

Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Amenability and Fan–Glicksberg theorem for set-valued mappings

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ABSTRACT.

In this paper, we begin by discussion of some well known results on the existence of left invariant means in the spaces: $LUC(S)$, $AP(S)$ and $WAP(S)$ with Hahn-Banach extension theorem. We then give a new and precise proof of the well known Fan–Glicksberg fixed point theorem. This is then followed by a discussion on some related open problems.

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