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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Cyclic permutations and crossing numbers of join products of two symmetric graphs of order six

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ABSTRACT. The main purpose of this article is broaden known results concerning crossing numbers for join of graphs of order six. We give the crossing number of the join product $G + D_n$, where the graph G consists of one 5-cycle and of one isolated vertex, and D_n consists on n isolated vertices. The proof is done with the help of software that generates all cyclic permutations for a given number k, and creates a new graph COG for calculating the distances between all $(k-1)!$ vertices of the graph. Finally, by adding some edges to the graph G, we are able to obtain the crossing numbers of the join product with the discrete graph D_n and with the path P_n on n vertices for other two graphs.

1. INTRODUCTION

It is well known that a computing of the crossing number of a given graph in general case is NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Research of the problem of reducing the number of crossings in the graph was studied in a lot of areas, and the most researched area is Very Large Scale Integration technology. Further, the problem of reducing the number of crossings in the graph is studied not only in the graph theory, but also by computer scientists. The exact values of the crossing numbers are known only for some graphs or some families of graphs.

In this article are used notations and definitions of the crossing numbers of graphs like in [7]. We will often use the Kleitman's result [5] on crossing numbers of the complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor \Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor, \quad \text{for} \quad m \leq 6.$$

Using Kleitman's result [5], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [7]. Moreover, the exact values for crossing numbers of $G + D_n$ and of $G + P_n$ for all graphs G of order at most four are given in [12]. It is also important to note that, the crossing numbers of the graphs $G + D_n$ are known for few graphs G of order five and six in [2], [6], [10], [11], [14], [15], [16], [17], and [18]. In all these cases, the graph G is mostly connected and contains also mostly at least one cycle. Further, the exact values for the crossing numbers $G + P_n$, and $G + C_n$ have been also investigated for some graphs G of order five and six in [6], [11], [13], and [19].

The methods presented in the paper are new, and they are based on multiple combinatorial properties of the cyclic permutations. The similar methods were partially used first time in the papers [4], and [14]. In [2], [3], [15], and [17], the properties of cyclic permutations are also verified by the help of software in [1]. According to our opinion the

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methods used in [6], [11], and [12], do not allow to establish the crossing number of the join product $G + D_n$. The proofs will be done with the help of software that generates all cyclic permutations in [1]. C++ version of the program is located also on the website `http://web.tuke.sk/fei-km/coga/`. The list with the short names of $6!/6 = 120$ cyclic permutations of six elements are collected in Table 1 of [15].

2. Cyclic permutations and configurations

Let G be the disconnected graph of order six consisting of one 5-cycle and of one isolated vertex. We will consider the join product of the graph G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. Thus, $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$G + D_n = G \cup K_{6,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

In the paper, we will use the same notation and definitions for cyclic permutations and the corresponding configurations for a good drawing D of the graph $G + D_n$ like in [15]. Let D be a drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i have been defined by Hernández-Vélez, Medina, and Salazar [4]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. We have to emphasize that a rotation is a cyclic permutation. We will separate all subgraphs T^i, $i = 1, \dots, n$, of the graph $G + D_n$ into three mutually-disjoint subsets depending on how many times the considered T^i crosses the edges of G in D. For $i = 1, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses the edges of G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given subdrawing of G in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

According to the arguments in the proof of the main Theorem 3.1, if we would like to obtain a drawing of $G + D_n$ with the smallest number of crossings, then the set R_D must be nonempty. Thus, we will deal with only drawings of the graph G with a possibility of an existence of a subgraph $T^i \in R_D$. Of course, there is only one drawing of G in which the edges of G do not cross each other. Since there is only one subdrawing of its subgraph isomorphic with the path P_4 with one crossing among its edges, then we obtain four next possibilities in which two remaining edges of the graph G are able to cross the edges of the fixed subgraph. Hence, there are only five possible drawings of G which are presented in Fig. 1.

