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Dedicated to Prof. Juan Nieto on the occasion of his 60^{th} anniversary

Convergence results for fixed point iterative algorithms in metric spaces

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ABSTRACT.
Let (X, d) be a metric space, $f, f_n : X \to X$, with $F_f = F_{f_n}, n \in \mathbb{N}$. For the fixed point equation

(1) $$x = f(x)$$

we consider the following iterative algorithm,

(2) $$x \in X,\ x_0 = x,\ x_{n+1}(x) = f_n(x_n(x)),\ n \in \mathbb{N}.$$

By definition, the algorithm (2) is convergent if,

$$x_n(x) \to x^*(x) \in F_f \text{ as } n \to \infty,\ \forall\, x \in X.$$

In this paper we give some conditions on f_n and f which imply the convergence of algorithm (2). In this way we improve some results given in [Rus, I. A., *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations*, Fixed Point Theory, **13** (2012), No. 1, 179–192]. In our results, in general we do not suppose that, $F_f \neq \emptyset$. Some research directions are formulated.

REFERENCES

[1] Alber, Y., Reich, S. and Yao, J.-C., *Iterative methods for solving fixed point problems with nonself-mappings in Banach spaces*, Abstract Appl. Anal., **4** (2003), 193–216
[2] Aoyama, K., Eshita, K. and Takahashi, W., *Iteration processes for nonexpansive mappings in convex metric spaces* In: Proc. Int. Conf. Nonlinear Anal. Convex Anal., Okinava, 2005, 31–39
[3] Bachar, M., Dehaish, B. A. B. and Khamsi, M. A., *Approximate Fixed Points* In: Fixed Point Theory and Graph Theory, 99-138, Elsevier, 2016
[4] Bauschke, H. H. and Borwein, J. M., *On projection algorithms for solving convex feasibility problems*, SIAM Review, **38** (1996), No. 3, 367–426
[5] Berinde, V., *Iterative Approximation of Fixed Points*, Springer, 2007
[6] Berinde, V., *Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator*, Carpathian J. Math., **29** (2013), No. 1, 9–18
[7] Berinde, V., Khan, A. R. and Păcurar, M., *Convergence theorems for admissible perturbations of pseudocontractive operators*, Miskolc Math. Notes, **15** (2014), No. 2, 27–37
[8] Berinde, V., Măruşter, Şt. and Rus, I. A., *An abstract point of view on iterative approximation of fixed points of nonself operators*, J. Nonlinear Convex Anal., **15** (2014), No. 5, 851–865
[9] Berinde, V., Păcurar, M. and Rus, I. A., *From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators*, Carpathian J. Math., **30** (2014), No. 3, 283–292
[10] Berinde, V., Petruşel, A., Rus, I. A. and Şerban, M. A., *The retraction-displacement condition in the theory of fixed point equation with a convergent iterative algorithm*, In: Mathematical Analysis, Approximation Theory and Their Applications, Springer, 2016, 75–106
[11] Berinde, V. and Rus, I. A., *Caristi-Browder operator theory in distance spaces*, In: Fixed Point Theory and Graph Theory, 1-28, Elsevier, 2016

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[12] Borwein, J., Reich, S. and Shafrir, I., *Krasnoselski-Mann iterations in normed spaces*, Canad. Math. Bull., **35** (1992), No. 1, 21–28

[13] Browder, F. E., *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Zeitschr.,**100** (1967), 201–225

[14] Browder, F. E. and Petryshyn, W. V., *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72** (1966), No. 3, 571–575

[15] Browder, F. E. and Petryshyn, W. V., *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), No. 2, 197–228

[16] Bruck, R. E., *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32** (1979), 107–116

[17] Bruck, R. E., *Random products of contractions in metric and Banach spaces*, J. Math. Anal. Appl., **88** (1982), 319–332

[18] Bruck, R. E., *Asymptotic behavior of nonexpansive mappings*, Contemporary Math., **18** (1983), 1–47

