

# On linear independence of trigonometric numbers

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## ABSTRACT.

A necessary and sufficient condition is established for  $1$ ,  $\cos(\pi r_1)$ , and  $\cos(\pi r_2)$  to be rationally independent, where  $r_1, r_2$  are rational numbers. The elementary computational argument yields linear independence over larger number fields as well.

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