Let us assume first a good drawing D of the graph $G + D_n$ in which the edges of G do not cross each other. In this case, without loss of generality, we can choose the vertex notation of the graph in such a way as shown in Fig. 1(a). Our aim shall be to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Since there is only one subdrawing of $F^i \setminus \{v_6\}$ represented by the rotation (12345), there are five possibilities for how to obtain the subdrawing of F^i depending on in which region the edge t_iv_6 is placed. These five possibilities under our consideration will be denoted by A_k, for $k = 1, \dots, 5$. As for our considerations, it does not play a role in which of the regions is unbounded; assume the drawings shown in Fig. 2. In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in

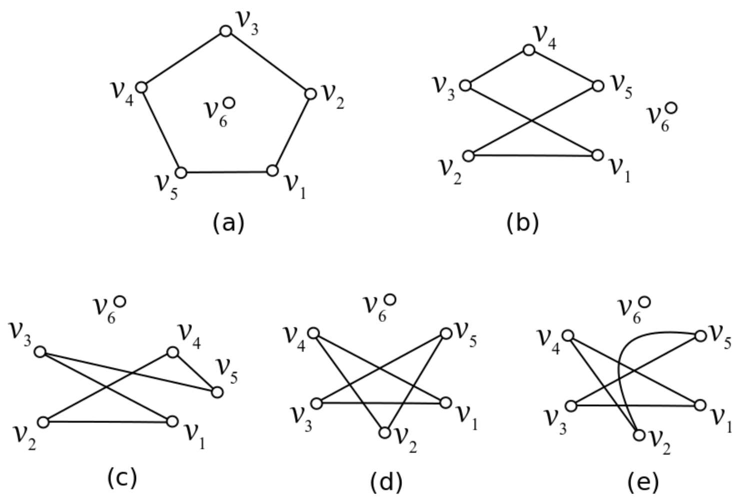

FIGURE 1. Five possible drawings of the graph G.

the first position. Thus, the configurations A_1, A_2, A_3, A_4, and A_5 are represented by the cyclic permutations (123456), (123645), (162345), (123465), and (126345), respectively. Of course, in a fixed drawing of the graph $G + D_n$, some configurations from $\mathcal{M}$ need not appear. We denote by $\mathcal{M}_D$ the subset of $\mathcal{M} = \{A_1, A_2, A_3, A_4, A_5\}$ consisting of all configurations that exist in the drawing D.

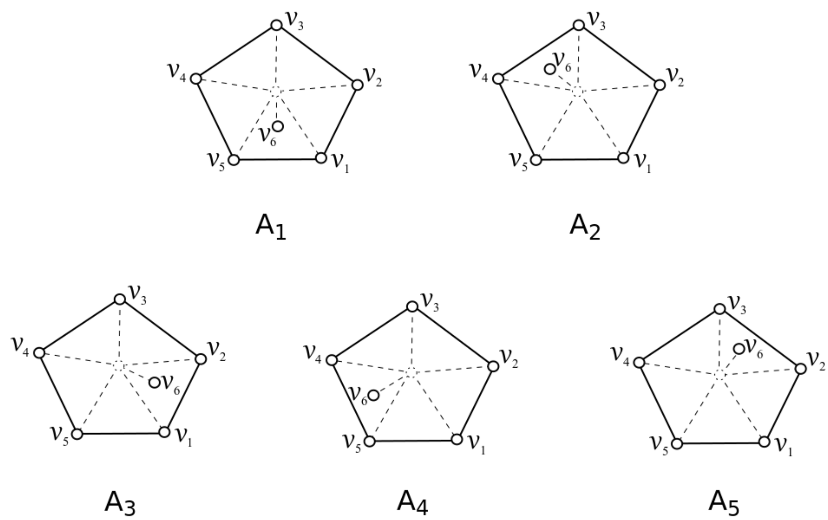

FIGURE 2. Drawings of five possible configurations of the subgraph F^i.

We remark that if two different subgraphs F^i and F^j with configurations from $\mathcal{M}_D$ cross in a drawing D of $G + D_n$, then only the edges of T^i cross the edges of T^j. Thus, we will deal with the minimum numbers of crossings between two different subgraphs F^i and F^j depending on their configurations. Let X, Y be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(X, Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X,Y) = \min\{\mathrm{cr}_D(X,Y)\}$ over all good drawings of the graph $G+D_n$ with $X, Y \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(X,Y)$ for all pairs $X, Y \in \mathcal{M}$.