[19] Bunlue, N. and Suantai, S., *Convergence theorems of fixed point iterative methods defined by admissible functions*, Thai J. Math., **13** (2015), No. 3, 527–537

[20] Ceng, L.-C., Petruşel, A., Yao, J.-C. and Yao, Y., *Hybrid viscosity extragradient method for systems of variational inequalities, fixed point of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19** (2018), No. 2, 487–502

[21] Chaoha, P. and Chanthorn, P. C., *Fixed point sets through iterationi schemes*, J. Math. Anal. Appl., **386** (2012), 273–277

[22] Chidume, C., *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer, 2009

[23] Chidume, C. E. and Măruşter, Şt., *Iterative methods for the computation of fixed points of demicontractive mappings*, J. Comput. Appl. Math., **234** (2010), 861–882

[24] Datson, W. G., *Fixed points of quasinonexpansive mappings*, J. Austral. Math. Soc., **13** (1972), 167–170

[25] Edelstein, M., *A remark on a theorem of M.A. Krasnoselski*, Amer. Math. Monthly, **73** (1966), 509–510

[26] Eldred, A. A. and Praveen, A., *Convergence of Mann's iteration for relatively nonexpansive mappings*, Fixed Point Theory, **18** (2017), No. 2, 545–554

[27] Glăvan, V., *Private communication*, 2012

[28] Goebel, K. and Kirk, W. A., *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, 1990

[29] Goebel, K. and Reich, S., *Uniform convexity, Hyperbolic Geometry and Nonexpansive Mapping*, Marcel Dekker, 1984

[30] Hicks, T. L. and Kubicek, J. D., *On the Mann iteration process in a Hilbert space*, J. Math. Anal. Appl., **59** (1977), 498–504

[31] Ishikawa, S., *Fixed point and iteration of a non-expansive mapping in a Banach space*, Proc. Amer. Math. Soc., **59** (1976), 65–71

[32] Kirk, W. A., *Krasnoselskii's iteration process in hyperbolic space*, Num. Funct. Anal. Optimiz., **4** (1981-82), 371–381

[33] Kirk, W. A., *Approximate fixed points of nonexpansive maps*, Fixed Point Theory, **10** (2009), No. 2, 275–288

[34] Kohlembach, U., *Some computational aspect of metric fixed point theory*, Nonlinear Anal., **61** (2005), 823–837

[35] Krichen, B. and O'Regan, D., *On the class of relatively weakly demicompact nonlinear operators*, Fixed Point Theory, **19** (2018), No. 2, 625–630

[36] Latif, A., Alofi, A. S. M., Al-Mazroofi, A. E. and Yao, J.-C., *General composite iterative methods for general systems of variational inequalities*, Fixed Point Theory, **19** (2018), No. 1, 287–300

[37] Leuştean, L., *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, Contemporary Math., **513** (2010), 193–209

[38] Lin, L.-J. and Takahashi, W., *Attractive point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanesse J. Math., **16** (2012), No. 5, 1763–1779

[39] Liu Z., Feng, C., Kang, S. M. and Ume, J. S., *Approximating fixed points of nonexpansive mappings in hyperspaces*, Fixed Point Theory and Appl., 2007, ID50596, 9 pp.

[40] Măruşter, Şt., *The solution by iteration of nonlinear equations in Hilbert spaces*, Proc. Amer. Math. Soc., **63** (1977), No. 1, 69–73

[41] Măruşter, Şt. and Rus, I. A., *Kannan contractions and strongly demicontractive mappings*, Creative Math. Inf., **24** (2015), No. 2, 171–180

[42] Ortega, J. M. and Rheinboldt, W. C., *Iterative Solution of Nonlinear Equation in Several Variables*, Acad. Press, New York, 1970

[43] Petruşel, A. and Rus, I. A., *An abstract point of view on iterative approximation schemes of fixed points for multivalued operators*, J. Nonlinear Sci. Appl., **6** (2013), 97–107

[44] Petruşel, A., Rus, I. A. and Şerban, M. A., *Nonexpansive operators as graphic contractions*, J. Nonlinear Convex Anal., **17** (2016), No. 7, 1409–1415

