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On nonconvex retracts in normed linear spaces

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ABSTRACT. Let E be a real normed linear space. A subset $X \subset E$ is called a retract of E if there exists a continuous mapping $r : E \to X$, a retraction, satisfying $r(x) = x,\ x \in X$. It is well known that every nonempty closed convex subset of E is a retract of E. Nonconvex retracts are studied in this paper.

1. INTRODUCTION

Let E be a real normed linear space with the zero element denoted by θ. A nonempty convex closed set $P \subset E$ is called a cone if it satisfies the following two conditions: (i) $\lambda x \in P$ for $x \in P$ and $\lambda \geq 0$; (ii) $\pm x \in P$ implies $x = \theta$. For the properties of cones we refer to [4, 5]. A functional $\gamma : P \to \mathbb{R}$ is *convex* if $\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$ for any $x,\ y \in P$ and $t \in [0, 1]$; γ is *concave* if $-\gamma$ is convex. The function γ is *bounded* if the image of any bounded set in P under γ is bounded as well. The open ball centered at θ with radius $R > 0$ is denoted by $B_R = \{x \in E \mid \|x\| < R\}$. Throughout this paper, the notations

$$D_1 = \{x \in P|\ \alpha(x) \leq R_1\},\ D_2 = \{x \in P|\ \beta(x) \leq R_2\}$$

and

$$D_1' = \{x \in P|\ \alpha(x) \geq R_1\},\ D_2' = \{x \in P|\ \beta(x) \geq R_2\}$$

are always used for the functionals $\alpha, \beta : P \to [0, +\infty)$ and the constants $R_1, R_2 > 0$.

A subset $X \subset E$ is called a retract of E if there exists a continuous mapping $r : E \to X$, a retraction, satisfying $r(x) = x,\ x \in X$. It is well known that every nonempty closed convex subset of E is a retract of E [1, 3]. By a theorem due to Dugundji [1], $D = \{x \in E \mid \|x\| \geq R\}(R > 0)$ is a nonconvex retract in infinite dimensional spaces. The concept of retract plays a very important role in fixed point theory, see [4]–[8]. In [7] there are the following results for nonconvex retracts.

Theorem 1.1. *Let P be a cone in E, $\alpha : P \to [0, +\infty)$ be a continuous convex functional and $\beta : P \to [0, +\infty)$ be a bounded continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0,\ \beta(x) > 0$ for $x \neq \theta$, both $\{x \in P|\ \alpha(x) \leq R\}$ and $\{x \in P|\ \beta(x) \leq R\}$ be bounded for all $R > 0$. If*

$$\beta(\mu x) > \beta(x) \text{ for } \mu > 1,\ x \in P \backslash \{\theta\}, \tag{1.1}$$

then $D_1 \cap D_2$ is a retract of E.

Theorem 1.2. *Let P be a cone in E, $\alpha : P \to [0, +\infty)$ be a continuous functional and $\beta : P \to [0, +\infty)$ be a continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0,\ \beta(x) > 0$ for $x \neq \theta$. If (1.1) holds and*

$$\alpha(\lambda x) \leq \lambda\alpha(x) \text{ for } \lambda \in [0, 1],\ x \in P,$$

then $D_1' \cap D_2'$ is a retract of E.

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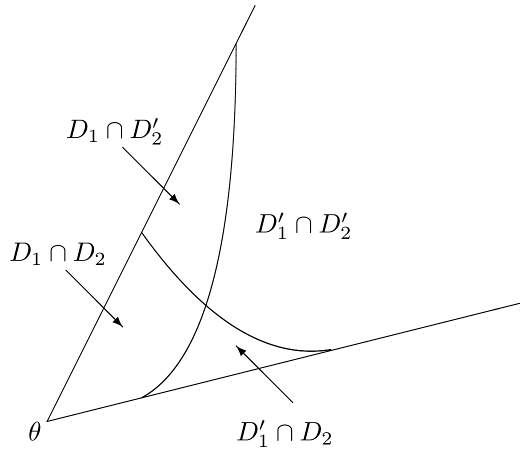

In this paper we prove that $D_1' \cap D_2$ is a retract(see the figure above) under some conditions, where $\alpha, \beta : P \to [0, +\infty)$ are respectively continuous convex and concave functionals. Two examples are given respectively in infinite and finite dimensional spaces to illustrate that $D_1' \cap D_2$ is nonconvex. As for $D_1 \cap D_2'$, it is obviously a retract if it is nonempty since it is closed convex.