Let $\overline{P_i}$ denotes the inverse cyclic permutation to the permutation P_i, for $i = 1, \dots, 120$, where the list with the short names of $6!/6 = 120$ cyclic permutations of six elements were collected in Table 1 of [15]. Woodall [20] have been defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (i. e. an ordered set with 6 elements). Hence, if $d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\mathrm{rot}_D(t_j)\text{"})$ denotes the distance between two vertices which correspond to the cyclic permutations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$ in the graph COG, then

(2.2) $$\mathrm{cr}_D(T^i, T^j) \geq Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\overline{\mathrm{rot}_D(t_j)}\text{"})$$

for any two different subgraphs T^i and T^j, where $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ as the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$ have been already defined in [15].

Now, we are ready to find the necessary numbers of crossings between subgraphs T^i and T^j for the corresponding configurations of F^i and F^j from $\mathcal{M}$. The configurations A_1 and A_2 are represented by the cyclic permutations $P_1 = (123456)$ and $P_{31} = (123645)$, respectively. Since $\overline{P_{31}} = (154632) = P_{116}$, we have $\mathrm{cr}(A_1, A_2) \geq 4$ using of $d_{COG}(\text{"}P_1\text{"}, \text{"}P_{116}\text{"}) = 4$. The same reason gives $\mathrm{cr}(A_1, A_3) \geq 5$, $\mathrm{cr}(A_1, A_4) \geq 5$, $\mathrm{cr}(A_1, A_5) \geq 4$, $\mathrm{cr}(A_2, A_3) \geq 4$, $\mathrm{cr}(A_2, A_4) \geq 5$, $\mathrm{cr}(A_2, A_5) \geq 5$, $\mathrm{cr}(A_3, A_4) \geq 4$, $\mathrm{cr}(A_3, A_5) \geq 5$, and $\mathrm{cr}(A_4, A_5) \geq 4$. Clearly, also $\mathrm{cr}(A_i, A_i) \geq 6$ for any $i = 1, \dots, 5$. Thus, all lower-bounds of number of crossing of two configurations from $\mathcal{M}$ are summarized in the symmetric Table 1 (here, A_k and A_l are configurations of the subgraphs F^i and F^j, where $k, l \in \{1, 2, 3, 4, 5\}$).

$-$	A_1	A_2	A_3	A_4	A_5
A_1	6	4	5	5	4
A_2	4	6	4	5	5
A_3	5	4	6	4	5
A_4	5	5	4	6	4
A_5	4	5	5	4	6

TABLE 1. The necessary number of crossings between T^i and T^j for the configurations A_k, A_l.

Assume a good drawing D of the graph $G + D_n$ with at least one crossing among edges of the graph G (in which there is a subgraph $T^i \in R_D$). In this case, without loss of generality, we can choose the vertex notations of the graphs in such a way as shown in Fig. 1(b), (c), (d), and (e). In all mentioned cases, we are able to use the same idea as above, i.e., we obtain the same configurations, and also the same corresponding lower-bounds of numbers of crossings between two configurations as in Table 1.

3. THE CROSSING NUMBER OF $G + D_n$

Two vertices t_i and t_j of $G + D_n$ are *antipodal* in a drawing of $G + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. In the proof of the main theorem, the following lemma related to some restricted drawing of the graph $G + D_2$ is needful.

Lemma 3.1. $\mathrm{cr}(G + D_2) = 1$.

Proof. In Fig. 3(a) it is easy to see that $\mathrm{cr}(G + D_2) \leq 1$. Thus, it remains to prove the reverse inequality. Let us suppose that there is a drawing D of the graph $G + D_2$ with no crossing. Then $\mathrm{cr}_D(G) = 0$, and the edges of both subgraphs T^1, T^2 do not cross the edges of G, i.e., $T^1, T^2 \in R_D$. Hence, the positive values in Table 1 force a contradiction. □

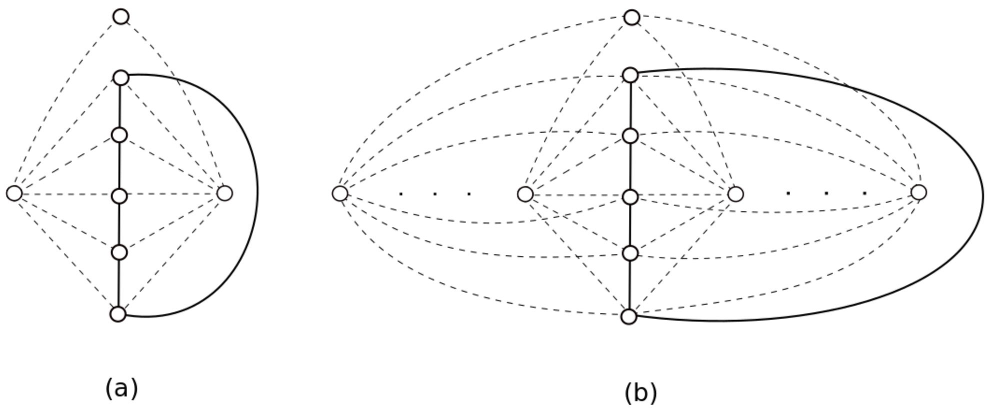

FIGURE 3. The good drawings of $G + D_2$ and of $G + D_n$.