[45] Petryshyn, W. V., *Construction of fixed points of demicompact mappings in Hilbert space*, J. Math. Anal. Appl., **14** (1966), 276–284

[46] Petryshyn, W. V. and Williamson, T. E., *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl., **43** (1973), 459–497

[47] Roux, D., *Applicazioni quasi non expansive: approssimazione dei punti fissi*, Rendiconti di Matematica, **10** (1977), 597–605

[48] Rus, I. A., *On a theorem of Dieudonné*, (V. Barbu, Ed.), Diff. Eq. and Control Theory, Longmann, 1991

[49] Rus, I. A., *Weakly Picard mappings*, Comment. Mat. Univ. Carolinae, **34** (1993), No. 4, 769–773

[50] Rus, I. A., *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001

[51] Rus, I. A., *Picard operators and applications*, Sci. Math. Jpn., **58** (2003), 191–219

[52] Rus, I. A., *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl., **292** (2004), No. 1, 259–261

[53] Rus, I. A., *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006

[54] Rus, I. A., *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations*, Fixed Point Theory, **13** (2012), No. 1, 179–192

[55] Rus, I. A., *Properties of the solutions of those equations for which the Krasnoselski iteration converges*, Carpathian J. Math., **28** (2012), No. 2, 329–336

[56] Rus, I. A., *Relevant clases of weakly Picard operators*, Analele Univ. Vest Timişoara, Mat. Inf., **54** (2016), No. 2, 3–19

[57] Rus, I. A., *Some problems in the fixed point theory*, Advances in the Theory of Nonlinear Analysis and its Applications,**2** (2018), No. 1, 1–10

[58] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*, Cluj Univ. Press, Cluj-Napoca, 2008

[59] Schott, D., *Basic properties of Fejer monotone sequences*, Rostock Math. Kolloq., **49** (1995), 57–74

[60] Senter, H. F. and Datson, W. G., *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44** (1974), No. 2, 375–380

[61] Shih, M.-H. and Takahashi, W., *Positive stochastic matrices as contraction maps*, J. Nonlinear Convex Anal., **14** (2013), No. 4, 649–650

[62] Singh, S. P. and Watson, B., *On convergence results in fixed point theory*, Rend. Sem. Mat. Univ. Politec. Torino, **51** (1993), No. 2, 73–91

[63] Smale, S., *On the efficiency of algorithms of analysis*, Bull. Amer. Math. Soc., **13** (1985), 87–121

[64] Şerban, M. A., *Fiber contraction principle with respect to an iterative algorithm*, J. Operators, 2013, ID408791, 6 pp.

[65] Takahashi, W., *A convexity in metric space and nonexpansive mappings*, Kodai Math. Sem. Rep., **22** (1970), 142–149

[66] Takahashi, W., *Nonlinear Functional Analysis. Fixed Point Theory and Applications*, Yokohama Publ., Yokohama, 2000

[67] Thole, R. L., *Iterative techniques for approximation of fixed points of certain nonlinear mappings in Banach spaces*, Pacific J. Math., **53** (1974), 259–266

[68] Timiş, I., *New stability results of Picard iteration for contractive type mappings*, Fasciculi Math., 2016, Nr. 56, DOI: 10.1515

[69] Toscano, E. and Vetro, C., *Admissible perturbations of α-ψ-pseudocontractive operators: convergence theorems*, Math. Methods Appl. Sci., **40** (2016), No. 5, 1438–1447

[70] Toscano, E. and Vetro, C., *Fixed point iterative schemes for variational inequality problems*, J. Convex Anal., **25** (2018), No. 2, 701–715

[71] Tricomi, F., *Un teorema sulla convergenza delle successioni formate delle successive iterate di una fuzione di una variabile reale*, Giorn. Mat. Battoglini, **54** (1916), 1–9

[72] Ţicală, C., *Approximating fixed points of asymptotically demicontractive mapping by iterative schemes defined as admissible perturbations*, Carpathian J. Math., **33** (2017), No. 3, 381–388

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