2. Main results

The following lemma in [2] is needed to prove the main theorem.

Lemma 2.1. *Let X and Y be topological spaces and $\{A_i \mid i = 1, 2, \cdots, n\}$ be a finite family of closed sets such that $X = \cup_{i=1}^n A_i$. If $f_i : A_i \to Y$ is continuous and $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$ for $i \neq j (i, j = 1, 2, \cdots, n)$, then there exists a unique continuous map $f : X \to Y$ such that $f|_{A_i} = f_i (i = 1, 2, \cdots, n)$.*

Theorem 2.3. *Let P be a cone in E, $\alpha : P \to [0, +\infty)$ be a uniformly continuous convex functional and $\beta : P \to [0, +\infty)$ be a bounded continuous concave functional with $\alpha(\theta) = \beta(\theta) = 0$ and $\alpha(x) > 0$, $\beta(x) > 0$ for $x \neq \theta$. Suppose that for $R > 0$, $\{x \in P \mid \beta(x) \leq R\}$ is bounded. If*

$$R_1 \beta(x) \leq R_2 \alpha(x) \text{ for } x \in D_1 \cap D_2 \tag{2.1}$$

and

$$\beta(x + \lambda y) \leq \beta(x + y) \text{ for } x, y \in P,\ \lambda \in [0, 1], \tag{2.2}$$

then $D_1' \cap D_2$ is a retract of E.

Proof. (I) We first prove that D_2 is a retract of E.

(i) It is clear that $D_2' \neq \emptyset$ since D_2 is bounded. Take $R > 0$ such that

$$D_2 \subset \{x \in P \mid \|x\| \leq R\} =: P_R$$

and $D_2' \cap P_R \neq \emptyset$. Since P_R is closed convex, there exists a retraction $g_1 : E \to P_R$.

(ii) Because $\beta(x)$ is a bounded functional, there exists a constant $M > R_2$ such that $\beta(x) \leq M$ for $x \in D_2' \cap P_R$. It follows from the boundedness of $D_{M+1} = \{x \in P \mid \beta(x) \leq M + 1\}$ that there exists $R' > R$ such that $\beta(x) > M + 1$ for $x \in P \cap \partial B_{R'}$. Since $\theta \notin D_2'$ we can define

$$g_2(x) = \frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)}(x - R'[x]) \text{ for } x \in D_2' \cap P_R,$$

here and later $[x]$ stands for $x/\|x\|$ for $x \in E \backslash \{\theta\}$. Obviously, g_2 is continuous on $D_2' \cap P_R$.

(iii) Consider the topological space $D_2' \cap P_R$ and denote

$$A_1 = \{x \in D_2' \cap P_R \mid \|g_2(x)\| \le R'\},\ A_2 = \{x \in D_2' \cap P_R \mid \|g_2(x)\| \ge R'\}$$

which are closed sets in $D_2' \cap P_R$. It is clear that $D_2' \cap P_R = A_1 \cup A_2$. Define

$$h_{A_1}(x) = g_2(x) + R'[x] \text{ for } x \in A_1 \text{ and } h_{A_2}(x) = \theta \text{ for } x \in A_2.$$

Both h_{A_1} and h_{A_2} are continuous.