Now we are able to prove the main results of the paper.

Theorem 3.1. $\mathrm{cr}(G + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

Proof. In Fig. 3(b) there is the drawing of $G + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. We prove the reverse inequality by induction on n. The graph $G + D_1$ is planar; hence, $\mathrm{cr}(G + D_1) = 0$. By Lemma 3.1, the result is true for $n = 2$. Suppose now that for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \tag{3.3}$$

and let

$$\mathrm{cr}(G + D_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \qquad \text{for any integer } m < n. \tag{3.4}$$

Let us first show that the considered drawing D must be antipodal-free. As a contradiction, suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using positive values in Table 1, one can easily verify that both subgraphs T^n and T^{n-1} are not from the set R_D, i.e., $\mathrm{cr}_D(G, T^n \cup T^{n-1}) \geq 1$. The known fact that $\mathrm{cr}(K_{6,3}) = 6$ implies that any T^k, $k = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D\,(G + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n)$$
$$+\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 6(n-2) + 1 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

This contradiction with the assumption (3.3) confirms that D must be an antipodal-free drawing. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (3.4) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor$ imply that, in D, there are at least $\left\lceil \frac{n}{2}\right\rceil + 1$ subgraphs T^i which do not cross the edges of G. More precisely

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{6,n}) \leq \mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) < \left\lfloor \frac{n}{2}\right\rfloor ,$$

i.e.,

$$s + 2(n-r-s) < \left\lfloor \frac{n}{2}\right\rfloor . \tag{3.5}$$

This forces that $r \geq \left\lceil \frac{n}{2}\right\rceil + 1 \geq 3$. Now, for $T^i \in R_D$, we will discuss the existence of possible configurations of subgraphs $F^i = G \cup T^i$ in the drawing D. Moreover, if $n = 3$ then $r = 3$, and $\mathrm{cr}_D(G + D_3) \geq \mathrm{cr}_D(T^1 \cup T^2 \cup T^3) \geq 12$ holds by summing of three minimal values of Table 1 in all possible drawings of G which are presented in Fig. 1. This contradiction with the assumption (3.3) confirms that $n \geq 4$.

Case 1: $\mathrm{cr}_D(G) = 0$.

Without loss of generality, we can choose the vertex notation of the graph G in such a way as shown in Fig. 1(a). Thus, we will deal with the configurations belonging to the nonempty set $\mathcal{M}_D$, i.e., we will discuss over all cardinalities of the set $\mathcal{M}_D$ in the following subcases:

a) $|\mathcal{M}_D| \geq 3$.
We will consider two subcases. Let us first assume that $\{A_i, A_j, A_k\} \subseteq \mathcal{M}_D$ with $i+2 \equiv j+1 \equiv k \pmod 5$. Without lost of generality, let us consider three different subgraphs T^{n-2}, T^{n-1}, $T^n \in R_D$ such that F^{n-2}, F^{n-1} and F^n have mentioned configurations A_i, A_j and A_k, respectively. Then $\mathrm{cr}_D(G \cup T^{n-2} \cup T^{n-1} \cup T^n, T^m) \geq 14$ for any $T^m \in R_D$ with $m \neq n-2, n-1, n$ by summing the values in all columns in the considered three rows of Table 1. Moreover, $\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n, T^m) \geq 4$ for any subgraph $T^m \notin R_D$ provided by there is no permutation P_l for some $l \in \{1, \ldots, 120\}$ with $Q\left(\mathrm{rot}_D(t_{n-2}), P_l\right) = Q\left(\mathrm{rot}_D(t_{n-1}), P_l\right) = Q\left(\mathrm{rot}_D(t_n), P_l\right) = 1$, for more see (2.2). Since $\mathrm{cr}_D(T^{n-2} \cup T^{n-1} \cup T^n) \geq 13$ holds by summing of three corresponding values of Table 1 between the mentioned configurations A_i, A_j and A_k, then by fixing the subgraph $G \cup T^{n-2} \cup T^{n-1} \cup T^n$,

