Lemma 2.4, $x_{\sigma(i+1)} z x_{\sigma(i+1)} = 0$, hence

$$x_1 \ldots x_n t x_{\sigma(1)} \ldots x_{\sigma(n)} = x_1 \ldots x_{\sigma(i+1)} z x_{\sigma(i+1)} x_{\sigma(i+2)} \ldots x_{\sigma(n)} = 0.$$

Hence

$$x_1 \ldots x_n t x_1 \ldots x_n = \sum_{\sigma \in S_n} x_1 \ldots x_n t x_{\sigma(1)} \ldots x_{\sigma(n)} = x_1 \ldots x_n t \sum_{\sigma \in S_n} x_{\sigma(1)} \ldots x_{\sigma(n)} = 0.$$

This shows that $x_1 \ldots x_n$ generates a nil left ideal in R and consequently $x_1 \ldots x_n = 0$. $\square$

Lemma 3.6. *Let R be an exchange NR ring of bounded index with $\mathrm{Nil}^*(R) = 0$. Then $\mathrm{Nil}(R) = 0$.*

Proof. First note that we may assume that R is radical. In fact, if R satisfies the hypotheses of the lemma, then $J(R)$ is clearly a radical NR ring (with $\mathrm{Nil}(J(R)) = \mathrm{Nil}(R)$) of bounded index. To see that $\mathrm{Nil}^*(J(R)) = 0$, take a nil left ideal I in $J(R)$. Then the left ideal in R generated by I, which is $I + RI$, is also nil. Indeed, since R is NR, it suffices to see that $ax \in \mathrm{Nil}(R)$ for every $a \in R$ and $x \in I$. Now, $axa \in J(R)$, hence $(ax)^2 = (axa)x \in \mathrm{Nil}(R)$, which gives $ax \in \mathrm{Nil}(R)$, as desired. Hence $J(R)$ indeed satisfies the hypotheses of the lemma, and therefore we may assume that R is a radical ring.

Let $x \in R$ such that $x^2 = 0$. We need to prove that $x = 0$. It suffices to prove that $qx \in \mathrm{Nil}(R)$ for every $q \in R$. Embed R into a unital ring S, then $u = 1 - q \in U(S)$, with $u^{-1} = 1 - r$, where $r \in R$ is taken such that $q \circ r = r \circ q = 0$. Let n denote the index of R, and, for each $i = 1, \ldots, n$, define $x_i = u^{i-1} x u^{1-i}$. Clearly, each x_i is a nilpotent of index 2. Moreover, $x_i \in R$ since it can be expressed in terms of x, q, r. Hence by Lemma 3.5, $x_1 \ldots x_n = 0$. But we have

$$x_1 \ldots x_n = x u x u^{-1} u^2 x u^{-2} \ldots u^{n-1} x u^{1-n} = (xu)^n u^{-n}.$$

Thus $(xu)^n = 0$ and hence $xu \in \mathrm{Nil}(R)$, which gives $x - qx = ux \in \mathrm{Nil}(R)$. Thus $qx \in \mathrm{Nil}(R)$, as desired. $\square$

Now the following is easy:

Theorem 3.2. *Let R be an exchange NR ring of bounded index. Then R is NI.*

Proof. Let $R' = R/\operatorname{Nil}^*(R)$. Then R' is an exchange NR ring of bounded index with $\operatorname{Nil}^*(R') = 0$, so that $\operatorname{Nil}(R') = 0$ by Lemma 3.6. It follows that $\operatorname{Nil}(R) = \operatorname{Nil}^*(R)$, as desired. □

Remark 3.5. The assumption that R is exchange is crucial in Theorem 3.2. For example, the ring $R = F\langle x, y\rangle/(x^2)$ is NR of bounded index 2 (in fact, $\operatorname{Nil}(R)$ has trivial multiplication), but it is not NI.