We will show that $h_{A_1} : A_1 \to P$. In fact, since $\|g_2(x)\| \le R'$, that is,

$$\left\| \frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)}(x - R'[x]) \right\| = \frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)}(R' - \|x\|) \le R', \tag{2.3}$$

we have

$$h_{A_1}(x) = \left(\frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)}(\|x\| - R') + R' \right)[x] \in P. \tag{2.4}$$

For $x \in A_1 \cap A_2 = \{y \in D_2' \cap P_R \mid \|g_2(y)\| = R'\}$, it follows from (2.3) and (2.4) that $h_{A_1}(x) = h_{A_2}(x) = \theta$. By Lemma 2.1 there is a unique continuous map $g_3 : D_2' \cap P_R \to P$ such that $g_3|_{A_1} = h_{A_1}$ and $g_3|_{A_2} = h_{A_2}$

(iv) Define

$$g_4(x) = \begin{cases} g_3(x), & x \in D_2' \cap P_R; \\ x, & x \in D_2. \end{cases}$$

For $x \in \{y \in P \mid \beta(y) = R_2\} \cap P_R$, we have that $g_2(x) = x - R'[x]$ and $\|g_2(x)\| = R' - \|x\| < R'$. Therefore, $g_3(x) = x$ and hence $g_4 : P_R \to P$ is well defined and continuous.

(v) Now we show that for $x \in D_2' \cap P_R$, $\beta(g_3(x)) \le R_2$, i.e., $g_4 : P_R \to D_2$.

In fact, when $\|g_2(x)\| \ge R'$, $\beta(g_3(x)) = 0 \le R_2$; when $\|g_2(x)\| \le R'$, it follows from $\beta(x) \ge R_2$ that

$$\frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)} \ge 1$$

and

$$g_3(x) = \frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)}(x - R'[x]) + R'[x],$$

$$x = \frac{\beta(R'[x]) - \beta(x)}{\beta(R'[x]) - R_2} g_3(x) + \left(1 - \frac{\beta(R'[x]) - \beta(x)}{\beta(R'[x]) - R_2}\right) R'[x].$$

By the concavity of β, we have

$$\beta(x) \ge \frac{\beta(R'[x]) - \beta(x)}{\beta(R'[x]) - R_2} \beta(g_3(x)) + \left(1 - \frac{\beta(R'[x]) - \beta(x)}{\beta(R'[x]) - R_2}\right) \beta(R'[x]),$$

$$\beta(g_3(x)) \le \frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)} \beta(x) - \left(\frac{\beta(R'[x]) - R_2}{\beta(R'[x]) - \beta(x)} - 1 \right) \beta(R'[x]) = R_2.$$

(vi) Let $f_1(x) = g_4(g_1(x))$ for $x \in E$, then $f_1 : E \to D_2$ is a retraction.

(II) In the following we prove step by step that $D_1' \cap D_2$ is a retract of E.

(i) Since α is uniformly continuous and β is continuous with $\alpha(\theta) = \beta(\theta) = 0$, there exists $x_0 \in D_2 \backslash \{\theta\}$ such that $\alpha(x_0) \le R_1/3, \beta(x_0) \le R_2/2$ and for $x \in P$,

$$|\alpha(x + x_0) - \alpha(x)| \le \frac{R_1}{3}. \tag{2.5}$$

(ii) Define

$$W = \left\{ x \in D_1 \cap D_2 \mid \alpha(x + x_0) \le \frac{R_1}{2}, \beta(x + x_0) \le \frac{R_2}{2} \right\}.$$

Clearly, $W \neq \emptyset$ due to $\theta \in W$. Now we show that for $x \in W$,

$$\alpha\left(x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0\right) > 0. \tag{2.6}$$

When $\alpha(x+x_0) = R_1/2$, we have

$$\alpha\left(x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0\right) = \alpha(x).$$

If $\alpha(x) = 0$, then $x = \theta$ and $\alpha(x+x_0) = \alpha(x_0) = R_1/2$ which contradicts $\alpha(x_0) \le R_1/3$. Hence $x \neq \theta$ and $\alpha(x) > 0$, that is, (2.6) holds.

When $\alpha(x+x_0) < R_1/2$, we have

$$x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0 \in P\backslash\{\theta\},$$

which implies that (2.6) holds.