$$\mathrm{cr}_D(G + D_n) \geq 6\left\lfloor \frac{n-3}{2}\right\rfloor \left\lfloor \frac{n-4}{2}\right\rfloor + 14(r-3) + 5s + 5(n-r-s) + 13$$
$$= 6\left\lfloor \frac{n-3}{2}\right\rfloor \left\lfloor \frac{n-4}{2}\right\rfloor + 5n + 9r - 29 \geq 6\left\lfloor \frac{n-3}{2}\right\rfloor \left\lfloor \frac{n-4}{2}\right\rfloor$$
$$+5n + 9\left(\left\lceil \frac{n}{2}\right\rceil + 1\right) - 29 > 6\left\lfloor \frac{n}{2}\right\rfloor \left\lfloor \frac{n-1}{2}\right\rfloor + \left\lfloor \frac{n}{2}\right\rfloor .$$

In addition, let us assume that $\mathcal{M}_D = \{A_i, A_j, A_k\}$ with $i+1 \equiv j \pmod 5$, $j+1 \not\equiv k \pmod 5$, and $k+1 \not\equiv i \pmod 5$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have mentioned configurations A_i and A_j, respectively. Then $\mathrm{cr}_D(G \cup T^{n-1} \cup T^n, T^m) \geq 10$ for any $T^m \in R_D$ with $m \neq n-1, n$ by Table 1. Hence, by fixing the subgraph $G \cup T^{n-1} \cup T^n$,

$$\mathrm{cr}_D(G + D_n) \geq 6\left\lfloor \frac{n-2}{2}\right\rfloor \left\lfloor \frac{n-3}{2}\right\rfloor + 10(r-2) + 3s + 4(n-r-s) + 4$$
$$= 6\left\lfloor \frac{n-2}{2}\right\rfloor \left\lfloor \frac{n-3}{2}\right\rfloor + 4n + 5r + (r-s) - 16 \geq 6\left\lfloor \frac{n-2}{2}\right\rfloor \left\lfloor \frac{n-3}{2}\right\rfloor$$

$$+4n+5\left(\left\lceil\frac{n}{2}\right\rceil+1\right)+0-16>6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

b) $|\mathcal{M}_D|=2$, i.e., $\mathcal{M}_D=\{A_i,A_j\}$ for some $i,j\in\{1,\dots,5\}$ with $i\neq j$.
Without lost of generality, let us consider two different subgraphs $T^{n-1},\ T^n\in R_D$ such that F^{n-1} and F^n have mentioned configurations A_i and A_j, respectively. Then $\mathrm{cr}_D(G\cup T^{n-1}\cup T^n,T^k)\geq 6+4=10$ for any $T^k\in R_D$ with $k\neq n-1,n$ by Table 1. Hence, by fixing the subgraph $G\cup T^{n-1}\cup T^n$, we are able to use the same inequalities as in the previous subcase.

c) $|\mathcal{M}_D|=1$, i.e., $\mathcal{M}_D=\{A_j\}$ for some $j\in\{1,\dots,5\}$.
Without lost of generality, let us assume that $T^n\in R_D$ with the configuration $A_j\in\mathcal{M}_D$ of the subgraph F^n for some $j\in\{1,\dots,5\}$. Hence, by fixing the subgraph $G\cup T^n$,

$$\mathrm{cr}_D(G+D_n)\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+6(r-1)+2s+3(n-r-s)+0$$

$$=6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+2r+(r-s)-6\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$$

$$+3n+2\left(\left\lceil\frac{n}{2}\right\rceil+1\right)+0-6>6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