We conclude the paper by providing an interesting application of Theorem 2.1 to the class of Diesl's *strongly nil clean* rings. These rings were defined by Diesl in [7], as rings in which every element can be written as the sum of an idempotent and a nilpotent that commute. Along with these, Diesl also defined the wider class of *nil clean* rings, which have the same definition without commutativity. It turns out that very little can be said about nil clean rings, according to many fundamental open questions stated in [7]. On the other hand, much more can be said about strongly nil clean rings. In fact, while probably not known to the author of [7] at that time, these rings had been completely characterized much earlier, back in 1988, by Hirano et al. [9]. Hirano et al. proved that strongly nil clean rings (of course, they were not called this way in [9]) are precisely those rings R in which the set of nilpotents $\operatorname{Nil}(R)$ forms an ideal and $R/\operatorname{Nil}(R)$ is a Boolean ring. Accordingly, these rings are just those which are Boolean modulo the upper nilradical.

In [9], the authors used Jacobson's structure theory for primitive rings to obtain their results. Recently, several new proofs of the above equivalence appeared, using different techniques (see [13, 11, 6]). In the following we provide another proof of that equivalence, which, we believe, is noteworthy because it is very short, self-contained, and it applies also to nonunital rings.

Proposition 3.2. *Given a ring R, the following are equivalent:*

(i) *R is strongly nil clean.*
(ii) *For every $a \in R$, $a - a^2 \in \operatorname{Nil}(R)$.*
(iii) *$R/\operatorname{Nil}^*(R)$ is Boolean.*

Proof. (iii) ⇒ (ii) is trivial.

(i) ⇔ (ii): First, let $a \in R$ be strongly nil clean, i.e. $a = e+q$ and $eq = qe$, where $e \in \operatorname{Id}(R)$ and $q \in \operatorname{Nil}(R)$. Then $a - a^2 = e + q - e^2 - 2eq - q^2 = q - 2eq - q^2$, which is clearly a nilpotent since e and q commute. Conversely, if $a \in R$ is such that $(a - a^2)^n = 0$, then, embedding R into a unital ring S and setting $e = (1 - (1 - a)^n)^n$, one easily checks that e is a multiple of a^n and $1 - e$ is a multiple of $(1 - a)^n$, so that $e(1 - e)$ and $(a - e)^n = a^n - a^n e + (ae - e)^n = a^n(1 - e) + (a - 1)^n e$ are both multiples of $a^n(1 - a)^n = 0$ and thus $e - e^2 = e(1 - e) = 0$ and $(a - e)^n = 0$. Note that also $e \in R$ and $ae = ea$. Hence $a = e + (a - e)$ is a strongly nil clean decomposition of a in R.

(i) and (ii) ⇒ (iii): This is the main part of the proof. First, observe that every ring R satisfying (ii) has $\operatorname{Nil}(R) = Q(R)$. Indeed, given any $q \in Q(R)$, then, taking $r \in R$ with $q \circ r = r \circ q = 0$, we easily see that $q = (q - q^2) - (q - q^2)r$, so that q must be a nilpotent since q and r commute and $q - q^2 \in \operatorname{Nil}(R)$. Now, by Corollary 2.2 R is a NR ring, so that $R' = R/\operatorname{Nil}^*(R)$ is NR and Abelian by Proposition 3.1. Accordingly, since every element in R' can be expressed as the sum of an idempotent and a nilpotent, $\operatorname{Nil}(R')$ forms an ideal in R', so that $\operatorname{Nil}(R') = 0$. Hence $\operatorname{Nil}(R) = \operatorname{Nil}^*(R)$, which immediately yields (iii). □

Remark 3.6. From the above proof we see that the equivalence (i) ⇔ (ii) holds also on the elementwise level, that is, an element a in a ring is strongly nil clean (i.e., the sum of an idempotent and a nilpotent) if and only if $a - a^2$ is a nilpotent. This shows that the condition "a is strongly clean" in [11, Theorem 2.1] is actually superfluous, and provides an easy self-evident argument for [11, Theorem 2.9].

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