(iii) Here we prove that $W \cap D_1' = \emptyset$. Otherwise, for $x_1 \in W \cap D_1'$, we have from $x_1 \in D_1'$ that $\alpha(x_1) \ge R_1$ and from $x_1 \in W$ that $\alpha(x_1) \le R_1$ with $\alpha(x_1+x_0) \le R_1/2$. Hence $\alpha(x_1) = R_1$ and $\alpha(x_1) - \alpha(x_1+x_0) \le R_1/3$ by (2.5). Consequently,

$$R_1 = \alpha(x_1) \le \frac{R_1}{3} + \alpha(x_1+x_0) \le \frac{R_1}{3} + \frac{R_1}{2} = \frac{5R_1}{6} < R_1.$$

The contradiction implies that $W \cap D_1' = \emptyset$ and $W \cap (D_1' \cap D_2) = \emptyset$.

(iv) Consider the topological space D_2. Denote $O = \{x \in D_2 \mid \alpha(x+x_0) < R_1/2\}$ which is open in D_2. Let $A_3 = (D_1 \cap D_2)\backslash O$ and $A_4 = D_1' \cap D_2$ which are closed sets in D_2. Obviously, $D_2 = W \cup A_3 \cup A_4$. Define

$$h_W(x) = R_1 \frac{x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0}{\alpha\left(x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0\right)} \text{ for } x \in W,$$

$$h_{A_3}(x) = R_1 \frac{x}{\alpha(x)} \text{ for } x \in A_3 \text{ and } h_{A_4}(x) = x \text{ for } x \in A_4$$

which are all continuous.

If $x \in W \cap A_3$, then $\alpha(x+x_0) = R_1/2$ and $h_W(x) = h_{A_3}(x)$; if $x \in A_3 \cap A_4$, then $\alpha(x) = R_1$ and $h_{A_3}(x) = h_{A_4}(x)$. By Lemma 2.1 there is a unique continuous map f_2 on D_2 such that $f_2|_W = h_W$, $f_2|_{A_3} = h_{A_3}$ and $f_2|_{A_4} = h_{A_4}$

(v) In this step we will show that $f_2 : D_2 \to D_1' \cap D_2$.

If $x \in A_3$, then $x = (\alpha(x)/R_1)f_2(x)$ and

$$\alpha(x) = \alpha\left(\frac{\alpha(x)}{R_1}f_2(x) + \left(1 - \frac{\alpha(x)}{R_1}\right)\theta\right) \le \frac{\alpha(x)}{R_1}\alpha(f_2(x)),$$

thus $\alpha(f_2(x)) \ge R_1$. Since

$$\beta(x) = \beta\left(\frac{\alpha(x)}{R_1}f_2(x) + \left(1 - \frac{\alpha(x)}{R_1}\right)\theta\right) \ge \frac{\alpha(x)}{R_1}\beta(f_2(x)),$$

we have that $\beta(f_2(x)) \le (R_1/\alpha(x))\beta(x)$ and $\beta(f_2(x)) \le R_2$ by (2.1).

If $x \in W$, since

$$x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0 = \frac{1}{R_1}\alpha\left(x + x_0 - \frac{2\alpha(x+x_0)}{R_1}x_0\right) f_2(x) \tag{2.7}$$

and

$$
(2.8)\qquad \begin{array}{rl}
& \alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)\\
= & \alpha\big(\big(1-\frac{2\alpha(x+x_0)}{R_1}\big)(x+x_0)+\frac{2\alpha(x+x_0)}{R_1}x\big)\\
\le & \big(1-\frac{2\alpha(x+x_0)}{R_1}\big)\alpha(x+x_0)+\frac{2\alpha(x+x_0)}{R_1}\alpha(x)\\
\le & \big(1-\frac{2\alpha(x+x_0)}{R_1}\big)\frac{R_1}{2}+\frac{2\alpha(x+x_0)}{R_1}R_1\\
= & \frac{R_1}{2}+\alpha(x+x_0)\le R_1,
\end{array}
$$

we have from the convexity of α that

$$
\begin{array}{rl}
& \alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)\\
= & \alpha\big(\frac{1}{R_1}\alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)f_2(x)\big)\\
\le & \frac{1}{R_1}\alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)\alpha(f_2(x))
\end{array}
$$

and thus $\alpha(f_2(x))\ge R_1$. It follows from (2.2) that

$$
\beta\left(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\right)\le\beta(x+x_0)\le\frac{R_2}{2}\le R_2,
$$

and hence $x+x_0-(2\alpha(x+x_0)/R_1)x_0\in D_1\cap D_2$ by (2.8). From (2.1) we have

$$
(2.9)\qquad R_1\beta\left(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\right)\le R_2\alpha\left(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\right).
$$