Case 2: $\mathrm{cr}_D(G)\geq 1$.
In all considered cases, we can choose the vertex notations of the graph G in such a way as shown in Fig. 1(b), (c), (d) or (e). According to $r\geq 1$, there is a subgraph $T^i\in R_D$. Without lost of generality, we can also assume that $T^n\in R_D$ with the configuration A_j of the subgraph $G\cup T^n=F^n$ for some $j\in\{1,\dots,5\}$. Since there is no region with at least four vertices of G on its boundary (in the subdrawing of F^n), then there is no subgraph $T^k\in S_D$ with $\mathrm{cr}_D(T^n,T^k)=1$, i.e., $\mathrm{cr}_D(G\cup T^n,T^k)\geq 3$ for any $T^k\in S_D$. Of course, $\mathrm{cr}_D(G\cup T^n,T^k)\geq 2+1=3$ for any $T^k\notin R_D\cup S_D$. Hence, by fixing the subgraph $G\cup T^n$,

$$\mathrm{cr}_D(G+D_n)\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4(r-1)+3s+3(n-r-s)+1=6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$$

$$+3n+r-3\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+\left(\left\lceil\frac{n}{2}\right\rceil+1\right)-3>6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor.$$

Thus, it was shown that there is no good drawing D of the graph $G+D_n$ with less than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$ crossings. This completes the proof of the main theorem. □

4. Two Other Graphs

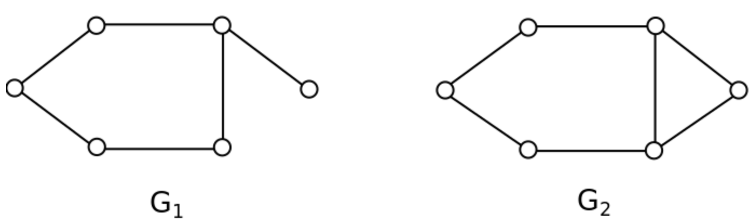

FIGURE 4. Two graphs G_1 and G_2 by adding new edges to the graph G.

In Fig. 3(b) we are able to add some edges to the graph G without additional crossings. So the drawing of the graphs $G_1 + D_n$ and $G_2 + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings is obtained. Thus, the next results are obvious.

Corollary 4.1. $\mathrm{cr}(G_i + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$, *where* $i = 1, 2$.

Remark that the crossing numbers of the graph $G_2 + D_n$ was obtained in [8] without using the vertex rotation.

5. The join product with paths

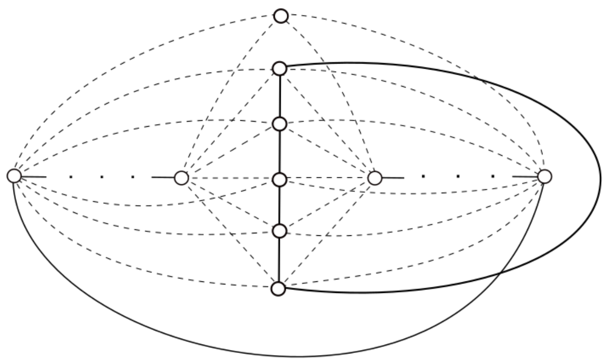

FIGURE 5. The good drawing of $G + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings.

Theorem 5.2. $\mathrm{cr}(G + P_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

Proof. In Fig. 5 there is the drawing of $G + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings. Thus, $\mathrm{cr}(G + P_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$. We prove the reverse inequality by induction on n. The graph $G + P_2$ contains a subdivision of the graph $(C_4 \cup \{v\}) + P_2$. It was proved in [19] that $\mathrm{cr}\left((C_4 \cup \{v\}) + P_2\right) = 2$. Thus, the result is true for $n = 2$. Suppose now that for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + P_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{5.6}$$

and let

$$\mathrm{cr}_D(G + P_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 1 \qquad \text{for any integer } m < n. \tag{5.7}$$

As the graph $G + P_n$ contains $G + D_n$ like a subgraph, by Theorem 3.1, $\mathrm{cr}_D(G + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, and therefore, no edge of the path P_n is crossed in D. Let us first show that the considered drawing D must be antipodal-free. As a contradiction, suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using positive values in Table 1, one can also easily verify that both subgraphs T^{n-1} and T^n are not from the set R_D, i.e., $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 1$. The known fact $\mathrm{cr}(K_{6,3}) = 6$ implies that any T^k, $k = 1, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G{+}P_n) = \mathrm{cr}_D\,(G + P_{n-2}){+}\mathrm{cr}_D(T^{n-1}{\cup}T^n){+}\mathrm{cr}_D(K_{6,n-2}, T^{n-1}{\cup}T^n){+}\mathrm{cr}_D(G, T^{n-1}{\cup}T^n)$$
$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + 1 + 0 + 6(n-2) + 1 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