By (2.7) and the concavity of β, we have

$$
\begin{array}{rl}
& \beta\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)\\
= & \beta\big(\frac{1}{R_1}\alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)f_2(x)\big)\\
\ge & \frac{1}{R_1}\alpha\big(x+x_0-\frac{2\alpha(x+x_0)}{R_1}x_0\big)\beta(f_2(x)).
\end{array}
$$

Therefore (2.9) leads to $\beta(f_2(x))\le R_2$.

(vi) Let $r(x)=f_2(f_1(x))$ for $x\in E$, then $r:E\to D_1'\cap D_2$ is a retraction, i.e., $D_1'\cap D_2$ is a retract of E. □

Remark 2.1. By Theorem 3.1 in [7] and references therein, the retracts in Banach Spaces can be applied to compute fixed point index in cones and to obtain the existence and the location of positive fixed points about nonlinear completely continuous operators.

3. Examples

In this section two examples are given respectively in infinite and finite dimensional spaces to illustrate that $D_1'\cap D_2$ is nonconvex.

Example 3.1. Let $E=C[0,1]$ with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$ for $x\in C[0,1]$ and

$$
P=\left\{x\in C[0,1]\mid x(t)\ge 0 \text{ for } t\in[0,1],\ \min_{t\in[1/3,2/3]}x(t)\ge\frac{1}{9}\|x\|\right\}.
$$

Define two functionals as

$$
\alpha(x)=\max_{t\in[1/3,2/3]}x(t)\text{ and }\beta(x)=\min_{t\in[1/3,2/3]}x(t)\text{ for }x\in P.
$$

Obviously, P is a cone in E, $\alpha:P\to[0,+\infty)$ is a uniformly continuous convex functional and $\beta:P\to[0,+\infty)$ is a bounded continuous concave functional with $\alpha(\theta)=\beta(\theta)=0$ and $\alpha(x)>0,\ \beta(x)>0$ for $x\ne\theta$, $D_R=\{x\in P\mid\beta(x)\le R\}$ is bounded for any $R>0$.

Let $R_1 = R_2 = 4/9$ and $x_1(t) = t,\ x_2(t) = (t-1)^2$ for $t \in [0,1]$. It is easy to see that $x_1, x_2 \in D_1' \cap D_2$. Clearly, (2.1) and (2.2) hold. Since

$$\alpha\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = \frac{7}{18} < R_1,$$

it follows that $D_1' \cap D_2$ is nonconvex.

Example 3.2. Let $E = R^2$ and $P = \{(x,y) \in R^2 \mid x \geq 0, y \geq 0\}$. For $(x,y) \in P$ define $\alpha(x,y) = x + y$ and $\beta(x,y) = \min(x+y, \sqrt{x} + \sqrt{y})$ which immediately shows that β is concave (as the minimum of concave functions) and satisfies (2.2). Obviously, P is a cone in E, $\alpha : P \to [0,+\infty)$ is a uniformly continuous convex functional and $\beta : P \to [0,+\infty)$ is a bounded continuous functional with $\alpha(0,0) = \beta(0,0) = 0$ and $\alpha(x,y) > 0,\ \beta(x,y) > 0$ for $(x,y) \neq (0,0)$, $D_R = \{(x,y) \in P \mid \beta(x,y) \leq R\}$ is bounded for any $R > 0$.

Let $R_1 = R_2 = 3$. It is clear that (2.1) is satisfied. Since $(0,9), (9,0) \in D_1' \cap D_2$ and $\beta((0,9)/2 + (9,0)/2) = \sqrt{4.5} + \sqrt{4.5} > 3 = R_2$, It follows that $D_1' \cap D_2$ is nonconvex.

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