This contradiction with the assumption (5.6) forces that D must be an antipodal-free drawing. Moreover, our assumption on D together with $\mathrm{cr}(K_{6,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ also imply that, in D, there are at least $\left\lceil\frac{n}{2}\right\rceil$ subgraphs T^i which do not cross the edges of G. More precisely

$$\mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) \le \left\lfloor\frac{n}{2}\right\rfloor. \tag{5.8}$$

This forces that $r \ge \left\lceil\frac{n}{2}\right\rceil \ge 2$, and $2r+s \ge 2n - \left\lfloor\frac{n}{2}\right\rfloor$. Now, for $T^i \in R_D$, we will discuss the existence of possible configurations of subgraphs $F^i = G \cup T^i$ in the drawing D.

Case 1: $\mathrm{cr}_D(G) = 0$.
Without loss of generality, we can choose the vertex notation of the graph G in such a way as shown in Fig. 1(a). We will discuss two possibilities over congruence n modulo 2.

- Let n be even, and let us also assume that $T^n \in R_D$ with the configuration $A_j \in \mathcal{M}_D$ of the subgraph F^n for some $j \in \{1,\dots,5\}$. Since no edge of the path P_n is crossed in D, then $\mathrm{cr}_D(G\cup T^n, T^k) \ge 3$ for any $T^k \in S_D$, see Fig. 2. Hence, by fixing the subgraph $G \cup T^n$,
$$\mathrm{cr}_D(G+P_n) \ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(r-1) + 3s + 3(n-r-s) + 0$$
$$= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + r - 4 \ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + \left\lceil\frac{n}{2}\right\rceil - 4$$
$$= 6\,\frac{n-2}{2}\,\frac{n-2}{2} + 3n + \frac{n}{2} - 4 > 6\,\frac{n}{2}\,\frac{n-2}{2} + \frac{n}{2}.$$
- Let n be odd, and let us also consider two different subgraphs $T^{n-1}, T^n \in R_D$ with some configurations from $\mathcal{M}_D$ of the subgraphs F^{n-1} and F^n (of course, the mentioned configurations can be same). Since no edge of the path P_n is crossed in D, then $\mathrm{cr}_D(G\cup T^{n-1}, T^k) \ge 3$ and $\mathrm{cr}_D(G\cup T^n, T^k) \ge 3$ for any $T^k \in S_D$, see Fig. 2. Moreover, $\mathrm{cr}_D(G \cup T^{n-1}\cup T^n, T^k) \ge 8$ for any $T^k \in R_D$ with $k \ne n-1, n$ by summing the values in all columns in the considered two rows of Table 1. Hence, by fixing the subgraph $G \cup T^{n-1} \cup T^n$,
$$\mathrm{cr}_D(G+P_n) \ge 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 8(r-2) + 5s + 4(n-r-s) + 4 = 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$
$$+4n + 2r + 2r + s - 12 \ge 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 4n + 2\left\lceil\frac{n}{2}\right\rceil + 2n - \left\lfloor\frac{n}{2}\right\rfloor - 12$$
$$= 6\,\frac{n-3}{2}\,\frac{n-3}{2} + 4n + 2\,\frac{n+1}{2} + 2n - \frac{n-1}{2} - 12 > 6\,\frac{n-1}{2}\,\frac{n-1}{2} + \frac{n-1}{2}.$$

Case 2: $\mathrm{cr}_D(G) \ge 1$.
In all considered cases, we can also choose the vertex notations of the graph G in such a way as shown in Fig. 1(b), (c), (d) or (e). Since we are able to use the same arguments like in Case 2 in the proof of Theorem 3.1, then by fixing the subgraph $G \cup T^n$ with $T^n \in R_D$,
$$\mathrm{cr}_D(G+P_n) \ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(r-1) + 3s + 3(n-r-s) + 1$$
$$= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + r - 3 \ge 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + \left\lceil\frac{n}{2}\right\rceil - 3 > 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

These contradictions with the assumption of less than $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 1$ crossings in D completes the proof. $\square$

The crossing number of the graph $G_2 + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings was established in [8]. Since $G_1 + P_n$ is a subgraph of $G_2 + P_n$ and $G + P_n$ is a subgraph of $G_1 + P_n$, the next result is also obvious.

Corollary 5.2. $\mathrm{cr}(G_1 + P_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

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